NORTH SYDNEY GIRLS HIGH SCHOOL

2015
TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in questions 11–16

Total Marks – 100

Section I Pages 3–8

10 Marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II Pages 9–19

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

NAME: ____________________ **TEACHER** ____________________

NUMBER: ____________________

QUESTION	MARK
1 – 10	/10
11	/15
12	/15
13	/15
14	/15
15	/15
16	/15
TOTAL	/100

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Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1. What is the remainder when the polynomial $z^3 - 2z^2 + 1$ is divided by $z + i$?

(A) $3 + i$

(B) $3 - i$

(C) $-1 + i$

(D) $-1 - i$

2. Which of the following is a focus of the ellipse $x^2 + 16y^2 = 4$?

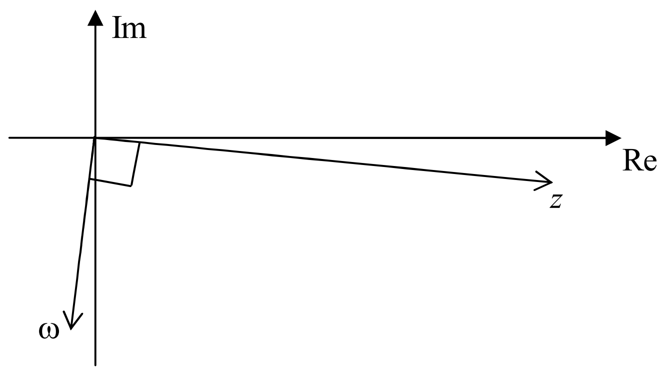
(A) $\left(\sqrt{15},\, 0\right)$

(B) $\left(\dfrac{\sqrt{15}}{2},\, 0\right)$

(C) $\left(\dfrac{\sqrt{255}}{4},\, 0\right)$

(D) $\left(\dfrac{\sqrt{255}}{8},\, 0\right)$

3. Consider the following diagram, drawn to scale, showing the complex numbers z and ω.



Which statement is false?

(A) $|z+\omega| = |z-\omega|$

(B) $\mathrm{Re}\left(\dfrac{\omega}{z}\right) = 0$

(C) $-\pi < \arg(z-\omega) < 0$

(D) $z^2 = k\omega^2$ for some real number k

4. ω is a non-real root of the equation $z^5 + 1 = 0$.

Which of the following is not a root of this equation?

(A) $\overline{\omega}$

(B) ω^2

(C) $\dfrac{1}{\omega}$

(D) ω^3

5. The cubic equation $P(x)=0$ has roots α, β and γ.

Which of the following cubic equations has roots $2\alpha+1$, $2\beta+1$ and $2\gamma+1$?

(A) $P(2x)+1=0$

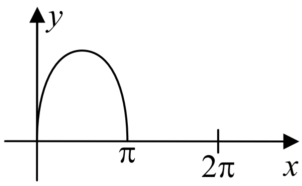
(B) $P(2x+1)=0$

(C) $P\left(\frac{x}{2}-1\right)=0$

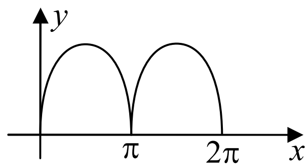
(D) $P\left(\frac{x-1}{2}\right)=0$

6. Which graph best represents the curve $y^2=\sin x$ over the domain $0\le x\le 2\pi$?

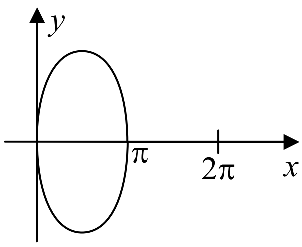
(A)



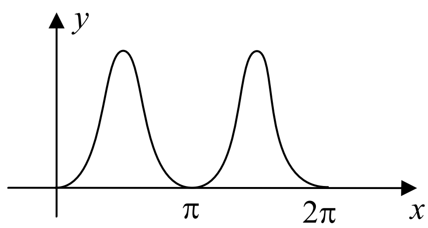
(B)



(C)



(D)



7. Which of the following integrals has the same value as $\displaystyle\int_0^{\sqrt{2}} \frac{x^2\,dx}{\sqrt{4-x^2}}$?

(A) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\cos^2\theta\; d\theta$

(B) $\displaystyle\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 4\cos^2\theta\; d\theta$

(C) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cot\theta\cos\theta\; d\theta$

(D) $\displaystyle\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 2\cot\theta\cos\theta\; d\theta$

8. Consider the equation $\left(\dfrac{2+i}{c}\right)^p = 1$ where c is a real constant and $p \neq 0$

For how many values of c will this equation have real solutions ?

(A) none

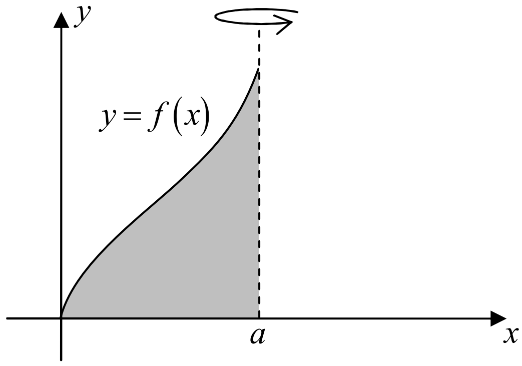
(B) one

(C) two

(D) four

9. The function $y = f(x)$ is monotonic increasing on the interval $0 \le x \le a$.

The region bounded by this function, the x-axis and the line $x = a$ is to be rotated about the line $x = a$ to form a solid of revolution.



Which of the following integrals represents the volume of this solid?

(A) $\pi \int_0^{f(a)} \left[a - f^{-1}(y)\right]^2 dy$

(B) $\pi \int_0^{a} \left[a - f(x)\right]^2 dx$

(C) $\pi \int_0^{a} \left[a - x\right]^2 dx$

(D) $\pi \int_0^{f(a)} \left[a - y\right]^2 dy$

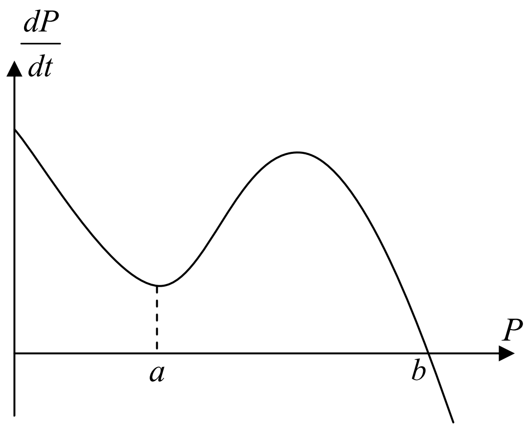
10. The following diagrams show the rates of change $\frac{dP}{dt}$ of four different rabbit populations as functions of the population P.

The values of a and b correspond to roots or turning points of each curve.

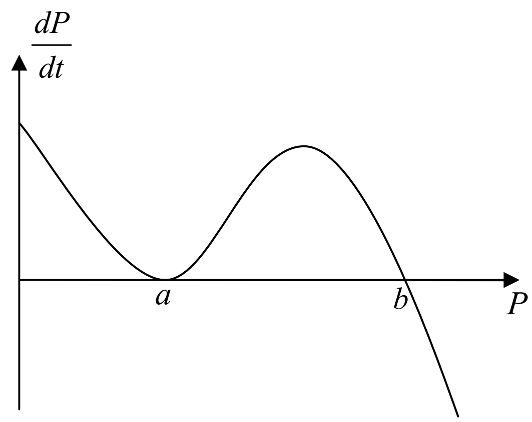
The number of rabbits in each population at a particular moment in time has the same value P_0.

In which rabbit population will the number of rabbits approach a value of b if $P_0 > a$, and a value of a if $P_0 < a$?

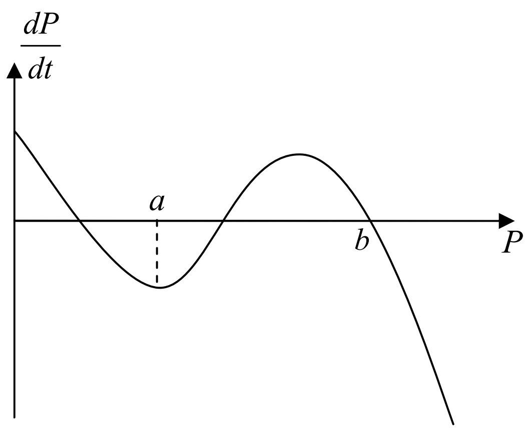
(A)



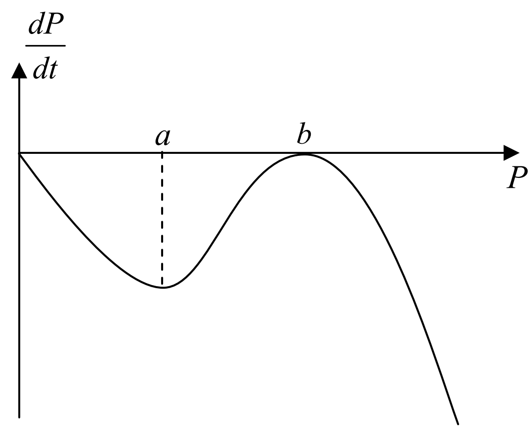
(B)



(C)



(D)



Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 5 - 2i$. Find the value of

(i) $|z|$ **1**

(ii) $\dfrac{\overline{z}}{z}$ in the form $a + ib$ **2**

(b) If $(x+y)^2 = y$, find $\dfrac{dy}{dx}$ in terms of x and y. **2**

(c) Find $\displaystyle\int \frac{dx}{x^2 - 4x + 6}$. **2**

(d) Use the substitution $u = \dfrac{\pi}{2} - x$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \sin 2x}\, dx$. **3**

