Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10.

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1 Which of the following is a primitive of $x\sin x$?

A. $-x\cos x+\sin x$

B. $x\cos x-\sin x$

C. $-x\cos x-\sin x$

D. $x\cos x+\sin x$

2 If $\mathbf{u}=-3\underset{\sim}{i}+\underset{\sim}{j}+2t\underset{\sim}{k}$ and $\mathbf{v}=\begin{pmatrix}1\\t\\-1\end{pmatrix}$ are perpendicular, what is the value of t ?

A. -3

B. -2

C. $\frac{2}{3}$

D. 1

3 Which of the following equations represents a sphere of radius 2 units?

A. $(x+y+z)^2=4$

B. $x^2+y^2+z^2=4$

C. $(x+y+z)^3=8$

D. $x^3+y^3+z^3=8$

4 Given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 2$ and $\mathbf{a} \cdot \mathbf{b} = 5$, what is the value of $\mathbf{a} \cdot (\mathbf{a} + \mathbf{b})$?

A. 11

B. 14

C. 15

D. 21

5 What is the Cartesian equation of the curve defined by $\underset{\sim}{r}(t) = \begin{pmatrix} \sin t \\ \cos t \\ \cos 2t \end{pmatrix}$?

A. $x^2 + y^2 = 1$

B. $z = 2y^2 - 1$

C. $z = 1 - 2x^2$

D. $z = y^2 - x^2$

6 A particle is undergoing simple harmonic motion and takes 3 seconds to travel between the extremes of the motion at $x = -4$ and $x = 4$. Which of the following is a possible equation for the velocity of the particle?

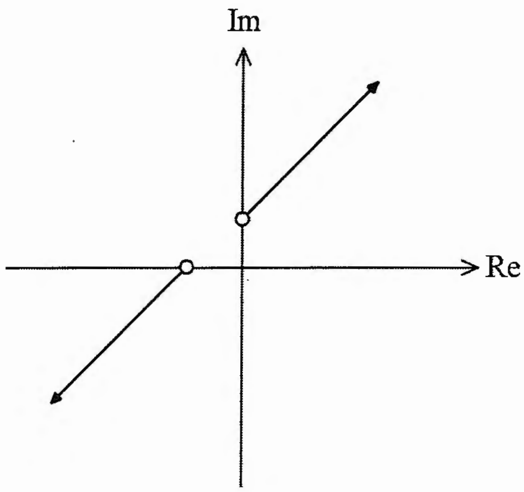
A. $v = 4\sin\left(\frac{2\pi t}{3}\right)$

B. $v = \frac{8\pi}{3}\sin\left(\frac{2\pi t}{3}\right)$

C. $v = 4\sin\left(\frac{\pi t}{3}\right)$

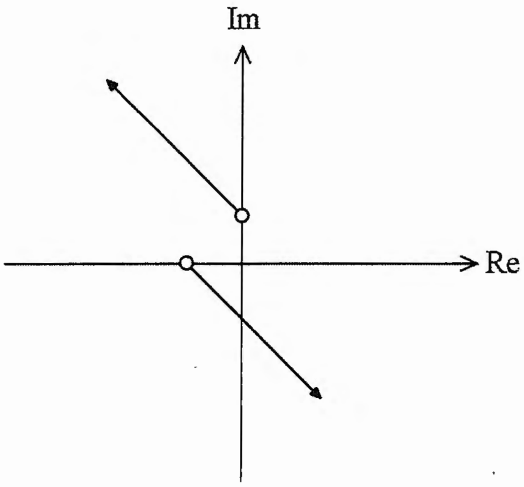
D. $v = \frac{4\pi}{3}\sin\left(\frac{\pi t}{3}\right)$

7 The diagram below shows the locus of the complex number z.

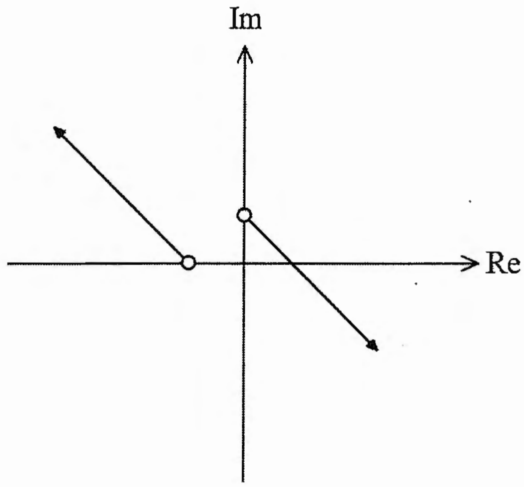


Which of the following could represent the locus of iz ?

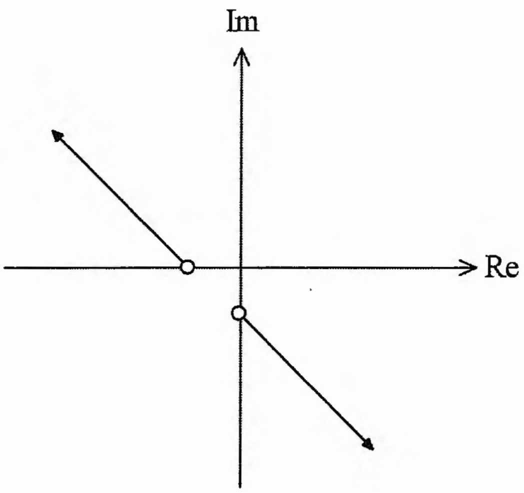
A.



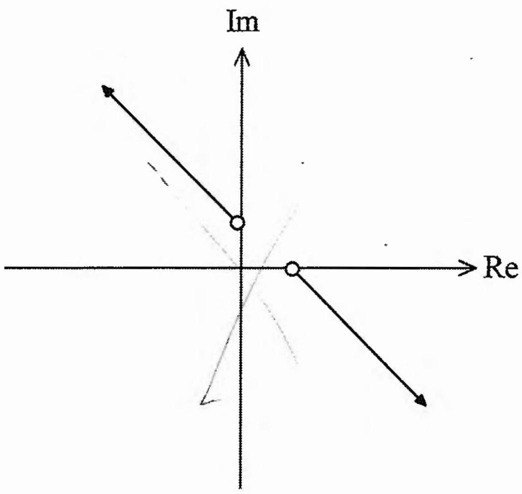
B.



C.



D.



8 What is the contrapositive of the following statement?

"If both m and n are divisible by d then $m-n$ is divisible by d."

A. If $m-n$ is divisible by d, then both m and n are divisible by d

B. If $m-n$ is not divisible by d, then both m and n are not divisible by d

C. If $m-n$ is not divisible by d, then m or n is not divisible by d

D. If both m and n are not divisible by d then $m-n$ is not divisible by d

9 Consider the statement "There is no integer which is the largest integer". Which of the following statements is equivalent to this statement?

A. $\exists y : (\forall x,\ y > x)$

B. $\forall x,\ (\exists y : y > x)$

C. $\exists x : (\forall y\ ,\ y > x)$

D. $\forall y,\ (\exists x : y > x)$

10 If a, b and c are real numbers such that $0 < a < b < c$ which of the following is NOT necessarily true?

A. $\frac{1}{a}+\frac{1}{b}-\frac{1}{c}>0$

B. $\frac{1}{c+a}+\frac{1}{b+c}-\frac{1}{a+b}>0$

C. $\frac{1}{ab}-\frac{1}{bc}>0$

D. $\frac{c}{ab}+\frac{a}{bc}-\frac{b}{ac}>0$

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Let $z=2-3i$ and $w=-1+i$. **2**

Find $z^2-\overline{w}$ in the form $x+iy$ where x and y are real numbers.

(b) The complex numbers $1+3i$ and $4+2i$ are denoted by u and v respectively.

(i) Find $\dfrac{u}{v}$ in the form $x+iy$. **2**

(ii) Hence, justify why $\tan^{-1}3-\tan^{-1}\dfrac{1}{2}=\dfrac{\pi}{4}$. **1**

(c) Consider $f(x)=x^4+2x^3+2x^2+26x+169$. **3**

Given $f(2+3i)=0$, fully factorise $f(x)$ over the set of complex numbers.

(d) By first writing $\sqrt{3}-i$ in exponential form, find the roots of the equation $z^4=\sqrt{3}-i$. **3**
Give your answers in exponential form using principal arguments.