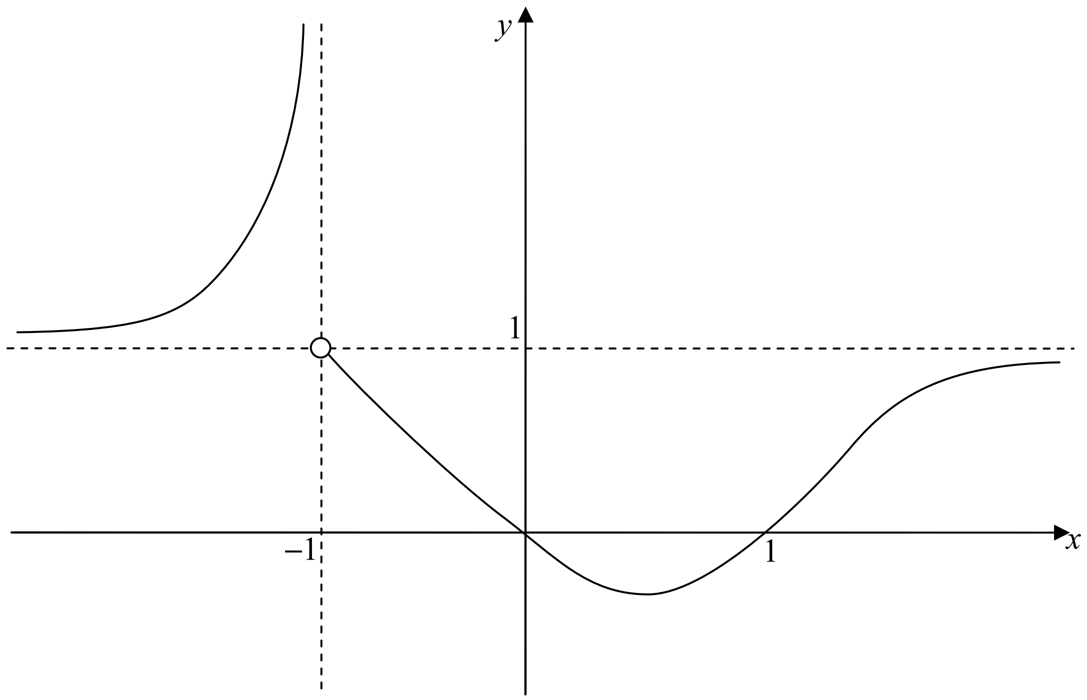
(e) Evaluate $\displaystyle\int_1^{e^2} x^2 \log_e x \, dx$. **2**

(f) Sketch the region in the Argand plane whose points satisfy **3**

$$|z - 3i| < 3 \quad \text{and} \quad |z + 3| > |z - 3i|$$

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows a graph of the function $y = f(x)$.



Draw separate half-page graphs for each of the following functions, showing all important features:

(i) $y = f(|x|)$ **2**

(ii) $y = \log_e f(x)$ **2**

(b) Sketch the hyperbola with parametric equations **3**

$$x = 3\sec\theta$$
$$y = 4\tan\theta$$

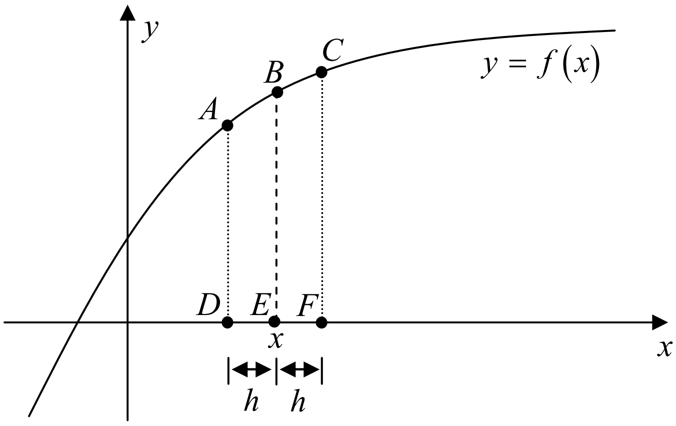
indicating the vertices and foci, and the equations of the directrices and asymptotes.

(c) $1+i$ is a root of the quadratic equation $z^2 + \omega z - i = 0$. **3**

Find the value of ω.

Question 12 continues on page 11

(d) (i) **2**



Use the diagram above to explain why an alternative formula for the derivative is

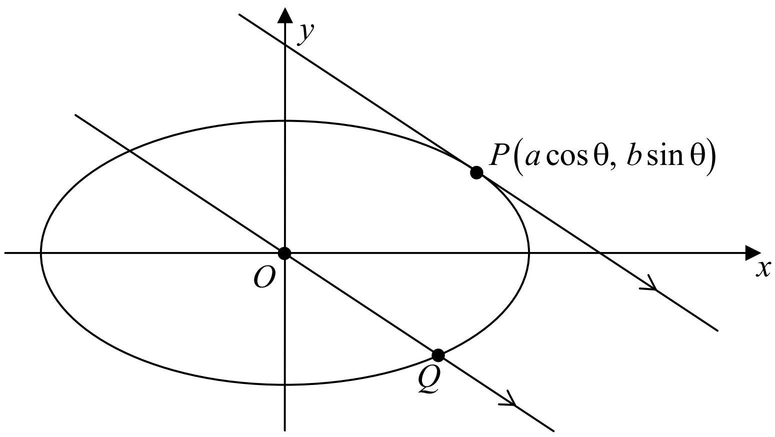
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

(ii) Use this formula to differentiate $f(x) = \tan x$ by first principles. **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) In the diagram, $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, such that $0<\theta<\dfrac{\pi}{2}$. O is the origin, and OQ is parallel to the tangent to the ellipse at P, where Q is a point on the ellipse in the fourth quadrant.



(i) Show that the equation of line OQ is $bx\cos\theta + ay\sin\theta = 0$. **2**

(ii) Show that Q has coordinates $(a\sin\theta, -b\cos\theta)$. **2**

(iii) Find the area of ΔOPQ, and show that this area is independent of the choice of P. **3**

(b) (i) Find $\displaystyle\int_0^1 (x+1)^{-2}\,dx$. **1**

(ii) Find real values a and b such that $\dfrac{x}{(x+1)^2} = \dfrac{a}{x+1} + \dfrac{b}{(x+1)^2}$ for all x. **2**

(iii) Hence find $\displaystyle\int_0^1 \frac{x}{(x+1)^2}\,dx$. **1**

(iv) Given $I_n = \displaystyle\int_0^1 \frac{x^n}{(x+1)^2}\,dx$ for $n \ge 0$, show that **2**

$$I_n + 2I_{n-1} + I_{n-2} = \frac{1}{n-1} \quad \text{for } n \ge 2.$$

(v) Hence evaluate $\displaystyle\int_0^1 \frac{x^3}{(x+1)^2}\,dx$. **2**

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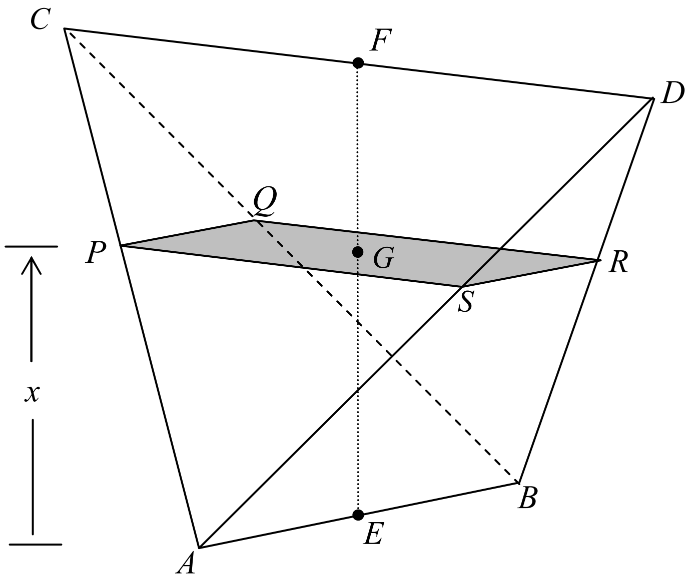
Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) In the following diagram, AB and CD are perpendicular horizontal skew lines. E and F are the midpoints of AB and CD respectively, and EF is vertical.

$EF = h$ cm, $AB = a$ cm and $CD = b$ cm.

$EG = x$ cm where G is the centre of a typical horizontal rectangular slice $PQRS$ of the solid shown, where the solid has vertices A, B, C and D.



(i) Using the fact that PS and SR both vary linearly with x, or otherwise, show that **2**

$$PS = \frac{bx}{h} \text{ and } SR = a\left(1 - \frac{x}{h}\right).$$

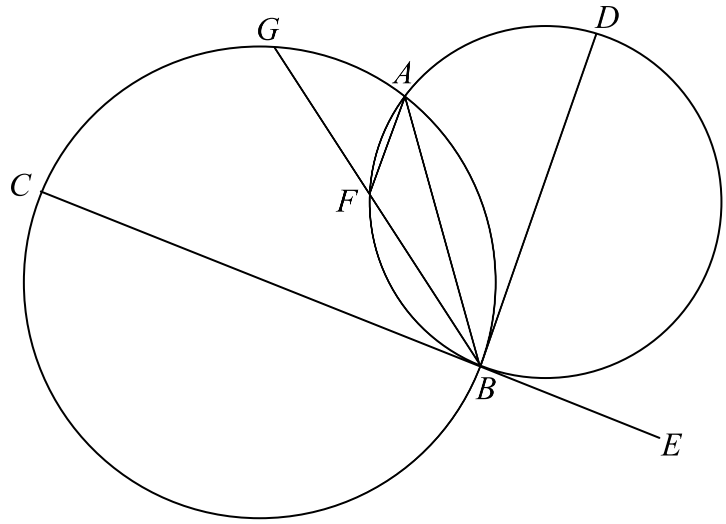
(ii) Hence show that the volume of this solid is given by $V = \dfrac{abh}{6}$. **2**

(b) The polynomial $P(x) = x^5 + 2x^2 + mx + n$ has a double zero at $x = -2$. **3**

Find the product of the other three zeros.

Question 14 continues on page 15

(c) In the diagram, BC and BD are chords (not necessarily diameters) of one circle and tangents of the other. BFG is a straight line.



(i) Prove that $\angle CBA = \angle GFA$ **2**

(ii) Prove that $\angle EBD = \angle FAG$ **2**

(d) Consider the function $f(t) = t - \dfrac{a^2}{t} - 2a\log_e\left(\dfrac{t}{a}\right)$ where $a > 0$.

(i) Evaluate $f(a)$. **1**

(ii) By finding $f'(t)$, show that $f(t) \geq 0$ for $t \geq a$. **3**

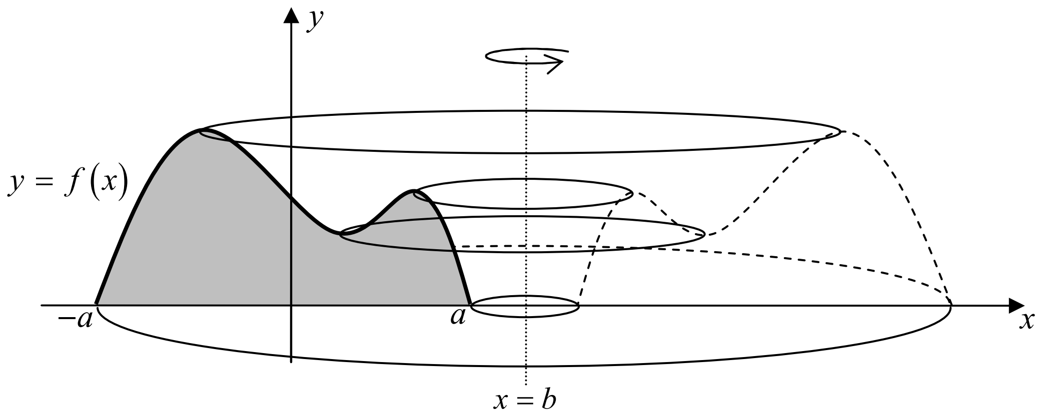
End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The function $f(x)$ is defined over the domain $-a \le x \le a$, and is positive for $-a < x < a$.

The region bounded by the curve $y = f(x)$ and the x-axis is shaded and has area A.

This region is rotated $360°$ about the line $x = b$, where $b > a$, to form a solid of revolution.



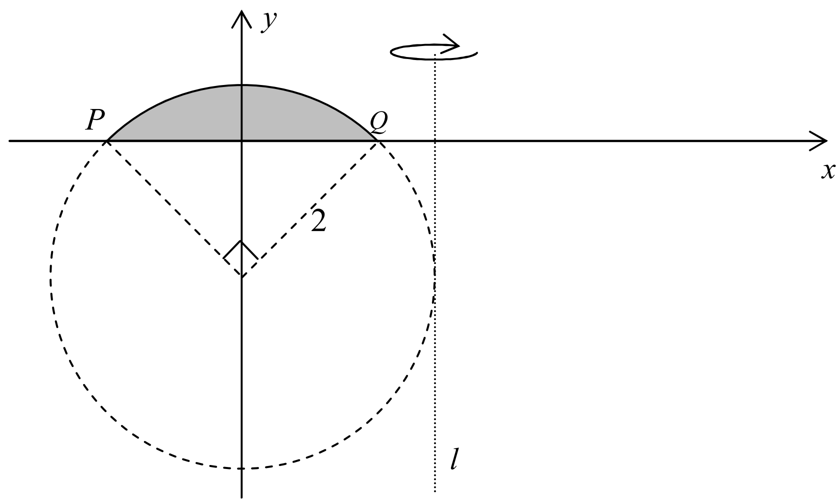
(i) Use the method of cylindrical shells to show that the volume of this solid is given by **3**

$$V = 2\pi bA - 2\pi \int_{-a}^{a} x\, f(x)\, dx$$

(ii) Prove that for any even function $f(x)$, the function $g(x) = x \cdot f(x)$ is odd. **1**

(iii) In the following diagram, the chord PQ lies on the x-axis, and subtends a right angle at the centre of a circle of radius 2 units. **3**

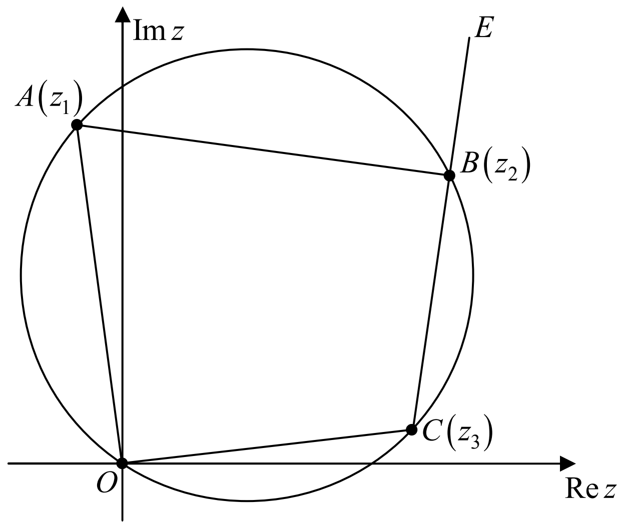
The chord cuts off a minor segment, which is rotated about the vertical tangent l to the circle to form a solid of revolution.



Without finding the equation of the circle, use the results of parts (i) and (ii) to find the volume of the solid formed.

Question 15 continues on page 17

(b) The diagram shows three points A, B and C representing complex numbers z_1, z_2 and z_3 arranged on a circle through the origin O so that O, A, B and C are in clockwise order.



You may assume that the arguments of all complex numbers z lie in the range $0 \le \arg z < 2\pi$.

(i) Explain why $\angle ABE = \arg(z_1 - z_2) - \arg(z_2 - z_3)$. **1**

(ii) Hence show that $z_1(z_2 - z_3) = kz_3(z_1 - z_2)$ where k is a positive real number. **3**

(c) The Fibonacci numbers f_n are defined by:

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \text{ for } n \ge 3$$

(i) Prove by mathematical induction that $f_n < \left(\frac{5}{3}\right)^n$ for all positive integers n. **3**

(ii) Based on your proof in part (i), find the least value of a for which $f_n \le a^n$ for all positive integer values of n. **1**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Using the identity $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$ or otherwise, solve for $0 \le x \le 2\pi$: **2**

$$\sin 3x + \sin 2x = 0$$

(b) (i) Show that if a and b are real, then $a^2 + b^2 \ge 2ab$. **1**

(ii) Hence show that if a, b, c and d are real, then $a^4 + b^4 + c^4 + d^4 \ge 4abcd$. **2**

(iii) Consider the polynomial equation $x^4 + Ax^2 + Bx + C = 0$, where A, B and C are real.

Let the roots of this equation be α, β, γ and δ.

Show that:

1. $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -2A$ **1**

2. $\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = 2A^2 - 4C$. **2**

(iv) Hence show that the equation $x^4 + Ax^2 + Bx + C = 0$ cannot have four real roots if $A^2 < 4C$. **2**

(c) (i) Show that the equation of the normal to the hyperbola $xy = c^2$ at the point $\left(ct, \frac{c}{t}\right)$ is **2**

$$ty - t^3x = c\left(1 - t^4\right)$$

(ii) A point $P(x_0, y_0)$ is chosen randomly in the number plane. **1**

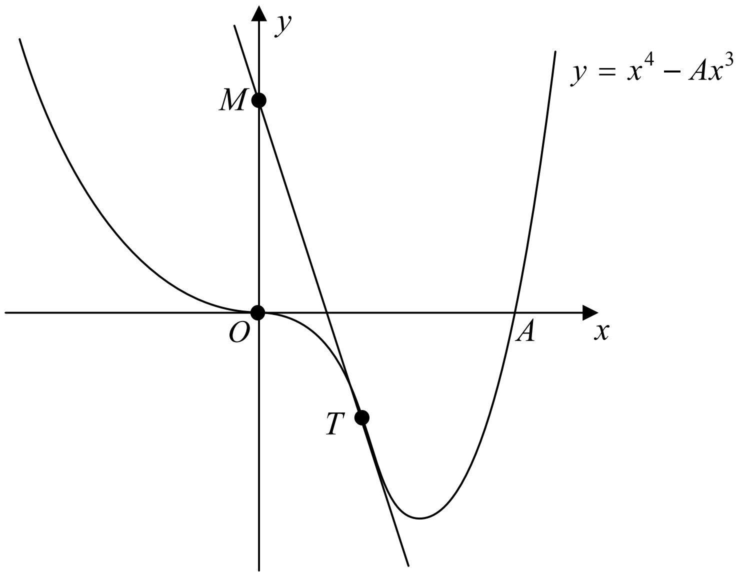
The number of normals which can be drawn to the hyperbola $xy = c^2$ from P depends on the choice of P.

Given an ideal choice of P, what is the greatest number N of such normals which can be drawn to the hyperbola from P ?

Briefly justify your answer.

Question 16 continues on page 19

(iii) Following is the graph of the function $y = x^4 - Ax^3$ for $A > 0$.



O is the origin, T is the other point of inflexion, TM is the inflexional tangent, and M is the point where this tangent intersects the y-axis. **2**

You are given that M has coordinates $\left(0, \dfrac{A^4}{16}\right)$. **(DO NOT SHOW THIS)**

Use this graph and the equation from part (i) to show that for it to be possible to draw N normals to the hyperbola $xy = 1$ from the point $P(x_0, y_0)$, a minimum requirement is that $|x_0| > 2$.

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Extension 2 Trial 2015 – Solutions

Multiple Choice

Summary of Answers

1.	A	**2.**	A	**3.**	C	**4.**	B	**5.**	D
6.	C	**7.**	A	**8.**	C	**9.**	A	**10.**	B

Working

1.
$$\begin{aligned} P(-i) &= (-i)^3 - 2(-i)^2 + 1 \\ &= i + 2 + 1 \\ &= 3 + i \qquad \text{(A)} \end{aligned}$$

2. $\dfrac{x^2}{4} + \dfrac{y^2}{1/4} = 1$

$$\begin{aligned} b^2 &= a^2\left(1-e^2\right) \\ \frac{1}{4} &= 4\left(1-e^2\right) \\ 1-e^2 &= \frac{1}{16} \\ e &= \frac{\sqrt{15}}{4} \end{aligned}$$

$ae = 4 \times \dfrac{\sqrt{15}}{4} = \sqrt{15}$ (A)

3. (A) is true – diagonals of a rectangle are equal

(B) is true – Since the arguments of ω and z differ by $\dfrac{\pi}{2}$, $\operatorname{Arg}\left(\dfrac{\omega}{z}\right)$ is $\dfrac{\pi}{2}$ or $-\dfrac{\pi}{2}$,

so $\dfrac{\omega}{z}$ is imaginary, so $\operatorname{Re}\left(\dfrac{\omega}{z}\right) = 0$

(D) is true – Since $z = ic\omega$, $z^2 = i^2c^2\omega^2 = -c^2\omega^2 = k\omega^2$ $\left[\text{where } k = -c^2\right]$

(C) is false - $\arg(z-\omega)$ is positive

4. (A) is a root – conjugate root theorem

(C) is a root - $\left(\omega^{-1}\right)^5 + 1 = \omega^{-5} + 1 = \dfrac{1+\omega^5}{\omega^5} = \dfrac{0}{\omega^5} = 0$

(D) is a root - $\left(\omega^3\right)^5 + 1 = \left(\omega^5\right)^3 + 1 = (-1)^3 + 1 = 0$

(B) is not a root - $\left(\omega^2\right)^5 + 1 = \left(\omega^2\right)^3 + 1 = (-1)^2 + 1 \neq 0$

5. Equation with roots $2\alpha, 2\beta, 2\gamma$ is $P\left(\frac{x}{2}\right) = 0 = Q(x)$

Equation with roots $2\alpha+1$, $2\beta+1$ and $2\gamma+1$ is $Q(x-1) = 0$

$$P\left(\frac{x-1}{2}\right) = 0 \qquad \text{(D)}$$

6. (A) is $y = \sqrt{\sin x}$ (B) is $y = \sqrt{|\sin x|}$ (C) is the answer (D) is $y = \sin^2 x$

7. Letting $x = 2\cos\theta$: $dx = -2\sin\theta\, d\theta$ $x = 0 \Rightarrow \theta = \frac{\pi}{2}$

$x = \sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$

$$\int_0^{\sqrt{2}} \frac{x^2\, dx}{\sqrt{4-x^2}} = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{4\cos^2\theta \cdot -2\sin\theta\, d\theta}{2\sin\theta}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\cos^2\theta\, d\theta \qquad \text{(A)}$$

8. For this to have solutions, we require $\left|\frac{2+i}{c}\right|$ to equal 1, ie. $|c| = |2+i| = \sqrt{5}$.

Since c is real, there are two such values, $c = \pm\sqrt{5}$ (C)

9. By shells, no options are correct.

By slices, $r = a - x = a - f^{-1}(y)$ (A)

10. (A) population is increasing for $x < b$ and decreasing for $x > b$, so will always settle at b

(B) If $x < a$, the population is increasing, but it stops inceasing at $x = a$, so that is where it settles.
If $x > a$, it is the same scenario as (A).
So this is the answer.