(e) By first writing $\dfrac{5x^2+2}{x\left(x^2+1\right)}$ in the form $\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}$, find $\displaystyle\int\frac{5x^2+2}{x\left(x^2+1\right)}\,dx$. **4**

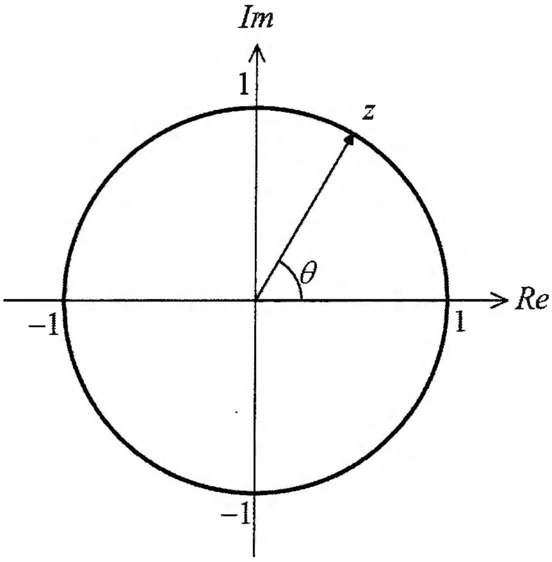
Question 12 (17 marks) Use a SEPARATE writing booklet

(a) Find $\int \frac{2-x}{\sqrt{4-x^2}}\,dx$. **2**

(b) (i) On an Argand diagram draw the locus $\arg(z-4i) = \pm\frac{\pi}{4}$. **2**

(ii) Find the value of k for which the circle $|z-6-4i| = k$ touches this locus. **1**

(c) Consider the complex number $z = e^{i\theta}$.



(i) Use a geometric argument to explain why $\frac{z-1}{z+1}$ is purely imaginary. **2**

(ii) Show that $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and find a similar result for $\sin\theta$. **2**

(iii) Use the results from (ii) to show that $\frac{z-1}{z+1} = i\tan\frac{\theta}{2}$. **2**

(d) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + 2\sin x + 3}\,dx$. **4**

(e) Prove that $x + y < 1 + xy$, given that $0 < x < 1$ and $0 < y < 1$. **2**

Question 13 (16 marks) Use a SEPARATE writing booklet

(a) Given that $u_{k+1} = 2u_k + (-1)^{k+1}$ and $u_1 = 1$, use mathematical induction to prove that $u_n = \frac{1}{3}\left(2^{n+1} + (-1)^n\right)$, where $n \in \mathbb{Z}^+$. **3**

(b) The line l_1 passes through the points $A(4,-3,-3)$ and $B(5,2,2)$.

(i) Find a vector equation of line l_1. **2**

(ii) The equation of line l_2 is given by $\frac{x-1}{k} = \frac{y}{-1} = \frac{z+3}{1}$. **3**

Find the value of k for which l_1 and l_2 intersect.

(c) (i) Use a suitable substitution to show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **1**

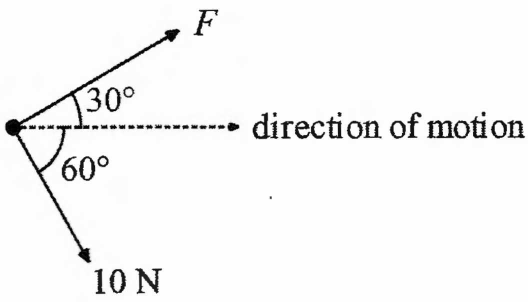
(ii) Hence, or otherwise, show that $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x + \sin x}\,dx = \frac{\pi}{8} - \frac{1}{4}\ln 2$. **3**

(d) A particle of mass 2 kg moving in a straight line is acted on by a force $F = 4x^3 + 16x$, where x is the displacement of the particle in metres, measured from a fixed origin, and F is measured in Newtons. **4**

If the particle is initially two metres to the left of the origin moving at a velocity of 8 m/s, find the displacement function $x(t)$.

Question 14 (14 marks) Use a SEPARATE writing booklet

(a) An object with a mass of 2 kg is acted on by forces of 10 N and F N at 60° and 30° respectively to the direction of motion as shown in the diagram below. **2**
Find the acceleration of the object.



(b) The line l has the equation $\underset{\sim}{r} = \begin{pmatrix} 13 \\ 8 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$, where λ is a scalar parameter.

The point $A(3,-2,6)$ lies on the line. The position vector of a point P is $-p\underset{\sim}{i} + 2p\underset{\sim}{k}$, where p is a constant and $\overrightarrow{PA}$ is perpendicular to the line l.

(i) Deduce that $p = 1$. **1**

(ii) Given that B is a point on l such that $\angle BPA = 45°$, find the coordinates of two possible locations for B. **3**

(c) A particle moves in a straight line in simple harmonic motion about the origin O.
Its displacement is x metres and its acceleration is given by $\ddot{x} = -n^2 x$.

(i) Show that $\dot{x}^2 = n^2\left(A^2 - x^2\right)$, where A is the amplitude of the motion. **2**

The point P is 1.6 m to the left of O and Q is 1.2 m to the right of O.
The ratio of the speed of the particle at P to its speed at Q is 3:4.

(ii) Show that the amplitude of the motion is 2. **2**

(iii) Given that the particle's maximum speed during the motion is $\frac{\pi}{3}$ ms^{-1}, find the time taken to travel directly from P to Q. **4**

Question 15 (14 marks) Use a SEPARATE writing booklet

(a) (i) Use mathematical induction to prove that $\dfrac{1}{n!} < \dfrac{1}{e^n}$ for $n \geq 6, n \in \mathbb{Z}^+$. **3**

(ii) Hence or otherwise show that $\dfrac{1}{6!} + \dfrac{1}{7!} + \dfrac{1}{8!} + \ldots < \dfrac{1}{e^6 - e^5}$. **2**

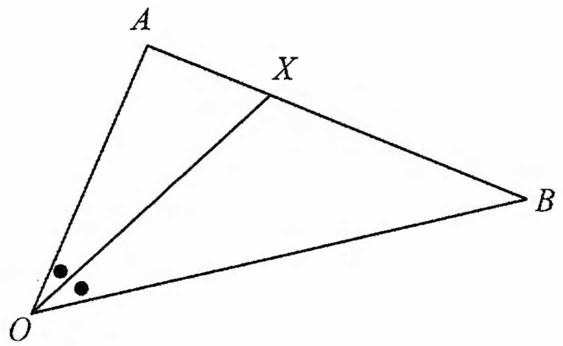
(b) Let $I_n = \displaystyle\int_0^1 \frac{1}{\left(1+x^2\right)^n}\, dx$, $n \geq 1$.

(i) Use integration by parts to show that $I_n = \dfrac{1}{2^n} + 2n\displaystyle\int_0^1 \frac{x^2}{\left(1+x^2\right)^{n+1}}\, dx$. **2**

(ii) Hence, or otherwise, show that $I_{n+1} = \dfrac{1}{n2^{n+1}} + \dfrac{2n-1}{2n} I_n$. **2**

(iii) Evaluate $\displaystyle\int_0^1 \frac{1}{\left(1+x^2\right)^2}\, dx$. **1**

(c) Consider two points A and B whose position vectors are $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively. The angle bisector of $\angle AOB$ meets AB at X.



(i) Explain why $\overrightarrow{OX} = \lambda\left(\hat{\underset{\sim}{a}} + \hat{\underset{\sim}{b}}\right)$ where $\hat{\underset{\sim}{a}}$ and $\hat{\underset{\sim}{b}}$ are the corresponding unit vectors for $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **1**

(ii) Let $\overrightarrow{AX} = \mu\, \overrightarrow{AB}$, where μ is a scalar and $0 < \mu < 1$. **3**
Use vector methods to show that OX divides AB in the ratio of the lengths of the adjacent sides. i.e. $AX : XB = OA : OB$.

Question 16 (14 marks) Use a SEPARATE writing booklet

(a) Construct a proof by contradiction to prove the following statement. **3**

$$\forall a,b \in \mathbb{Z}^+,\ a^2 - 4b \neq 2$$

(b) (i) Show that $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$. **2**

(ii) Hence or otherwise prove that $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \ldots + \dfrac{1}{n^2} < 2 \ \ \forall n \in \mathbb{Z}^+$. **2**

(c) (i) Use De Moivre's theorem to show that, if $\sin\theta \neq 0$, then **2**

$$\frac{(\cot\theta + i)^{2n+1} - (\cot\theta - i)^{2n+1}}{2i} = \frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta}$$, where n is a positive integer.

(ii) Deduce that the solutions to the equation **3**

$$\binom{2n+1}{1}x^n - \binom{2n+1}{3}x^{n-1} + \binom{2n+1}{5}x^{n-2} - \ldots + (-1)^n = 0$$

are $x = \cot^2\left(\dfrac{m\pi}{2n+1}\right), m = 1,2,3,\ldots,n$.