(C) This will settles at $x = a$ if the population starts to the left of the root between a and b, otherwise it will settle at $x = b$.

(D) If $x < b$, the population with decrease to zero, it $x > b$ it will decrease to $x = b$

Section II

Question 11

(a) Let $z = 5 - 2i$. Find the value of

(i) $|z|$ **1**

$$|z|^2 = 5^2 + (-2)^2$$
$$|z| = \sqrt{29}$$

(ii) $\dfrac{\overline{z}}{z}$ in the form $a + ib$ **2**

$$\frac{\overline{z}}{z} \times \frac{\overline{z}}{\overline{z}} = \frac{\overline{z}^2}{|z|^2}$$
$$= \frac{(5+2i)^2}{29}$$
$$= \frac{21}{29} + \frac{20}{29}i$$

(b) If $(x+y)^2 = y$, find $\dfrac{dy}{dx}$ in terms of x and y. **2**

$$2(x+y)\left(1+\frac{dy}{dx}\right) = \frac{dy}{dx}$$
$$2(x+y) + 2(x+y)\frac{dy}{dx} = \frac{dy}{dx}$$
$$2(x+y) = \left[1-2(x+y)\right]\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{2(x+y)}{1-2(x+y)}$$

(c) Find $\displaystyle\int \frac{dx}{x^2 - 4x + 6}$. **2**

$$\int \frac{dx}{x^2-4x+6} = \int \frac{dx}{(x-2)^2+2}$$
$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{x-2}{\sqrt{2}} + c$$

(d) Use the substitution $u = \frac{\pi}{2} - x$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin 2x}\, dx$. **3**

$$u = \frac{\pi}{2} - x \qquad x = 0, \quad u = \frac{\pi}{2}$$

$$du = -\ dx \qquad x = \frac{\pi}{2}, \quad u = 0$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin 2x}\, dx = \int_{\frac{\pi}{2}}^{0} \frac{\cos\left(\frac{\pi}{2}-u\right) - \sin\left(\frac{\pi}{2}-u\right)}{1+\sin\left(\pi - 2u\right)} (-du)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin 2x}\, dx$$

$$= -\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin 2x}\, dx$$

$$2\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin 2x}\, dx = 0$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin 2x}\, dx = 0$$

(e) Evaluate $\int_1^{e^2} x^2 \log_e x\ dx$. **2**

$$\int_1^{e^2} x^2 \log_e x\ dx = \int_1^{e^2} \ln x \cdot d\left(\frac{1}{3}x^3\right)$$

$$= \frac{1}{3}\left[x^3 \ln x\right]_1^{e^2} - \frac{1}{3}\int_1^{e^2} x^3 \cdot d\left(\ln x\right)$$

$$= \frac{1}{3}\left(2e^6 - 0\right) - \frac{1}{3}\int_1^{e^2} x^2\, dx$$

$$= \frac{2}{3}e^6 - \frac{1}{9}\left[x^3\right]_1^{e^2}$$

$$= \frac{2}{3}e^6 - \frac{1}{9}\left(e^6 - 1\right)$$

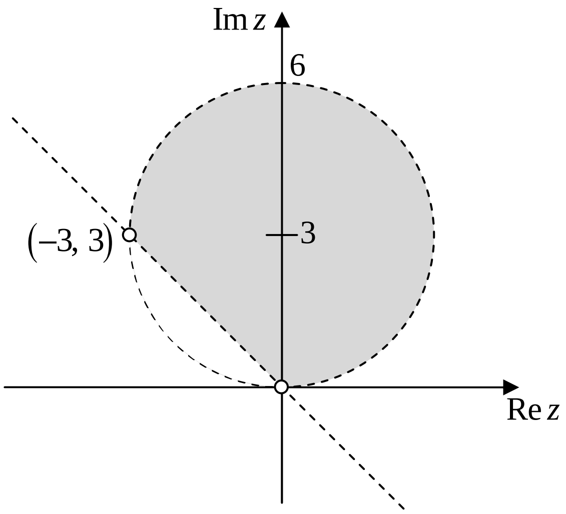
$$= \frac{1}{9}\left(5e^6 + 1\right)$$

(f)	Sketch the region in the Argand plane whose points satisfy $$\left|z-3i\right|<3 \quad \text{and} \quad \left|z+3\right|>\left|z-3i\right|$$	**3**

$\left|z-3i\right|<3$ is the region inside the circle with centre $(0,3)$ and radius 3.

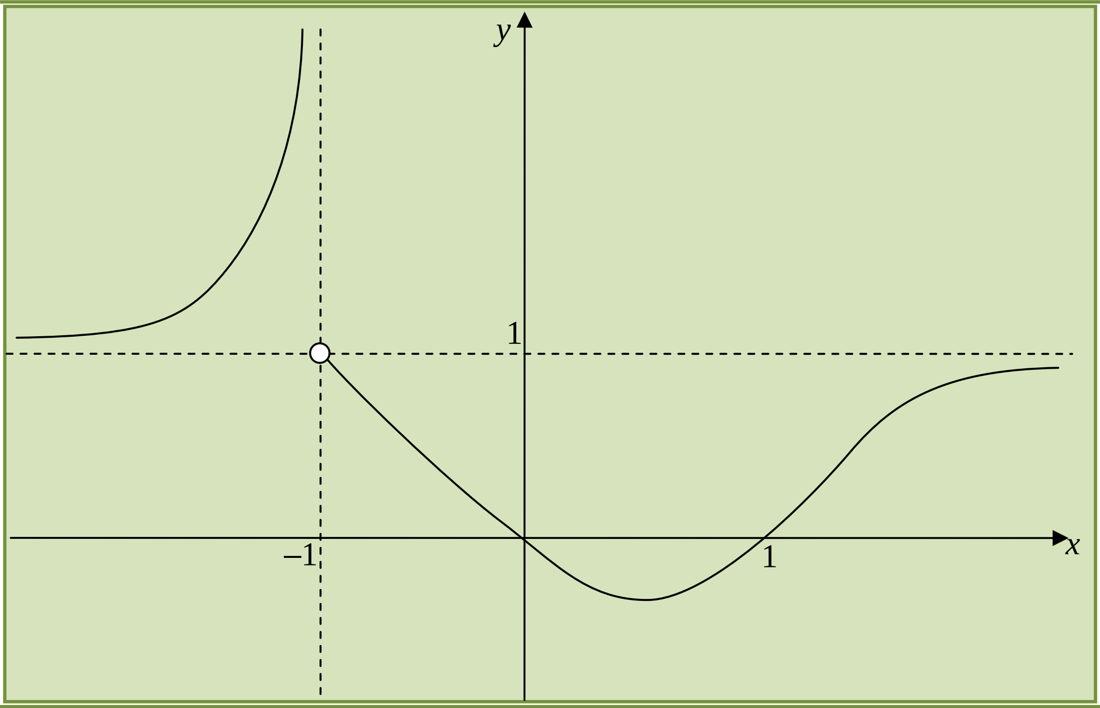
$\left|z+3\right|>\left|z-3i\right|$ is the region containing the points which are further from $(-3,0)$ than from $(0,3)$.

This is the region above the line $y=-x$.



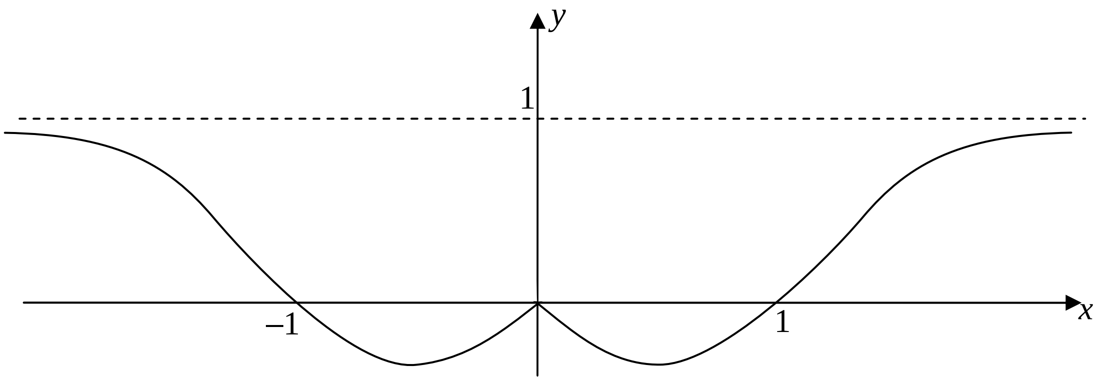
Question 12

(a) The diagram shows a graph of the function $y = f(x)$.