(iii) Hence find an expression for $\displaystyle\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right)$ and show that **2**

$$\cot^2\left(\frac{\pi}{17}\right) + \cot^2\left(\frac{2\pi}{17}\right) + \ldots + \cot^2\left(\frac{8\pi}{17}\right) = 40.$$

End of paper

North Sydney Girls High School

2020 **HSC TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 11)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

NAME: SOLUTIONS **TEACHER:** ______

STUDENT NUMBER: | | | | | | | | |

Question	1-10	11	12	13	14	15	16	Total
Mark	/10	/15	/17	/16	/14	/14	/14	/100

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10.

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1 Which of the following is a primitive of $x\sin x$?

A. $-x\cos x + \sin x$

B. $x\cos x - \sin x$

C. $-x\cos x - \sin x$

D. $x\cos x + \sin x$

$\int x\sin x = \int x\, d(-\cos x)$

$= -x\cos x + \int \cos x\, dx$

$= -x\cos x + \sin x$

2 If $\mathbf{u} = -3\underset{\sim}{i} + \underset{\sim}{j} + 2t\underset{\sim}{k}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ t \\ -1 \end{pmatrix}$ are perpendicular, what is the value of t?

A. -3

B. -2

C. $\frac{2}{3}$

D. 1

$\begin{pmatrix} -3 \\ 1 \\ 2t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ t \\ -1 \end{pmatrix} = 0$

$-3 + t - 2t = 0$

$t = -3$

3 Which of the following equations represents a sphere of radius 2 units?

A. $(x+y+z)^2 = 4$

B. $x^2 + y^2 + z^2 = 4$

C. $(x+y+z)^3 = 8$

D. $x^3 + y^3 + z^3 = 8$

4 Given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 2$ and $\mathbf{a} \cdot \mathbf{b} = 5$, what is the value of $\mathbf{a} \cdot (\mathbf{a} + \mathbf{b})$?

A. 11

B. 14

C. 15

D. 21

$\underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b}$
$= |a|^2 + \underset{\sim}{a} \cdot \underset{\sim}{b}$
$= 9 + 5$
$= 14$

5 What is the Cartesian equation of the curve defined by $\underset{\sim}{r}(t) = \begin{pmatrix} \sin t \\ \cos t \\ \cos 2t \end{pmatrix}$?

A. $x^2 + y^2 = 1$

B. $z = 2y^2 - 1$

C. $z = 1 - 2x^2$

D. $z = y^2 - x^2$

$\cos 2t = \cos^2 t - \sin^2 t$

6 A particle is undergoing simple harmonic motion and takes 3 seconds to travel between the extremes of the motion at $x = -4$ and $x = 4$. Which of the following is a possible equation for the velocity of the particle?

A. $v = 4\sin\left(\frac{2\pi t}{3}\right)$

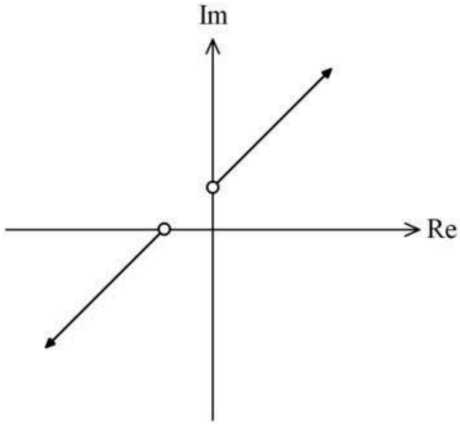
B. $v = \frac{8\pi}{3}\sin\left(\frac{2\pi t}{3}\right)$

C. $v = 4\sin\left(\frac{\pi t}{3}\right)$

D. $v = \frac{4\pi}{3}\sin\left(\frac{\pi t}{3}\right)$

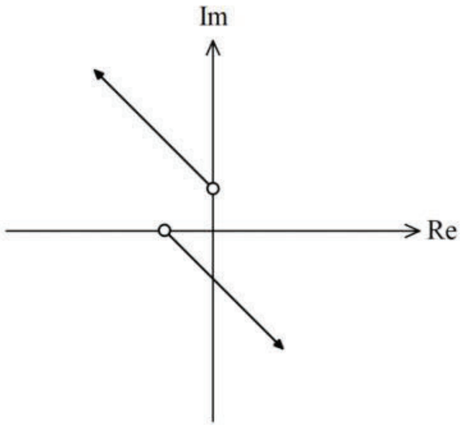
$A = 4$
$\frac{2\pi}{n} = 6 \Rightarrow n = \frac{\pi}{3}$
$v_{max} = An$
$= \frac{4\pi}{3}$ $\therefore D$

7 The diagram below shows the locus of the complex number z.

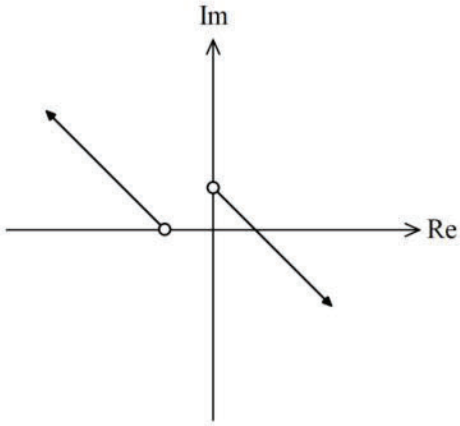


Which of the following could represent the locus of iz ?

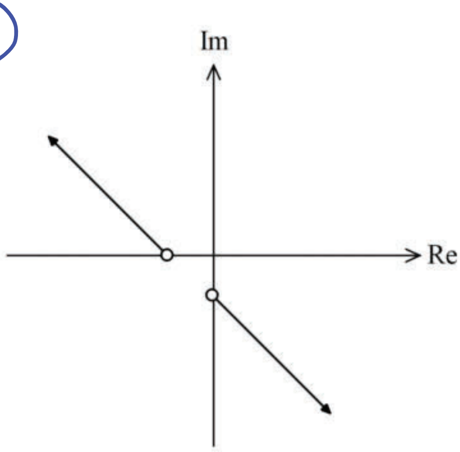
A.



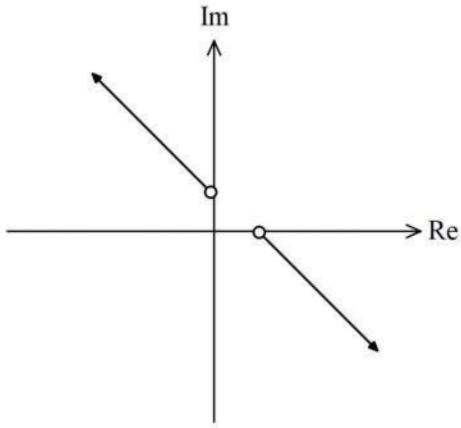
B.



C.



D.



8 What is the contrapositive of the following statement?

"If both m and n are divisible by d then $m-n$ is divisible by d."

A. If $m-n$ is divisible by d, then both m and n are divisible by d

B. If $m-n$ is not divisible by d, then both m and n are not divisible by d

C. If $m-n$ is not divisible by d, then m or n is not divisible by d

D. If both m and n are not divisible by d then $m-n$ is not divisible by d

9 Consider the statement "There is no integer which is the largest integer".
Which of the following statements is equivalent to this statement?

A. $\exists y : (\forall x,\ y > x)$

B. $\forall x,\ (\exists y : y > x)$

C. $\exists x : (\forall y\ ,\ y > x)$

D. $\forall y,\ (\exists x : y > x)$

10 If a, b and c are real numbers such that $0 < a < b < c$ which of the following is NOT necessarily true?

A. $\frac{1}{a}+\frac{1}{b}-\frac{1}{c}>0$

$b<c$

$\therefore \frac{1}{b}>\frac{1}{c}$ so $\frac{1}{b}-\frac{1}{c}>0$ so $\frac{1}{a}+\frac{1}{b}-\frac{1}{c}>0$ $\therefore$ not A

B. $\frac{1}{c+a}+\frac{1}{b+c}-\frac{1}{a+b}>0$

$c>b$

$c+a>a+b$

$\frac{1}{c+a}<\frac{1}{a+b}$

Also $c>a$

$b+c>a+b$

$\frac{1}{b+c}<\frac{1}{a+b}$

eg $a=1, b=2, c=5$

$\frac{1}{6}+\frac{1}{7}-\frac{1}{3}<0$

C. $\frac{1}{ab}-\frac{1}{bc}>0$

$a<c$

$\therefore ab<bc$ so $\frac{1}{ab}>\frac{1}{bc}$ so $\frac{1}{ab}-\frac{1}{bc}>0$ $\therefore$ not C