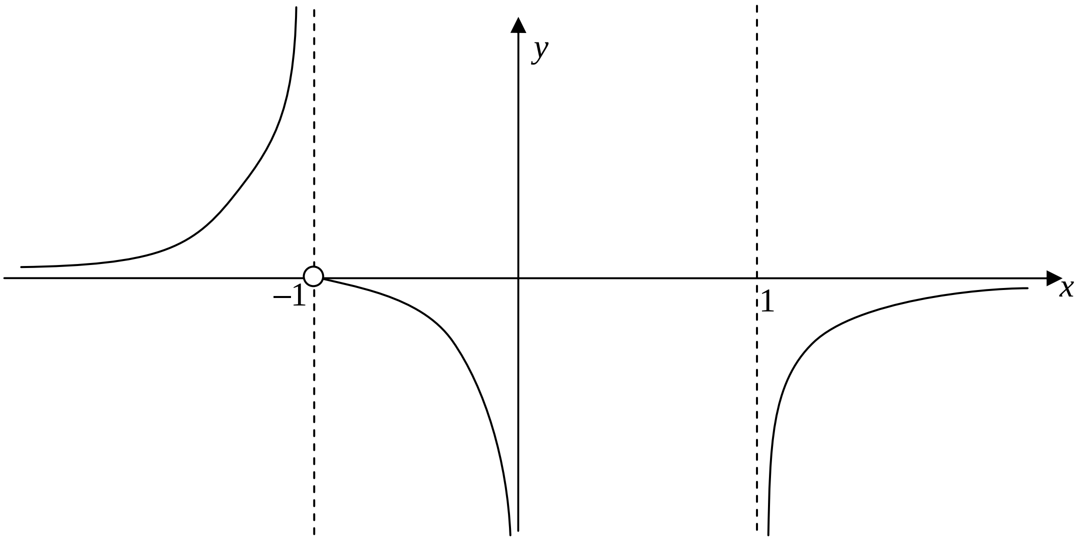


Draw separate half-page graphs for each of the following functions, showing all important features:

(i) $y = f(|x|)$ **2**



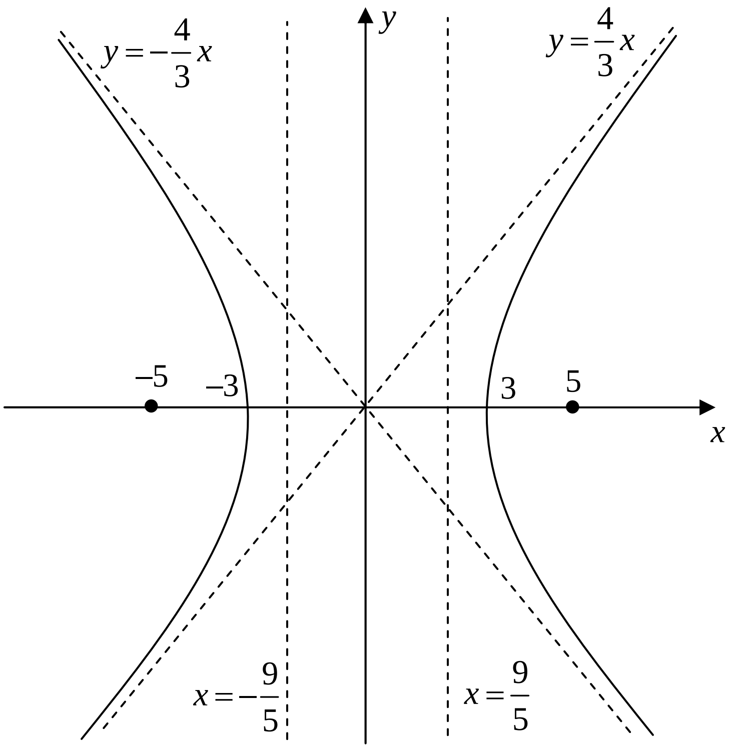
(ii) $y = \log_e f(x)$ **2**



(b)	Sketch the hyperbola with parametric equations $x = 3\sec\theta$ $y = 4\tan\theta$ indicating the vertices and foci, and the equations of the directrices and asymptotes.	**3**

$a = 3,\ b = 4$ $\qquad 4^2 = 3^2\left(e^2 - 1\right)$ $\qquad ae = 5$ $\qquad \dfrac{a}{e} = \dfrac{9}{5}$ $\qquad \dfrac{b}{a} = \dfrac{4}{3}$

$e = \dfrac{5}{3}$



(c)	$1+i$ is a root of the quadratic equation $z^2 + \omega z - i = 0$. Find the value of ω.	**3**

Let $\omega = a + ib$

$$\begin{aligned} (1+i)^2 + (a+ib)(1+i) - i &= 0 \\ 1 + 2i - 1 + a + ai + bi - b - i &= 0 \\ (a-b) + i(1+a+b) &= 0 \end{aligned}$$

$$a - b = 0 \qquad 1 + a + b = 0$$

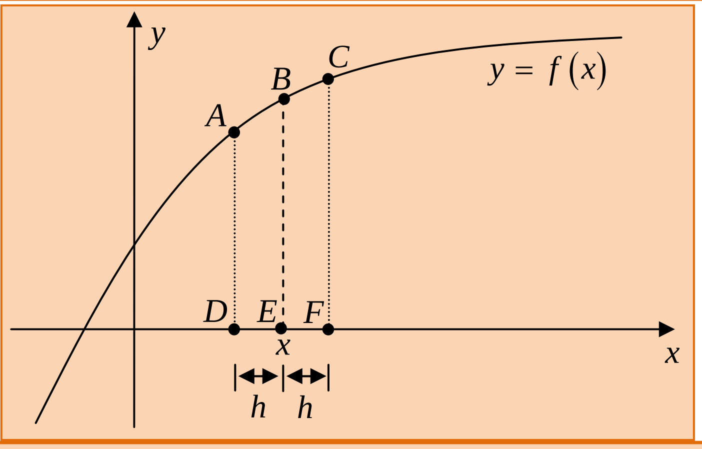
$$a = b \qquad 1 + 2a = 0$$

$$a = -\frac{1}{2} = b$$

$$\therefore \quad \omega = -\frac{1}{2} - \frac{1}{2}i$$

(d)

(i) **2**



Use the diagram above to explain why an alternative formula for the derivative is

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$$

Gradient of secant AC: $$m_{AC} = \frac{CF-AD}{DF} = \frac{f(x+h)-f(x-h)}{2h}$$

To find gradient of tangent, let A and C approach B, ie. let $h\to 0$

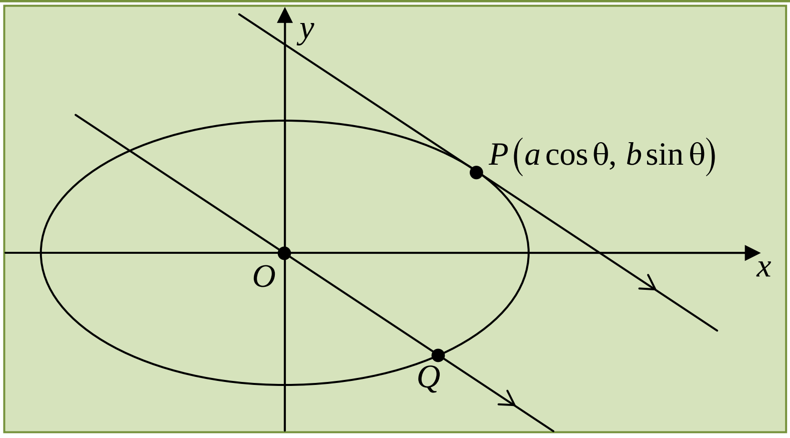
$$\therefore\quad f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$$

(ii) Use this formula to differentiate $f(x) = \tan x$ by first principles. **3**

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \lim_{h\to 0}\frac{\tan(x+h)-\tan(x-h)}{2h}\\
&= \lim_{h\to 0}\frac{\dfrac{\tan x+\tan h}{1-\tan x\tan h}-\dfrac{\tan x-\tan h}{1+\tan x\tan h}}{2h}\times\frac{(1-\tan x\tanh)(1+\tan x\tanh)}{(1-\tan x\tan h)(1+\tan x\tan h)}\\
&= \lim_{h\to 0}\frac{(\tan x+\tan h)(1+\tan x\tan h)-(\tan x-\tan h)(1-\tan x\tan h)}{2h(1-\tan x\tan h)(1+\tan x\tan h)}\\
&= \lim_{h\to 0}\frac{\cancel{\tan x}+\tan^2 x\tan h+\tan h+\cancel{\tan x\tan^2 h}-\cancel{\tan x}+\tan^2 x\tan h+\tan h-\cancel{\tan x\tan^2 h}}{2h(1-\tan x\tan h)(1+\tan x\tan h)}\\
&= \lim_{h\to 0}\frac{\cancel{2}\tan h(\tan^2 x+1)}{\cancel{2}h(1-\tan x\tan h)(1+\tan x\tan h)}\\
&= \lim_{h\to 0}\frac{\tan h}{h}\cdot\frac{\sec^2 x}{(1-\tan x\tan h)(1+\tan x\tan h)}\\
&= 1\cdot\frac{\sec^2 x}{(1-0)(1+0)}\\
&= \sec^2 x
\end{aligned}$$

Question 13

(a) In the diagram, $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, such that $0<\theta<\dfrac{\pi}{2}$. O is the origin, and OQ is parallel to the tangent to the ellipse at P, where Q is a point on the ellipse in the fourth quadrant.



(i) Show that the equation of line OQ is $bx\cos\theta + ay\sin\theta = 0$. **2**

$$x = a\cos\theta \qquad y = b\sin\theta \qquad \frac{dy}{d\theta} = \frac{b\cos\theta\cdot d\theta}{-a\sin\theta\cdot d\theta}$$

$$dx = -a\sin\theta\cdot d\theta \qquad dy = b\cos\theta\cdot d\theta \qquad = -\frac{b\cos\theta}{a\sin\theta}$$

Since OQ is parallel to tangent, equation is

$$y = -\frac{b\cos\theta}{a\sin\theta}x$$

$$ay\sin\theta = -bx\cos\theta$$

$$bx\cos\theta + ay\sin\theta = 0$$

(ii) Show that Q has coordinates $(a\sin\theta, -b\cos\theta)$. **2**

$$\frac{x^2}{a^2} + \frac{1}{\cancel{b^2}}\cdot\frac{\cancel{b^2}\cos^2\theta}{a^2\sin^2\theta}x^2 = 1 \qquad \frac{\cancel{a^2}\sin^2\theta}{\cancel{a^2}} + \frac{y^2}{b^2} = 1$$

$$\left[\times a^2\sin^2\theta\right] \quad x^2\sin^2\theta + x^2\cos^2\theta = a^2\sin^2\theta \qquad y^2 = b^2\left(1-\sin^2\theta\right)$$

$$x = \pm a\sin\theta \qquad y = \pm b\cos\theta$$

But $0<\theta<\dfrac{\pi}{2}$ and Q is in the 4th quadrant, so Q has coordinates $(a\sin\theta, -b\cos\theta)$.