D. $\frac{c}{ab}+\frac{a}{bc}-\frac{b}{ac}>0$

$\frac{c}{ab}+\frac{a}{bc}-\frac{b}{ac}=\frac{c^2+a^2-b^2}{abc}$

As $c^2>b^2$

$c^2-b^2>0$

$c^2+a^2-b^2>0$ $\therefore$ not D

Question 11

a) $z^2 - \overline{w} = (2-3i)^2 - (-1-i)$

$= 4 - 12i - 9 + 1 + i$

$= -4 - 11i$

b) (i) $\dfrac{u}{v} = \dfrac{1+3i}{4+2i} \times \dfrac{4-2i}{4-2i}$

$= \dfrac{4 - 2i + 12i + 6}{16 + 4}$

$= \dfrac{10 + 10i}{20}$

$= \frac{1}{2} + \frac{1}{2} i$

(ii) $\arg\left(\frac{u}{v}\right) = \frac{\pi}{4}$

$\arg u = \tan^{-1} 3$ & $\arg v = \tan^{-1}\left(\frac{2}{4}\right)$

$= \tan^{-1}\left(\frac{1}{2}\right)$

Now $\arg\left(\frac{u}{v}\right) = \arg u - \arg v$

$\therefore \quad \frac{\pi}{4} = \tan^{-1} 3 - \tan^{-1} \frac{1}{2}$

Question 11

c) $f(2+3i) = 0$, so $2+3i$ is a root (factor theorem)

$2-3i$ is also a root as $f(x)$ has real coeffs, so complex roots occur in conjugate pairs

$\therefore \quad (x - (2+3i))(x - (2-3i))$ is a factor

$(x^2 - 4x + 13)$ is a factor.

So $f(x) = (x^2 - 4x + 13)(x^2 + 6x + 13)$ by inspection

OR

$f(x) = (x^2 - 4x + 13)(x^2 + bx + 13)$

x^3 coeff. $\quad 2 = b - 4$

$b = 6$

$\therefore \quad f(x) = (x^2 - 4x + 13)(x^2 + 6x + 13)$

$x^2 + 6x + 13 = 0 \Rightarrow x = \dfrac{-6 \pm \sqrt{36 - 4(13)}}{2}$

$= \dfrac{-6 \pm \sqrt{-16}}{2} = -3 \pm 2i$

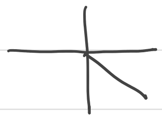
$\therefore$ roots are $2 \pm 3i, \; -3 \pm 2i$

$\therefore f(x) = (x - 2 + 3i)(x - 2 - 3i)(x + 3 + 2i)(x + 3 - 2i)$

Question 11

d) $(\sqrt{3})^2 + (-1)^2 = 4$

$|\sqrt{3} - i| = 2$



$\tan\theta = \frac{-1}{\sqrt{3}}$

$\theta = -\frac{\pi}{6}$

$\therefore \sqrt{3} - i = 2e^{-\frac{i\pi}{6}}$

$z^4 = \sqrt{3} - i$

Let $z = re^{i\theta}$

$(re^{i\theta})^4 = 2e^{i\left(-\frac{\pi}{6} + 2k\pi\right)} \qquad k \in \mathbb{Z}$

$r^4 e^{4i\theta} = 2e^{i\left(-\frac{\pi}{6} + 2k\pi\right)}$

$r^4 = 2 \qquad 4\theta = = -\frac{\pi}{6} + 2k\pi$

$r = 2^{1/4} \qquad \theta = -\frac{\pi}{24} + \frac{k\pi}{2} = \frac{(12k-1)\pi}{24}$

$z = \sqrt[4]{2}\, e^{-\frac{i\pi}{24}}, \ \sqrt[4]{2}\, e^{i\frac{11\pi}{24}}, \ \sqrt[4]{2}\, e^{i\frac{23\pi}{24}}, \ \sqrt[4]{2}\, e^{-i\frac{13\pi}{24}}$

$k=0 \qquad k=1 \qquad k=2 \qquad k=-1$

Question 11

e) $\frac{5x^2+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$= \frac{A(x^2+1) + (Bx+C)x}{x(x^2+1)}$

$5x^2 + 2 \equiv A(x^2+1) + (Bx+C)x$

x coeff: $0 = C$

$x = 0$: $2 = A$

x^2 coeff : $5 = A + B$ so $B = 3$

$\therefore \frac{5x^2+2}{x(x^2+1)} = \frac{2}{x} + \frac{3x}{x^2+1}$

$\int \frac{5x^2+2}{x(x^2+1)}\,dx = \int\left(\frac{2}{x} + \frac{3x}{x^2+1}\right)dx$

$= 2\ln|x| + \frac{3}{2}\ln|x^2+1| + C$

Question 11 Markers Feedback

a) Some silly errors. Please avoid rushing when completing easy questions.

b) (i) Read the question carefully and answer in correct form.

(ii) This was marked generously as the justification was often not well argued. (Please look at the solution)
The question states ‘hence’ which means students needed to use the result from (i) in order to obtain the mark. You could not just show that tan(LHS) = tan (RHS).

c) Some confusion over language here. 2+3i is a root while $(x-2-3i)$ is a factor.
Some students found the roots but did not answer the question by fully factorising $f(x)$ over $\mathbb{C}$.
The most common method used by students when writing a solution involved sum and product of roots.

d) To gain the first mark students needed to write $\sqrt{3}+i$ in exponential form as asked in the question. It was clear that some students had trouble working in exponential form.

e) Very well done.

Question 12

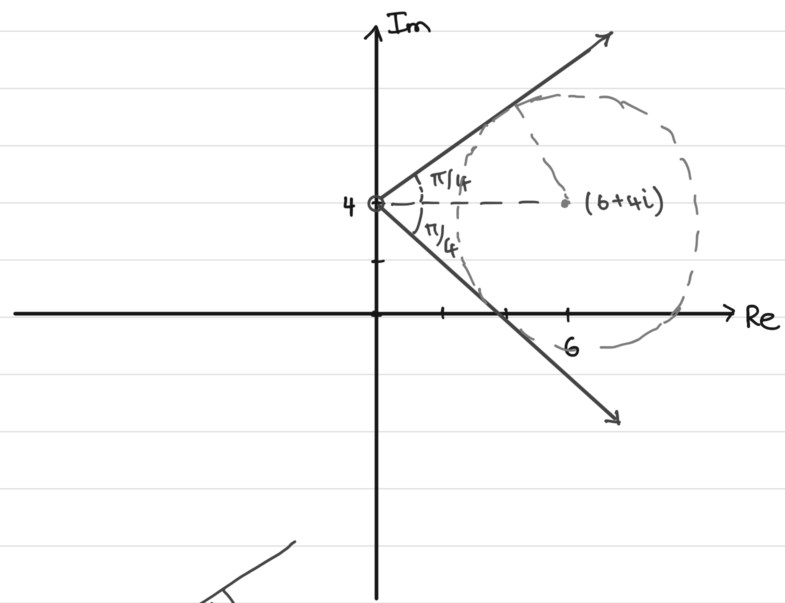
a) $\displaystyle\int \frac{2-x}{\sqrt{4-x^2}}\,dx = \int \frac{2\,dx}{\sqrt{4-x^2}} - \int \frac{x\,dx}{\sqrt{4-x^2}}$

$\displaystyle= 2\int \frac{dx}{\sqrt{2^2-x^2}} + \frac{1}{2}\int -2x\left(4-x^2\right)^{-\frac{1}{2}}dx$

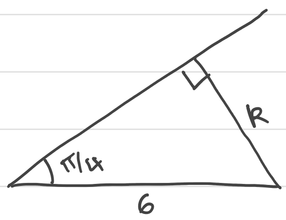
$\displaystyle= 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{\cancel{2}}\,\frac{\left(4-x^2\right)^{\frac{1}{2}}}{\frac{1}{\cancel{2}}} + C$

$\displaystyle= 2\sin^{-1}\left(\frac{x}{2}\right) + \sqrt{4-x^2} + C$

b)



(ii)



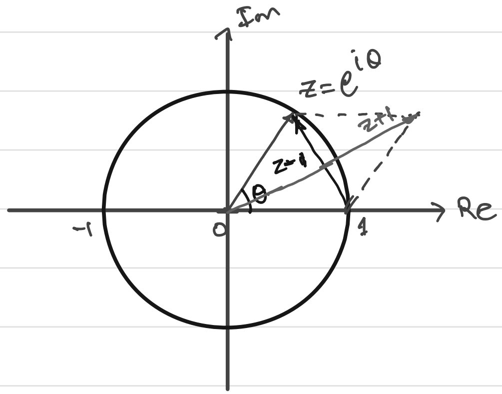
$\sin\frac{\pi}{4} = \frac{k}{6}$ so $k = 6\sin\frac{\pi}{4}$

$= \frac{6}{\sqrt{2}}$

$k = 3\sqrt{2}$

Question 12

c) i)



complete the parallelogram of vectors with z and 1 as the adjacent sides. As $|z|=1$, this parallelogram is a rhombus.