(iii)	Find the area of ΔOPQ, and show that this area is independent of the choice of P.	**3**

$$OQ^2 = a^2\sin^2\theta + b^2\cos^2\theta$$

$$\begin{aligned}\text{perp dist of } P \text{ from } OQ &= \frac{\left|(b\cos\theta)(a\cos\theta)+(a\sin\theta)(b\sin\theta)+0\right|}{\sqrt{(b\cos\theta)^2+(a\sin\theta)^2}} \\ &= \frac{\left|ab\left(\cos^2\theta+\sin^2\theta\right)\right|}{OQ} \\ &= \frac{ab}{OQ} \qquad (\text{since } a,b>0)\end{aligned}$$

$$\begin{aligned}\text{Area } \Delta OPQ &= \frac{1}{2}\times OQ\times\frac{ab}{OQ} \\ &= \frac{ab}{2} \qquad (\text{which is independent of } \theta \text{ and hence } P)\end{aligned}$$

(b)

(i)	Find $\int_0^1 (x+1)^{-2}\,dx$.	**1**

$$\begin{aligned}\int_0^1 (x+1)^{-2}\,dx &= -\left[\frac{1}{x+1}\right]_0^1 \\ &= -\left(\frac{1}{2}-1\right) \\ &= \frac{1}{2}\end{aligned}$$

(ii)	Find real values a and b such that $\dfrac{x}{(x+1)^2} = \dfrac{a}{x+1} + \dfrac{b}{(x+1)^2}$ for all x.	**2**

$$\begin{aligned}\frac{x}{(x+1)^2} &= \frac{a}{x+1}+\frac{b}{(x+1)^2} \\ x &= a(x+1)+b\end{aligned}$$

$(x=-1)$ $\quad -1=b$

$(x=0)$ $\quad 0=a+b$

$a=1$

(iii) Hence find $\displaystyle\int_0^1 \frac{x}{(x+1)^2}\,dx$.	**1**

$$\int_0^1 \frac{x}{(x+1)^2}\,dx = \int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2}$$

$$= \left[\ln(x+1)\right]_0^1 - \frac{1}{2} \quad \text{[from part i]}$$

$$= \ln 2 - \frac{1}{2}$$

(iv) Given $\displaystyle I_n = \int_0^1 \frac{x^n}{(x+1)^2}\,dx$ for $n \geq 0$, show that $\displaystyle I_n + 2I_{n-1} + I_{n-2} = \frac{1}{n-1}$ for $n \geq 2$.	**2**

$$I_n + 2I_{n-1} + I_{n-2} = \int_0^1 \left[\frac{x^n}{(x+1)^2} + \frac{2x^{n-1}}{(x+1)^2} + \frac{x^{n-2}}{(x+1)^2}\right] dx$$

$$= \int_0^1 \frac{x^{n-2}\cancel{\left(x^2+2x+1\right)}}{\cancel{(x+1)^2}}\,dx$$

$$= \frac{1}{n-1}\left[x^{n-1}\right]_0^1$$

$$= \frac{1}{n-1}$$

(v) Hence evaluate $\displaystyle\int_0^1 \frac{x^3}{(x+1)^2}\,dx$.	**2**

$$I_2 + 2I_1 + I_0 = 1$$

$$I_2 + 2\left(\ln 2 - \frac{1}{2}\right) + \frac{1}{2} = 1$$

$$I_2 = \frac{3}{2} - 2\ln 2$$

$$I_3 + 2I_2 + I_1 = \frac{1}{2}$$

$$I_3 + 2\left(\frac{3}{2} - 2\ln 2\right) + \left(\ln 2 - \frac{1}{2}\right) = \frac{1}{2}$$

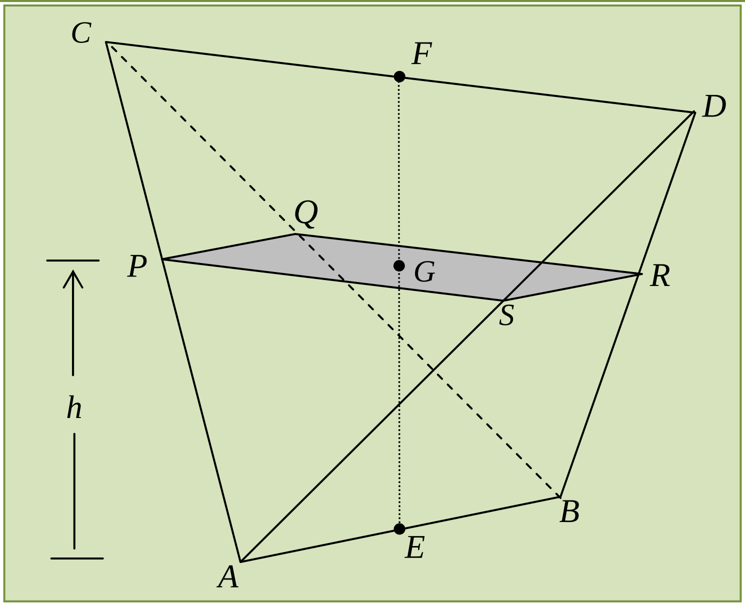
$$I_3 = 3\ln 2 - 2$$

Question 14

(a) In the following diagram, AB and CD are perpendicular horizontal skew lines. E and F are the midpoints of AB and CD respectively, and EF is vertical. **4**

$EF = H$ cm, $AB = a$ cm and $CD = b$ cm.

$EG = h$ cm where G is the centre of a typical horizontal rectangular slice $PQRS$ of the solid shown, where the solid has vertices A, B, C and D.



(i) Given that PS and SR both vary linearly with h, or otherwise, show that **2**

$$PS = \frac{bh}{H} \quad \text{and} \quad SR = a\left(1 - \frac{h}{H}\right).$$

$$PS = m_1 h + c_1 \qquad\qquad SR = m_2 h + c_2$$

$(h = 0,\ PS = 0)$ $\quad 0 = c_1$ $\qquad (h = 0,\ PS = a)$ $\quad a = c_2$

$$PS = m_1 h \qquad\qquad SE = m_2 h + a$$

$(h = H,\ PS = b)$ $\quad b = m_1 H$ $\qquad (h = H,\ PS = 0)$ $\quad 0 = m_2 H + a$

$$m_1 = \frac{b}{H} \qquad\qquad m_2 = -\frac{a}{H}$$

$$PS = \frac{bh}{H} \qquad\qquad SR = -\frac{ah}{H} + a$$

$$= a\left(1 - \frac{h}{H}\right)$$

(ii) Hence show that the volume of this solid is given by $V = \dfrac{abH}{6}$. **2**

Volume of slice $\delta V = PS \cdot SR \cdot \delta h$

$$= \frac{bh}{H} \cdot a\left(1 - \frac{h}{H}\right) \cdot \delta h$$

$$= \frac{ab}{H^2} \cdot h(H - h) \cdot \delta h$$

$$V = \lim_{\delta h \to 0} \sum_{h=0}^{H} \frac{ab}{H^2} \cdot h(H-h) \cdot \delta h$$

$$= \frac{ab}{H^2} \int_0^H \left(Hh - h^2\right) dh$$

$$= \frac{ab}{H^2} \left[\frac{Hh^2}{2} - \frac{h^3}{3} \right]_0^H$$

$$= \frac{ab}{H^2} \left(\frac{H^3}{2} - \frac{H^3}{3} \right)$$

$$= \frac{abH}{6}$$

(b) The polynomial $P(x) = x^5 + 2x^2 + mx + n$ has a double zero at $x = -2$. **3**

Find the product of the other three zeros.

$$P(x) = x^5 + 2x^2 + mx + n$$
$$P'(x) = 5x^4 + 4x + m$$

Since $P(x)$ has a double root at $x = -2$, $\quad P'(-2) = 0$

$$80 - 8 + m = 0$$
$$m = -72$$

and $\quad P(-2) = 0$

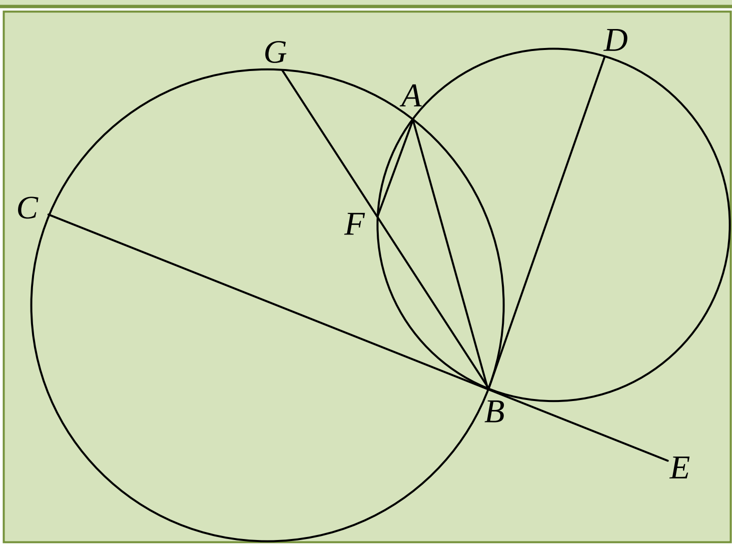
$$-32 + 8 + 144 + n = 0$$
$$n = -120$$

Let the zeros be $-2, -2, \alpha, \beta, \gamma$.

Product of roots: $\quad 4\alpha\beta\gamma = 120$

$$\alpha\beta\gamma = 30$$

(c) In the diagram, BC and BD are chords (not necessarily diameters) of one circle and tangents of the other. BFG is a straight line.



(i) Prove that $\angle CBA = \angle GFA$	**2**

Construct AD

$\angle CBA = \angle ADB$ (alternate segment theorem in circle $DAFB$)

$= \angle AFG$ (exterior angle of cyclic quad $DAFB$ = opposite interior angle)

(ii) Prove that $\angle EBD = \angle FAG$	**2**

Construct AG

$\angle ADB = \angle AFG$ (from part i)

$\angle ABD = \angle AGF$ (alternate segment theorem in circle $ABCG$)

$\therefore \angle FAG = \angle BAD$ (angles sums of triangles AGF and ABD)

$= \angle EBD$ (alternate segment thm in circle $DAFB$)

(d) Consider the function $f(t) = t - \dfrac{a^2}{t} - 2a\log_e\left(\dfrac{t}{a}\right)$ where $a > 0$.

(i) Evaluate $f(a)$.	**1**

$$f(a) = a - a - 2a\ln 1$$
$$= 0$$

(ii) By finding $f'(t)$, show that $f(t) \geq 0$ for $t \geq a$.	**3**

$$f'(t) = 1 + \frac{a^2}{t^2} - \frac{2a}{t}$$
$$= \left(1 - \frac{a}{t}\right)^2$$
$$\geq 0 \quad \text{for all } t$$

ie. $f(t)$ is increasing for all t

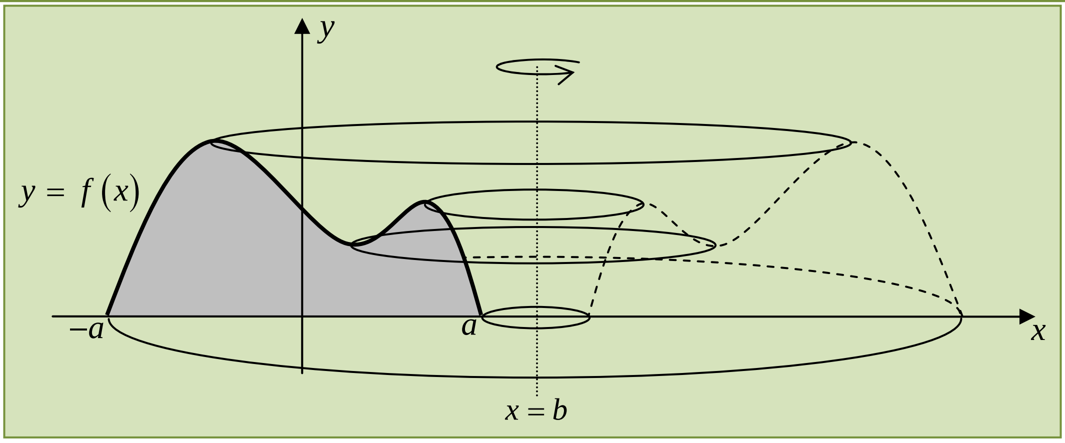
Since $f(a) = 0$, $f(t) \geq 0$ for $t \geq a$

Question 15

(a) The function $f(x)$ is defined over the domain $-a \le x \le a$, and is positive for $-a < x < a$.