$z-1$ and $z+1$ are the two diagonals of this rhombus (as shown).

As diagonals of the rhombus meet at right angles $z-1$ is perpendicular to $z+1$

ie. $z-1=(ki)(z+1)$ where $k\in\mathbb{R}$

so $\dfrac{z-1}{z+1}=ki$ which is purely imaginary.

Question 12

c) ii)

$e^{i\theta}=\cos\theta+i\sin\theta$ ①

$e^{-i\theta}=\cos(-\theta)+i\sin(-\theta)$

$=\cos\theta-i\sin\theta$ ② as cos is even, sin is odd

$\therefore e^{i\theta}+e^{-i\theta}=2\cos\theta$

$\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$

① − ②: $e^{i\theta}-e^{-i\theta}=2i\sin\theta$

$\sin\theta=\frac{1}{2i}(e^{i\theta}-e^{-i\theta})$

(iii)

$$\frac{z-1}{z+1}=\frac{e^{i\theta}-1}{e^{i\theta}+1}$$

$$=\frac{\cancel{e^{i\theta/2}}(e^{i\theta/2}-e^{-i\theta/2})}{\cancel{e^{i\theta/2}}(e^{i\theta/2}+e^{-i\theta/2})}$$

$$=\frac{\cancel{2}i\sin\theta/2}{\cancel{2}\cos\theta/2}$$

$=i\tan\theta/2$ as reqd.

d) $t=\tan\frac{x}{2}$ $\quad\cos x=\frac{1-t^2}{1+t^2}$; $\sin x=\frac{2t}{1+t^2}$

$x=2\tan^{-1}t$

$dx=\frac{2dt}{1+t^2}$

x	0	$\pi/2$
t	0	1

Question 12

d)
$$\int_0^{\pi/2} \frac{1}{\cos x + 2\sin x + 3}\,dx = \int_0^1 \frac{1}{\frac{1-t^2}{1+t^2} + \frac{2(2t)}{1+t^2} + 3}\;\frac{2\,dt}{1+t^2}$$

$$= \int_0^1 \frac{2dt}{1-t^2+4t+3(1+t^2)} = \int_0^1 \frac{2dt}{2t^2+4t+4}$$

$$= \int_0^1 \frac{dt}{t^2+2t+2} = \int_0^1 \frac{dt}{(t+1)^2+1}$$

$$= \left[\tan^{-1}(t+1)\right]_0^1 = \tan^{-1}2 - \tan^{-1}1$$
$$= \tan^{-1}2 - \frac{\pi}{4}$$

e) RTP $x+y < 1+xy$

consider $x+y-1-xy$
$$= x - xy - 1 + y$$
$$= x(1-y) - 1(1-y)$$
$$= (x-1)(1-y)$$
< 0 as $x-1<0$ and $1-y>0$

because $0<x<1$ & $0<y<1$

So $x+y-1-xy<0$

$\therefore\ x+y < 1+xy$ as reqd.

Question 12 Markers Feedback

(a) Generally well done although the reverse chain rule not always recognised leading to some circuitous methods.

(b) (i) Generally done well. Students are encouraged to mark the angle rather than rely on marking points – which is prone to error and relies on marker doing calculations to check you got the angle right. Don't forget the open circle.

(ii) Not well done. Many students thought the circle touches the line directly above the centre at (6,10) and got a radius of 4 units. Some thought it touches at the x-intercept (4,0) and obtained a radius of $2\sqrt{5}$. Several others used algebra and solved simultaneously with varying degrees of success. This is worth 1 mark so students should be looking for an easier approach – the most efficient way was to use some simple right angle trigonometry.

(c) (i) This type of geometric argument should have been familiar territory but for many it proved too hard. Diagrams were seldom drawn so even when a geometric argument was offered it was hard for the marker to follow. Some students did know how to proceed, however, despite the allocation of two marks some very brief responses were offered. The level of detail in responses should be commensurate with the allocated marks, making it easier for the marker to award credit.

(ii) This was bookwork and generally was attempted well although students often offered incorrect reasoning such as "cos is odd, sin is even".\

(iii) This part was not very well done. Most students struggled to apply the result from the previous part. Those who started with the RHS could see how to use the previous results and were generally more successful. Some abandoned any pretence of applying the previous result and proceeded to derive the required result from scratch by using $z = \cos\theta + i\sin\theta$ and using double angle trigonometric identities. These responses could not be awarded full credit.

(d) This was a standard question and was well done. It was evident that a small number of students have not adequately practised this substitution and made lamentably little progress on what should have been a straightforward integration problem. These students are advised to address this gap before the HSC.

(e) Not very well done. Students who used the technique of looking at the difference of the LHS and RHS were generally successful as the terms factorise to yield a product of two factors whose signs can be established in order to get the desired result. A number of students tried to unsuccessfully force an AM-GM result here. Other responses showed a lack of understanding of inequalities for eg $0<x<1; 0<y<1$ can be combined to get $0<x+y<2$ and $0<1+xy<2$ but this does not allow us to conclude that $x+y<1+xy$.

Question 13

a) $u_{k+1} = 2u_k + (-1)^{k+1}$ and $u_1 = 1$

RTP: $u_n = \frac{1}{3}\left(2^{n+1} + (-1)^n\right)$ for $n \in \mathbb{Z}^+$

Test $n=1$

$LHS = u_1 = 1$

$RHS = \frac{1}{3}\left(2^{1+1} + (-1)^1\right)$

$= \frac{1}{3}(4-1) = \frac{1}{3} \times 3 = 1$

$LHS = RHS$

True for $n=1$

Assume true for $n=k$, $k \in \mathbb{Z}^+$

so, $u_k = \frac{1}{3}\left(2^{k+1} + (-1)^k\right)$

Prove true for $n = k+1$

RTP: $u_{k+1} = \frac{1}{3}\left[2^{k+2} + (-1)^{k+1}\right]$

$LHS = u_{k+1} = 2u_k + (-1)^{k+1}$ using recursive relationship

$= 2\left[\frac{1}{3}\left(2^{k+1} + (-1)^k\right)\right] + (-1)^{k+1}$ using assumption

$= \frac{1}{3}2.2^{k+1} + \frac{2}{3}(-1)^k + (-1)^{k+1}$

$= \frac{1}{3}2^{k+2} + \frac{1}{3}(-1)^k\left[2 + 3(-1)\right]$

$= \frac{1}{3}2^{k+2} + \frac{1}{3}(-1)^k(2-3)$

$= \frac{1}{3}\left[2^{k+2} + (-1)^k(-1)\right]$

$= \frac{1}{3}\left[2^{k+2} + (-1)^{k+1}\right]$

$= RHS$

$\therefore$ true for $n=k+1$ when true for $n=k$

Hence, the statement is true by Mathematical Induction

Question 13

b) $A\ (4, -3, -3)$ $B\ (5, 2, 2)$

$\vec{AB} = \vec{OB} - \vec{OA}$

$= \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$

Eqn of l_1: $\underset{\sim}{r} = \begin{pmatrix} 4 \\ -3 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$

NB: Other answers possible

(ii) l_2: $\frac{x-1}{k} = \frac{y}{-1} = \frac{z+3}{1} = \mu$

For l_1: $x = 4 + \lambda$ (1) For l_2: $x = 1 + k\mu$ (2)

$y = -3 + 5\lambda$ (3) $y = -\mu$ (4)

$z = -3 + 5\lambda$ (5) $z = -3 + \mu$ (6)

From (5) and (6)

$\cancel{-3} + 5\lambda = \cancel{-3} + \mu$

$\mu = 5\lambda$

From (3) and (4)

$-3 + 5\lambda = -\mu$

so $-3 + 5\lambda = -5\lambda$

$10\lambda = 3$

$\lambda = \frac{3}{10}$ Then $\mu = 5\lambda = \frac{15}{10} = \frac{3}{2}$

Question 13

b) (ii) From (1) and (2)

$$4+\lambda = 1+k\mu$$

$$4+\frac{3}{10} = 1+k\cdot\frac{3}{2}$$

$$\frac{3k}{2} = 3+\frac{3}{10}$$

$$= \frac{33}{10}$$

$$k = \frac{\cancel{33}^{11}}{\cancel{10}_5} \times \frac{\cancel{2}}{\cancel{3}}$$

$$k = \frac{11}{5}$$

c) i) Let $u = a-x$

$x = a-u$

$dx = -du$

x	0	a
u	a	0

$$\int_0^a f(x)\,dx = \int_a^0 f(a-u)\cdot(-du)$$

$$= \int_0^a f(a-u)\,du \quad \text{limits swapped}$$

$$= \int_0^a f(a-x)\,dx \quad u \text{ is a dummy variable}$$

$$\therefore \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Question 13

c) (ii) $$\int_0^{\pi/4} \frac{\sin x}{\cos x+\sin x}\,dx$$

$$= \int_0^{\pi/4} \frac{\sin(\pi/4-x)}{\cos(\pi/4-x)+\sin(\pi/4-x)}\,dx \quad \text{using (i)}$$

$$= \int_0^{\pi/4} \frac{\sin\pi/4\cos x-\cos\pi/4\sin x}{\cos\pi/4\cos x+\sin\pi/4\sin x+\sin\frac{\pi}{4}\cos x-\cos\frac{\pi}{4}\sin x}\,dx$$

$$= \int_0^{\pi/4} \frac{\frac{1}{\sqrt{2}}(\cos x-\sin x)}{\frac{1}{\sqrt{2}}(\cos x+\cancel{\sin x}+\cos x-\cancel{\sin x})} \quad \text{as } \cos\frac{\pi}{4}=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

$$= \int_0^{\pi/4} \frac{\cos x-\sin x}{2\cos x}$$

$$= \frac{1}{2}\int_0^{\pi/4}\left(1-\frac{\sin x}{\cos x}\right)dx$$

$$= \frac{1}{2}\Big[x+\ln\cos x\Big]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4}+\ln\frac{1}{\sqrt{2}}\right)-(0+\ln 1)\right]$$

$$= \frac{\pi}{8}+\frac{1}{2}\ln\left(2^{-\frac{1}{2}}\right) = \frac{\pi}{8}-\frac{1}{4}\ln 2 \quad \text{as reqd.}$$

Question 13

d) $F = m\ddot{x}$ $\quad m = 2$

$$2\ddot{x} = 4x^3 + 16x$$

$$\ddot{x} = 2x^3 + 8x$$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 8x$$

Integrate wrt x

$$\frac{1}{2}v^2 = \frac{x^4}{2} + 4x^2 + C$$

$$v^2 = x^4 + 8x^2 + C_1$$

At $t=0$, $x = -2$, $v = 8$

$$64 = 16 + 32 + C_1$$

$$C_1 = 16$$

$$\therefore \quad v^2 = x^4 + 8x^2 + 16$$

$$= (x^2+4)^2$$

$$v = \pm(x^2+4)$$

Initially $v > 0$ so $v = x^2+4$ is valid

As $v \neq 0$ for any x, the particle never stops

hence never turns $\therefore$ $v \neq -(x^2+4)$

So $v = \dfrac{dx}{dt} = x^2 + 4$

$$\int_{-2}^{x} \frac{dx}{x^2+4} = \int_0^t dt \qquad \text{using separation of variables}$$

Question 13

(d) $$\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{x} = \left[t\right]_0^t$$

$$\frac{1}{2}\tan^{-1}\frac{x}{2} - \frac{1}{2}\tan^{-1}(-1) = t$$

$$\tan^{-1}\frac{x}{2} - \left(-\frac{\pi}{4}\right) = 2t$$

$$\tan^{-1}\frac{x}{2} = 2t - \frac{\pi}{4}$$

$$x = 2\tan^{-1}\left(2t - \frac{\pi}{4}\right)$$

Question 13 Markers Feedback

(a) Generally well done.

Some students wasted time by testing for two base cases. The recursive result depends only on one prior term.

(b) (i) Well done.

(ii) A second parameter needs to be introduced for the second line. The parameters in each vector equation will generally not have the same value at the point of intersection. It should be understood that an equation of this form is really $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=\lambda$, where λ is the parameter. Each of the three components can then be unravelled to write the vector equation of the line.

(c) (i) Generally well done. But it is not meaningful to say "Let $x=a-x$".

(ii) It is always wise to go with the 'hence', and use the 'otherwise' only when this fails or the other option appears obviously easier. Those who didn't take the hint generally did not succeed.

Some students wasted time re-deriving the result of part (i) for this specific case.

A few students didn't understand the restriction implied by this result - the two a's must be the same.

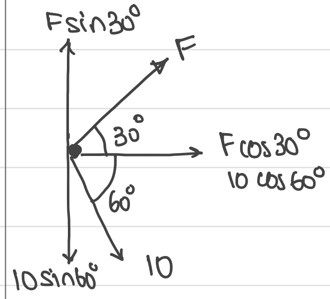
It was disappointing that a number of students could not perform the simple 2 unit integral that arose after completing the algebraic simplification.

(d) Quite well done.

There had to be a clear explanation of why the negative root was dropped. Simply claiming $v>0$, was not sufficient. Knowing that the velocity is initially positive is also not sufficient - think of SHM as an example.

Question 14

(a)



Vertically $F\sin 30^\circ = 10\sin 60^\circ$

$F \times \frac{1}{2} = 10 \times \frac{\sqrt{3}}{2}$

$F = 10\sqrt{3}$

Horizontally $m\ddot{x} = F\cos 30^\circ + 10\cos 60^\circ$

$m=2 \Rightarrow 2\ddot{x} = \cancel{10}^{5}\sqrt{3}\times\frac{\sqrt{3}}{\cancel{2}} + \cancel{10}^{5}\times\frac{1}{\cancel{2}}$

$2\ddot{x} = 15+5$

$= 20$

$\ddot{x} = 10$

The acceleration is 10 m/s in the direction of motion

b) $\underset{\sim}{r} = \begin{pmatrix}13\\8\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\-1\end{pmatrix}$ $A=(3,-2,6)$

$P = (-p, 0, 2p)$

$\overrightarrow{PA} = \begin{pmatrix}3+p\\-2\\6-2p\end{pmatrix}$ $PA \perp \ell$

so $\begin{pmatrix}3+p\\-2\\6-2p\end{pmatrix}\cdot\begin{pmatrix}2\\2\\-1\end{pmatrix} = 0$

$2(3+p)+(-2)(2)+(6-2p)(-1)=0$

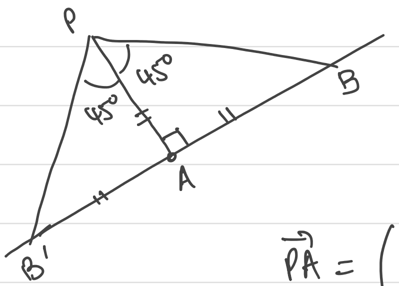
$6+2p-4-6+2p=0$

$4p = 4$

$p = 1$

Question 14

b) (iii)



ΔPAB is isosceles

$\therefore PA = AB$

$$\vec{PA} = \begin{pmatrix} 3+p \\ -2 \\ 6-2p \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$$

$$|\vec{PA}| = \sqrt{4^2 + (-2)^2 + 4^2}$$
$$= \sqrt{36}$$
$$= 6$$

PA is 6 units so AB = 6 units.

B lies 6 units either side of A on l.

Direction vector of l is $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ whose magnitude is 3

unit vector in this direction is $\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

So $\vec{OB} = \vec{OA} + \vec{AB}$

$$= \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} \pm 6 \times \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} \pm \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 4 \end{pmatrix} \text{ or } \begin{pmatrix} -1 \\ -6 \\ 8 \end{pmatrix}$$

$B = (7, 2, 4)$ or $(-1, -6, 8)$

Question 14

c) $\ddot{x} = -n^2 x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2 x$ Integrate wrt x

$$\frac{1}{2}v^2 = -n^2 \frac{x^2}{2} + C$$

$$v^2 = -n^2 x^2 + C_1$$

When $x = A$, $v = 0$ (at rest at extremes)

$$0 = -n^2A^2 + C_1$$
$$C_1 = n^2A^2$$
$$\therefore v^2 = -n^2x^2 + n^2A^2$$
$$= n^2(A^2 - x^2)$$
$$\dot{x}^2 = n^2(A^2 - x^2) \quad \text{as } v = \dot{x}$$

(ii) ratio of speeds at $x = -1.6$ & $x = 1.2$ is 3:4

ratio of v^2 is 9 : 16

$$\frac{\cancel{n^2}(A^2 - 1.6^2)}{\cancel{n^2}(A^2 - 1.2^2)} = \frac{9}{16}$$

$$16(A^2 - 1.6^2) = 9(A^2 - 1.2^2)$$
$$16A^2 - 16 \times 1.6^2 = 9A^2 - 9 \times 1.2^2$$
$$7A^2 = 16 \times 1.6^2 - 9 \times 1.2^2$$
$$A^2 = \frac{16 \times 1.6^2 - 9 \times 1.2^2}{7} = 4$$
$$A = 2$$

Question 14

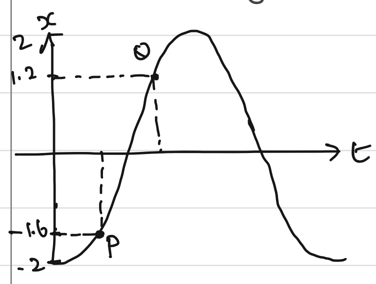
c) (iii) max speed is at centre of motion ie. $x = 0$

$v^2 = n^2(A^2 - x^2)$

$\left(\frac{\pi}{3}\right)^2 = n^2 \times 4$

$n^2 = \left(\frac{\pi}{3}\right)^2 \times \frac{1}{4}$

$n = \frac{\pi}{6}$



we need time taken to travel from P to Q.