The region bounded by the curve $y = f(x)$ and the x-axis is shaded and has area A.

This region is rotated $360°$ about the line $x = b$, where $b > a$, to form a solid of revolution.



(i) Use the method of cylindrical shells to show that the volume of this solid is given by **3**

$$V = 2\pi bA - 2\pi\int_{-a}^{a} x\, f(x)\, dx$$

Take a typical shell formed by rotating a vertical slice located at x about $x = b$, where the inner radius is $r = b - x$ the outer radius is $R = b - x + \delta x$, and the height is $h = f(x)$.

$$\begin{aligned} R + r &= 2(b-x) + \delta x \\ &\approx 2(b-x) \qquad \text{if } \delta x \text{ is sufficiently small in comparison to } 2(b-x) \end{aligned}$$

$$R - r = \delta x$$

$$\begin{aligned} \delta V &\approx \pi(R+r)(R-r)h \\ &\approx \pi\cdot 2(b-x)\cdot\delta x\cdot f(x) \\ &= 2\pi(b-x)\, f(x)\delta x \end{aligned}$$

$$\begin{aligned} V &= \lim_{\delta x\to 0}\sum_{x=-a}^{a} 2\pi(b-x)\, f(x)\delta x \\ &= 2\pi\int_{-a}^{a}(b-x)\, f(x)\, dx \\ &= 2\pi b\int_{-a}^{a} f(x)\, dx - 2\pi\int_{-a}^{a} x\, f(x)\, dx \\ &= 2\pi bA - 2\pi\int_{-a}^{a} x\, f(x)\, dx \end{aligned}$$

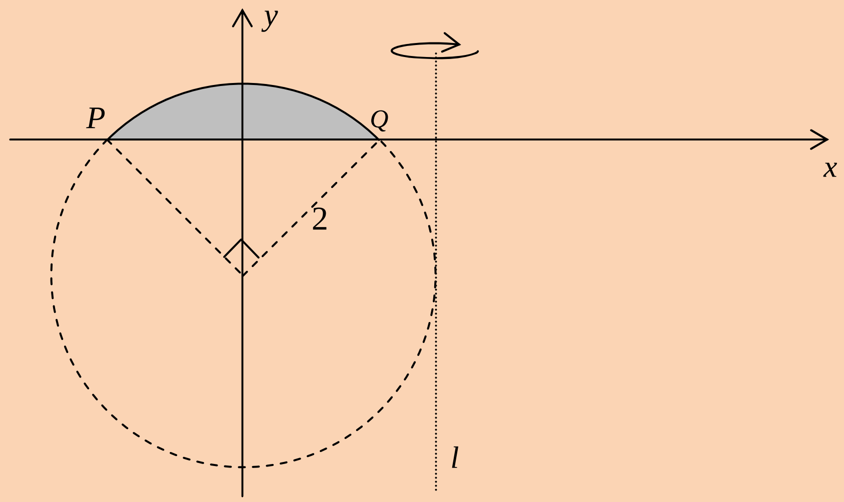
(ii) Prove that if $f(x)$ is an even function, then the function $g(x) = x \cdot f(x)$ is odd. **1**

$$\begin{aligned} g(-x) &= -x \cdot f(-x) \\ &= -x \cdot f(x) \quad [\text{since } f(x) \text{ is even}] \\ &= -g(x) \end{aligned}$$

$\therefore \quad g(x)$ is odd

(iii) In the following diagram, the chord PQ lies on the x-axis, and subtends a right angle at the centre of a circle of radius 2 units. **3**

The chord cuts off a minor segment, which is rotated about the vertical tangent l to the circle to form a solid of revolution.



Without finding the equation of the circle, use the results of parts (i) and (ii) to find the volume of the solid formed.

Since the function is positive over the domain $-a \le x \le a$ for some a, part (i) applies.

From part (ii), since the function is even, from part (ii) $x \cdot f(x)$ is odd.

Hence $\int_{-a}^{a} x\, f(x)\, dx$ is equal to zero.

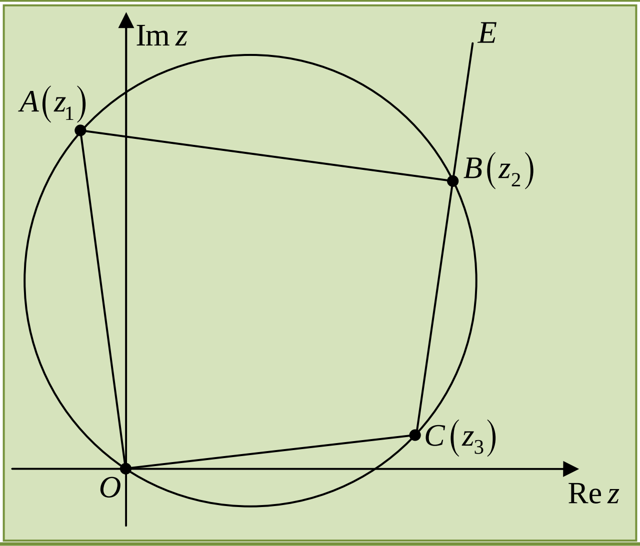
So $V = 2\pi b A$ where A is the area of the shaded segment.

$$\begin{aligned} A &= \frac{1}{4}\pi \times 2^2 - \frac{1}{2} \times 2 \times 2 \\ &= \pi - 2 \end{aligned}$$

b = radius of circle = 2

$$\begin{aligned} \therefore \quad V &= 2\pi \times 2 \times (\pi - 2) \\ &= 4\pi(\pi - 2) \text{ units}^3 \end{aligned}$$

(b) The diagram shows three points A, B, and C representing complex numbers z_1, z_2 and z_3 arranged on a circle through the origin O so that O, A, B and C are in clockwise order.



You may assume that the arguments of all complex numbers z lie in the range $0 \le \arg z < 2\pi$.

(i) Explain why $\angle ABE = \arg(z_1 - z_2) - \arg(z_2 - z_3)$. **1**

$\arg(z_1 - z_2)$ is the angle of rotation of the vector $\overrightarrow{BA}$ from the positive x-axis.

$\arg(z_2 - z_3)$ is the angle of rotation of the vector $\overrightarrow{CB}$ from the positive x-axis.

$\angle ABE$ is angle required to rotate $\overrightarrow{CB}$ to $\overrightarrow{BA}$, which is $\arg(z_1 - z_2) - \arg(z_2 - z_3)$.

(ii) Hence show that $z_1(z_2 - z_3) = kz_3(z_1 - z_2)$ where k is a positive real number. **3**

$$\angle ABE = \angle AOC \quad (\text{ext } \angle \text{ of cyclic quad } = \text{ opp int } \angle)$$

$$\arg(z_1 - z_2) - \arg(z_2 - z_3) = \arg z_1 - \arg z_3$$

$$\arg(z_1 - z_2) + \arg z_3 - \left[\arg(z_2 - z_3) + \arg z_1\right] = 0$$

$$\arg \frac{z_3(z_1 - z_2)}{z_1(z_2 - z_3)} = 0$$

$\dfrac{z_3(z_1 - z_2)}{z_1(z_2 - z_3)}$ is a positive real number, say k

$$z_1(z_2 - z_3) = kz_3(z_1 - z_2)$$

(c) The Fibonacci numbers f_n are defined by: $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$

(i) Prove by mathematical induction that $f_n < \left(\frac{5}{3}\right)^n$ for all positive integers n. **3**

Test $n = 1$ and $n = 2$: $f_1 = 1 < \left(\frac{5}{3}\right)^1$ $\quad f_2 = 1 < \left(\frac{5}{3}\right)^2$

$\therefore$ true for $n = 1$ and $n = 2$

Assume true for $n = k$ and $n = k+1$: ie. $f_k < \left(\frac{5}{3}\right)^k$ and $f_{k+1} < \left(\frac{5}{3}\right)^{k+1}$

Prove true for $n = k+2$:

$$\begin{aligned} f_{k+2} &= f_{k+1} + f_k \\ &< \left(\frac{5}{3}\right)^{k+1} + \left(\frac{5}{3}\right)^k \qquad [\text{by assumptions}] \\ &= \left(\frac{5}{3}\right)^{k+2}\left[\frac{3}{5} + \left(\frac{3}{5}\right)^2\right] \qquad [*] \\ &= \left(\frac{5}{3}\right)^{k+2} \cdot \frac{24}{25} \qquad [**] \\ &< \left(\frac{5}{3}\right)^{k+2} \qquad \left[\text{since } \frac{24}{25} < 1\right] \end{aligned}$$

$\therefore$ true for $n = k+2$ when true for $n = k$ and $n = k+1$

$\therefore$ by mathematical induction, $f_n < \left(\frac{5}{3}\right)^n$ for all positive integers n.

(ii) Based on your proof in part (i), show that the least value of a for which $f_n \leq a^n$ for all positive integer values of n is $a = \frac{1+\sqrt{5}}{2}$. **1**

From [*] and [**], require $\frac{1}{a} + \frac{1}{a^2} \leq 1$, so required value of a satisfies $\frac{1}{a} + \frac{1}{a^2} = 1$.