Let the displacement eqn be $x = -2\cos\frac{\pi t}{6}$

At P: $-2\cos\frac{\pi t}{6} = -1.6$

$\cos\frac{\pi t}{6} = 0.8$

$\frac{\pi t}{6} = 0.6435..$

$t = 1.22899.$ s

At Q: $-2\cos\frac{\pi t}{6} = 1.2$

$\cos\frac{\pi t}{6} = -0.6$

$\frac{\pi t}{6} = 2.21429..$

$t = 4.22899...$

Time taken from P to Q $= 4.2289 - 1.2289$

$= 3$ seconds.

Question 14 Markers Feedback

(a) Not well done. There was no reference to gravity in the question, nor was there anything to suggest gravity would be involved.

There were two valid ways to interpret the question. Either the object continued indefinitely to the right, in which case you could find values for F and the acceleration, or the given direction was only instantaneous, in which case the acceleration would be a function of F. In either interpretation, forces needed to be resolved both vertically and horizontally.

Many students mixed up sin and cos for horizontal and vertical components. Draw triangles as an aid if you can't visualise it.

Many students did not not consider that the vertical components of the two forces were in opposite directions, and did not attribute a negative to one of those components. Others did not consider vertical components at all, and could be awarded little.

When two forces F_1 and F_2 cancel, do not begin by saying $F_1 = F_2$. Start with $F_1 - F_2 = 0$ to indicate that the net force is zero, then rearrange. (This was not penalised.)

A large number of students thought vertical and horizontal components could be added numerically instead of vectorially. 5N vertically plus 5N horizontally does not equate to a net force of 10N. Please use $\underset{\sim}{i}$ and $\underset{\sim}{j}$ in your working to avoid this error.

Many students either did not convert the force to an acceleration, or multiplied by 2 instead of dividing. Please do the vector calculations on forces, and find the acceleration at the end.

(b) (i) Please use correct notation. l is not a vector, so it is not meaningful to refer to $\overline{PA} \cdot l$.

Also, be careful when translating from component form to column form, noting when a component is missing, in this case the $\underset{\sim}{j}$ component.

(ii) Naming $\overline{OB}$ $x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ was not helpful as it does not constrain B to be on the line. If it wasn't linked to the line at some point using the parameter, it was difficult to award marks.

(c) (i) Generally well done, but a number did not know what to do. This is book work.

Start from the supplied information to derive the result. Students who differentiated the result or introduced a trigonometric displacement function to differentiate could not be awarded full marks.

(ii) Many students thought the ratio of 3:4 corresponded to fractions of $\frac{3}{7}$ and $\frac{4}{7}$.

There is no quantity which is being divided in that ratio. It is not correct to substitute the values in the ratio into the result of part (i). They are not actual velocities, and such a statement is correct only after taking the ratios of the velocities.

Many lost the square in the result of part (i).

(iii) Read the question - you were asked for a time difference, not the distance travelled.

A number of students did not find the time for direct travel.

Remember to use radian mode on your calculator. As many students made the mistake of using degree mode, answers left in exact form had to be penalised.

There was no need to introduce an α in your displacement equation. The time **difference** is not affected by shifting the graph left or right.

Question 15

a) RTP $\frac{1}{n!} < \frac{1}{e^n}$ $n \geq 6$

Test $n=6$

LHS $= \frac{1}{6!}$ RHS $= \frac{1}{e^6}$

$= \frac{1}{720}$ $= 0.00247..$

$= 0.0013..$ LHS < RHS $\therefore$ true for $n=6$

Assume true for $n=k$, $k \geq 6$

ie. $\frac{1}{k!} < \frac{1}{e^k}$

Prove true for $n=k+1$

RTP $\frac{1}{(k+1)!} < \frac{1}{e^{k+1}}$

LHS $= \frac{1}{(k+1)!}$

$= \frac{1}{(k+1)\,k!} = \frac{1}{k+1} \times \frac{1}{k!}$

$< \frac{1}{k+1} \cdot \frac{1}{e^k}$ As $k+1 > e$ for $k \geq 6$ $\frac{1}{k+1} < \frac{1}{e}$

$< \frac{1}{e} \cdot \frac{1}{e^k}$

$= \frac{1}{e^{k+1}} =$ RHS

LHS < RHS $\therefore$ true for $n=k+1$ when true for $n=k$

Hence, stmt is true by Mathematical Induction

Question 15

(a) (ii)

$$\frac{1}{6!} < \frac{1}{e^6}$$
$$\frac{1}{7!} < \frac{1}{e^7}$$
$$\frac{1}{8!} < \frac{1}{e^8}$$
$$\vdots \qquad \vdots$$

Adding we get

$$\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}+\dots < \underbrace{\frac{1}{e^6}+\frac{1}{e^7}+\frac{1}{e^8}+\dots}_{\text{geometric series } a=\frac{1}{e^6},\ r=\frac{1}{e}}$$

As $0<\frac{1}{e}<1$ S_∞ exists

$$= \frac{\frac{1}{e^6}}{1-\frac{1}{e}}$$

$$= \frac{1}{\cancel{e^6}_{\,e^5}} \times \frac{\cancel{e}}{e-1}$$

$$= \frac{1}{e^6-e^5}$$

$$\therefore \frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}+\dots < \frac{1}{e^6-e^5}$$

Question 15

(b) (i) $I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx$

$u = \frac{1}{(1+x^2)^n}$ $\quad v' = 1$

$u' = \frac{-2nx}{(1+x^2)^{n+1}}$ $\quad v = x$

$$= \left[\frac{x}{(1+x^2)^n}\right]_0^1 + 2n\int_0^1 \frac{x^2\,dx}{(1+x^2)^{n+1}}$$

$$= \left(\frac{1}{2^n} - 0\right) + 2n\int_0^1 \frac{x^2\,dx}{(1+x^2)^{n+1}}$$

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$$

(ii) $I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{1+x^2-1}{(1+x^2)^{n+1}}\,dx$

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{1}{(1+x^2)^n} - \frac{1}{(1+x^2)^{n+1}}\,dx$$

$$I_n = \frac{1}{2^n} + 2n\,I_n - 2n\,I_{n+1}$$

$$2n\,I_{n+1} = \frac{1}{2^n} + (2n-1)\,I_n$$

$$I_{n+1} = \frac{1}{2n.2^n} + \frac{2n-1}{2n}\,I_n$$

$$I_{n+1} = \frac{1}{n\,2^{n+1}} + \frac{2n-1}{2n}\,I_n \quad \text{as reqd.}$$

Question 15

b)(iii) $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2}\,dx = I_2$

$$I_2 = \frac{1}{1.2^2} + \frac{1}{2} I_1$$

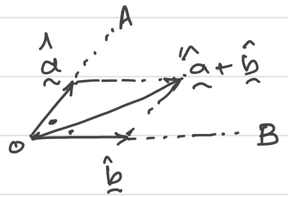
$$= \frac{1}{4} + \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{4} + \frac{1}{2}\left[\tan^{-1}x\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{2}\left(\tan^{-1}1 - \tan^{-1}0\right)$$

$$= \frac{1}{4} + \frac{\pi}{8}$$

(c)(i)



As $\hat{a}$ and $\hat{b}$ are unit vectors, they are equal in length so $\hat{a}+\hat{b}$ is the leading diagonal of the rhombus with $\hat{a}$ and $\hat{b}$ as adjacent sides.

Diagonals of a rhombus bisect vertex angles so $\hat{a}+\hat{b}$ bisects $A\hat{O}B$ ie. $\hat{a}+\hat{b}$ lies along the angle bisector $\overrightarrow{OX}$.