Rearranges to give: $a^2 - a - 1 = 0$

Solves to give: $a = \frac{1+\sqrt{5}}{2}$ $\quad$ [since $a > 0$]

Question 16

(a) Using the identity $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$ or otherwise, solve for $0 \le x \le 2\pi$: **2**

$\sin 3x + \sin 2x = 0$

$$\sin 3x + \sin 2x = 0$$

$$2\sin\frac{3x+2x}{2}\cos\frac{3x-2x}{2} = 0$$

$$\sin\frac{5x}{2} = 0 \qquad \text{or} \qquad \cos\frac{x}{2} = 0$$

$$\frac{5x}{2} = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi \qquad \frac{x}{2} = \frac{\pi}{2}$$

$$x = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, 2\pi \qquad x = \pi$$

(b)

(i) Show that if a and b are real, then $a^2 + b^2 \ge 2ab$. **1**

$$(a-b)^2 \ge 0$$

$$a^2 - 2ab + b^2 \ge 0$$

$$a^2 + b^2 \ge 2ab$$

(ii) Hence show that if a, b, c and d are real, then $a^4 + b^4 + c^4 + d^4 \ge 4abcd$. **2**

From (i), $a^4 + b^4 \ge 2a^2b^2$ and $c^4 + d^4 \ge 2c^2d^2$

Adding: $$a^4 + b^4 + c^4 + d^4 \ge 2a^2b^2 + 2c^2d^2$$

$$= 2\left[(ab)^2 + (cd)^2\right]$$

$$\ge 2\cdot[2\cdot ab\cdot cd] \qquad \text{[from i]}$$

$$= 4abcd$$

(iii) Consider the polynomial equation $x^4 + Ax^2 + Bx + C = 0$, where A, B and C are real.

Let the roots of this equation be α, β, γ and δ.

Show that:

1. $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -2A$ **1**

$$\left(\sum \alpha_i\right)^2 = \sum \alpha_i^2 + 2\sum \alpha_i\alpha_j$$

$$0^2 = \sum \alpha_i^2 + 2A$$

$$\sum \alpha_i^2 = -2A$$

2. $\alpha^4+\beta^4+\gamma^4+\delta^4 = 2A^2-4C$. **2**

Since α, β, γ and δ are roots, then:

$$\alpha^4 + A\alpha^2 + B\alpha + C = 0$$
$$\beta^4 + A\beta^2 + B\beta + C = 0$$
$$\gamma^4 + A\gamma^2 + B\gamma + C = 0$$
$$\delta^4 + A\delta^2 + B\delta + C = 0$$

Adding:
$$\sum\alpha_i^4 + A\cdot\sum\alpha_i^2 + B\cdot\sum\alpha_i + 4C = 0$$
$$\sum\alpha_i^4 + A(-2A) + B(0) + 4C = 0$$
$$\sum\alpha_i^4 = 2A^2-4C$$

(iv) Hence show that the equation $x^4+Ax^2+Bx+C=0$ cannot have four real roots if $A^2<4C$. **2**

If $A^2<4C$ then
$$2A^2 < 8C$$
$$2A^2-4C < 4C$$
$$\alpha^4+\beta^4+\gamma^4+\delta^4 < 4\alpha\beta\gamma\delta$$

$\therefore$ from part (ii), α, β, γ and δ cannot all be real

(c)

(i) Show that the normal to the hyperbola $xy=c^2$ at the point $\left(ct, \frac{c}{t}\right)$ **2**

has equation $ty-t^3x = c\left(1-t^4\right)$.

$$x = ct \qquad y = \frac{c}{t} \qquad \frac{dy}{dx} = \frac{-\frac{\cancel{c}}{t^2}\cdot\cancel{dt}}{\cancel{c}\cdot\cancel{dt}}$$
$$dx = c\cdot dt \qquad dy = -\frac{c}{t^2}\cdot dt \qquad = -\frac{1}{t^2}$$

$\therefore\ m_N = t^2$

Normal:
$$y-\frac{c}{t} = t^2(x-ct)$$
$$ty-c = t^3x-ct^4$$
$$ty-t^3x = c\left(1-t^4\right)$$

(ii) A point $P(x_0, y_0)$ is chosen randomly in the number plane. **1**

The number of normals which can be drawn to the hyperbola $xy = c^2$ from P depends on the choice of P.

Given an ideal choice of P, what is the greatest number N of such normals which can be drawn to the hyperbola from P ?

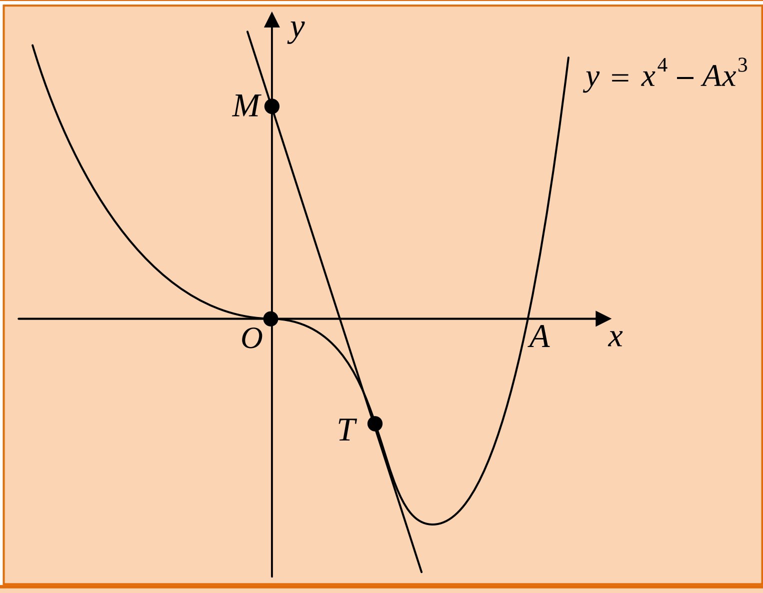
Briefly justify your answer.

Substituting $P(x_0, y_0)$ into equation of normal gives $ty_0 - t^3x_0 = c(1-t^4)$.

This is a quartic equation in t, and can have up to 4 real solutions.

If it has 4 real and distinct solutions, there will be four distinct points on the curve at which the normal passes through P. $\therefore\ N = 4$

(iii) Following is the graph of the function $y = x^4 - Ax^3$ for $A > 0$.



O is the origin, T is the other point of inflexion, TM is the inflexional tangent, and M is the point where this tangent intersects the y-axis. **2**

You are given that M has coordinates $\left(0, \dfrac{A^4}{16}\right)$. **(DO NOT SHOW THIS)**

Use this graph and the equation from part (i) to show that for it to be possible to draw N normals to the hyperbola $xy = 1$ from the point $P(x_0, y_0)$, a minimum requirement is that $|x_0| > 2$.

For $xy = 1$, $c = 1$, so the normal equation becomes $ty - t^3x = 1 - t^4$, which rearranges to $t^4 - t^3x = 1 - ty$. So if the normal passes through $P(x_0, y_0)$ then t satisfies:

$$t^4 - x_0t^3 = 1 - y_0t$$

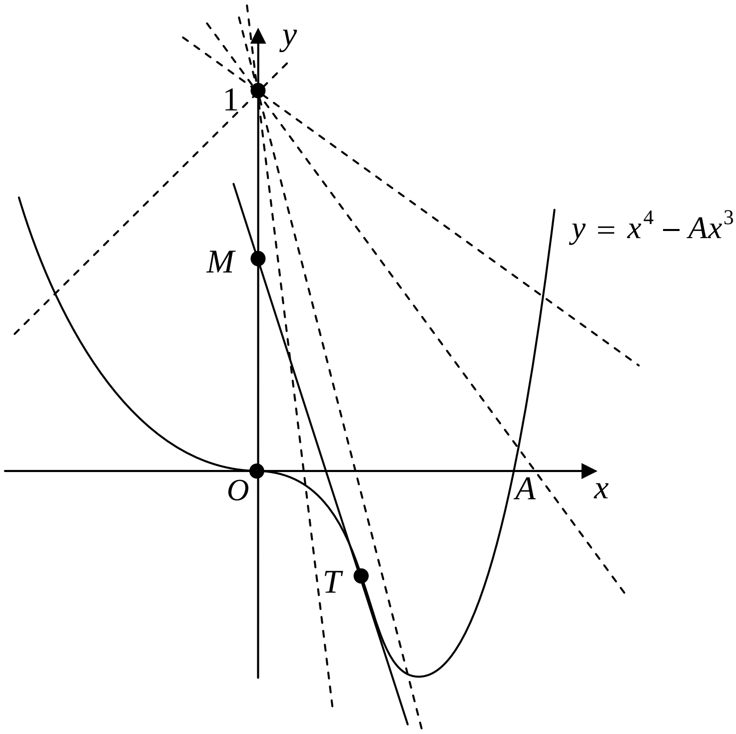
The number of solutions of this equation is the number of points of intersection of the graphs $y = t^4 - x_0t^3$ and $y = 1 - y_0t$.

Converting t to x, this is the number of points of intersection of the graphs $y = x^4 - Ax^3$ and $y = 1 - y_0x$, where $A = x_0$.

The diagram shows the situation for $A > 0$.

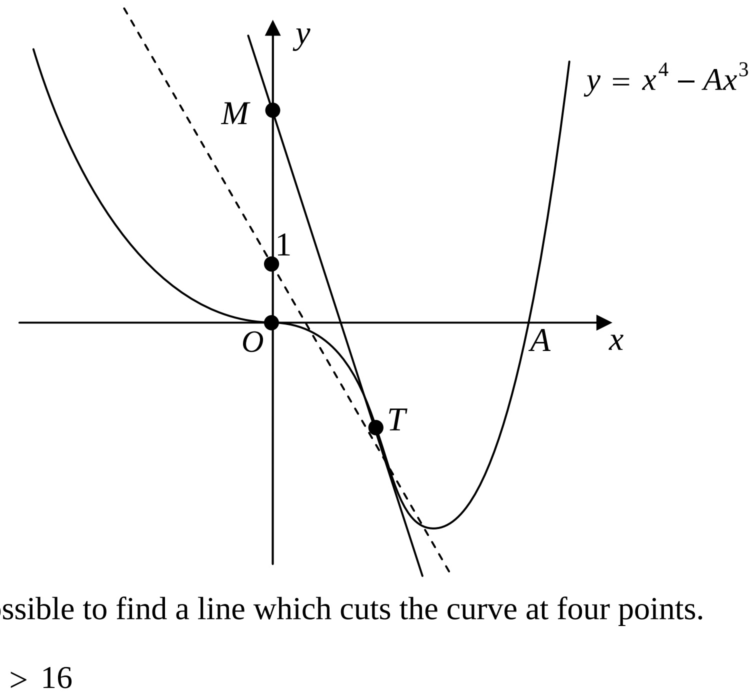
The first graph is given, the second is a straight line with y-intercept 1.

If $1 > \frac{A^4}{16}$:



The line can only intersect the curve at two points, regardless of the gradient. (2nd point of intersection out of range of diagram)

If $1 < \frac{A^4}{16}$:



It IS possible to find a line which cuts the curve at four points.

$$\text{ie.}\quad A^4 > 16$$
$$A > 2$$
$$x_0 > 2$$

If $A < 0$, the graph is reflected in the y-axis, and the same logic leads to $x_0 < -2$.

So $|x_0| > 2$