So $\overrightarrow{OX} = \lambda(\hat{a}+\hat{b})$

Question 15

c) (ii) $\overrightarrow{OX} = \lambda(\hat{a}+\hat{b})$ from (i)

Also $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$

$$= \underset{\sim}{a} + \mu\overrightarrow{AB}$$

$$= \underset{\sim}{a} + \mu(\underset{\sim}{b} - \underset{\sim}{a})$$

$$= \underset{\sim}{a}(1-\mu) + \mu\underset{\sim}{b}$$

Equating the two expressions for $\overrightarrow{OX}$

$$\lambda(\hat{a}+\hat{b}) = \underset{\sim}{a}(1-\mu) + \mu\underset{\sim}{b}$$

$$\lambda\hat{a} + \lambda\hat{b} = |\underset{\sim}{a}|(1-\mu)\hat{a} + \mu|\underset{\sim}{b}|\hat{b}$$

Equating components

$\lambda = |\underset{\sim}{a}|(1-\mu)$ and $\lambda = \mu|\underset{\sim}{b}|$

$1-\mu = \dfrac{\lambda}{|\underset{\sim}{a}|}$ $\qquad \mu = \dfrac{\lambda}{|\underset{\sim}{b}|}$

$$\frac{|\overrightarrow{AX}|}{|\overrightarrow{XB}|} = \frac{|\mu.\overrightarrow{AB}|}{|(1-\mu)\overrightarrow{AB}|} = \frac{\mu\,\cancel{|\overrightarrow{AB}|}}{(1-\mu)\,\cancel{|\overrightarrow{AB}|}}$$

$$= \frac{\mu}{1-\mu}$$

From above $\dfrac{\mu}{1-\mu} = \dfrac{\lambda/|\underset{\sim}{b}|}{\lambda/|\underset{\sim}{a}|}$

$$= \frac{|\underset{\sim}{a}|}{|\underset{\sim}{b}|} = \frac{|\overrightarrow{OA}|}{|\overrightarrow{OB}|}$$

$\dfrac{|\overrightarrow{AX}|}{|\overrightarrow{XB}|} = \dfrac{|\overrightarrow{OA}|}{|\overrightarrow{OB}|}$ $\quad\therefore\ AX : XB = OA : OB$

Question 15 Markers Feedback

(a) (i) Generally well done. Verifying the initial case is an easy first mark – students should ensure they gain this mark and not take liberties here. After using the assumption, the final inductive step of the proof needs to be argued carefully.

(ii) Most students were able to make the link with the result in part (i) to claim that $\frac{1}{6!}+\frac{1}{7!}+...<\frac{1}{e^6}+\frac{1}{e^7}+...$ and were able to gain the first mark. Recognising that the RHS is a geometric series was essential to obtain the required result.

(b) (i) Students who chose the correct parts were able to obtain the result easily.

(ii) The most direct way to obtain the result was to start with the result in (i) and use algebraic manipulation (writing x^2 as $1+x^2-1$) to evaluate the integral in the RHS of the result and many students did this successfully. A large number of students tried to use the result from (i) by substituting $n+1$ in place of *n*. This was not an efficient strategy and while it is possible to obtain the required result, it still required the algebraic manipulation. None of the attempts using this method were entirely successful. Students who showed the algebraic manipulation were able to gain part marks.

(iii) Too many students lost easy marks in this part by using $n=2$. As the formula is for I_{n+1}, to compute I_2 we need $n=1$.

(c) (i) Nearly all students who attempted this part claimed the diagonal of a parallelogram bisects the vertex angle. It is worth reviewing basic quadrilateral properties (and also some of the basic circle geometry theorems) as these can be presented for proof using vector methods.

(ii) Very few successful attempts here. Students who obtained two different expressions for the same vector are at least on their way to a solution. Common errors in notation use included writing ratios of vectors rather than of their lengths. *AB* is used to denote the name of interval *AB* and also its length and is not the same as $\overrightarrow{AB}$. Also $|\underset{\sim}{a}+\underset{\sim}{b}|\neq|\underset{\sim}{a}|+|\underset{\sim}{b}|$.

Question 16

(a) RTP: $\forall a,b \in \mathbb{Z}^+,\ a^2-4b \neq 2$

For the sake of contradiction, assume $\exists\, a,b \in \mathbb{Z}^+ : a^2-4b=2$

Then $a^2 = 4b+2$

$= 2(2b+1)$

$\therefore a^2$ is even.

So $a=2k$ for some $k\in\mathbb{Z}^+$

$(2k)^2 = 2(2b+1)$

$2\cancel{4}k^2 = \cancel{2}(2b+1)$

$2k^2 = 2b+1$

LHS $=2k^2$ is even & RHS $=2b+1$ is odd we have a contradiction.

Hence our assumption is incorrect.

So $\forall a,b\in\mathbb{Z},\ a^2-4b\neq 2$.

(b)(i) $\displaystyle\sum_{k=1}^{n}\frac{1}{k(k+1)} = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{n(n+1)}$

$= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$= 1-\frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$

Question 16

b) (iii)

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}$$

$$=\frac{1}{1.1}+\frac{1}{2.2}+\frac{1}{3.3}+\dots+\frac{1}{n.n}$$

$$<1+\frac{1}{1.2}+\frac{1}{2.3}+\dots+\frac{1}{(n-1)(n)}$$

$$=1+\frac{n-1}{n}\qquad \text{using (i)}$$

$$<1+1\qquad \text{as } \frac{n-1}{n}<1$$

$$=2$$

$$\therefore\ \frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{n^2}<2$$

c) i)

$$\text{LHS}=\frac{(\cot\theta+i)^{2n+1}-(\cot\theta-i)^{2n+1}}{2i}$$

$$=\frac{1}{2i}\left[\left(\frac{\cos\theta}{\sin\theta}+i\right)^{2n+1}-\left(\frac{\cos\theta}{\sin\theta}-i\right)^{2n+1}\right]$$

$$=\frac{1}{2i}\left[\left(\frac{\cos\theta+i\sin\theta}{\sin\theta}\right)^{2n+1}-\left(\frac{\cos\theta-i\sin\theta}{\sin\theta}\right)^{2n+1}\right]$$

$$=\frac{1}{2i}\left[\left(\frac{\text{cis}\,\theta}{\sin\theta}\right)^{2n+1}-\left(\frac{\text{cis}(-\theta)}{\sin\theta}\right)^{2n+1}\right]$$

Question 16

c) (i)

$$\text{LHS}=\frac{1}{2i}\left[\frac{\text{cis}(2n+1)\theta}{\sin^{2n+1}\theta}-\frac{\text{cis}(-(2n+1)\theta)}{\sin^{2n+1}\theta}\right]$$

$$=\frac{1}{2i\sin^{2n+1}\theta}\left(\text{cis}(2n+1)\theta-\overline{\text{cis}(2n+1)\theta}\right]\qquad \text{cis}(-\alpha)=\overline{\text{cis}\,\alpha}$$

$$=\frac{1}{\cancel{2i}\sin^{2n+1}\theta}\,\cancel{2i}\,\sin(2n+1)\theta\qquad \text{cis}\,\alpha-\overline{\text{cis}\,\alpha}=2i\sin\alpha$$

$$=\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta}$$

$$=\text{RHS}$$

(ii)

$$(\cot\theta+i)^{2n+1}=\cot^{2n+1}\theta+\binom{2n+1}{1}\cot^{2n}\theta\,.\,i+\binom{2n+1}{2}\cot^{2n-1}\theta\, i^2+\dots+\binom{2n+1}{2n}\cot\theta\, i^{2n}+i^{2n+1}$$

$$(\cot\theta-i)^{2n+1}=\cot^{2n+1}\theta-\binom{2n+1}{1}\cot^{2n}\theta\, i+\binom{2n+1}{2}\cot^{2n-1}\theta\, i^2+\dots+\binom{2n+1}{2n}\cot\theta\, i^{2n}-i^{2n+1}$$

$$(\cot\theta+i)^{2n+1}-(\cot\theta-i)^{2n+1}=2\binom{2n+1}{1}\cot^{2n}\theta\, i+2\binom{2n+1}{3}\cot^{2n-2}\theta\, i^3+\dots+2i^{2n+1}$$

$$=2\left[\binom{2n+1}{1}\cot^{2n}\theta\, i-\binom{2n+1}{3}\cot^{2n-2}\theta\, i+\dots+(-1)^n i\right]$$

by simplifying powers of i

$$=2i\left[\binom{2n+1}{1}\cot^{2n}\theta-\binom{2n+1}{3}\cot^{2n-2}\theta+\dots+(-1)^n\right]$$

$$\frac{(\cot\theta+i)^{2n+1}-(\cot\theta-i)^{2n+1}}{2i}=\binom{2n+1}{1}\cot^{2n}\theta-\binom{2n+1}{3}\cot^{2n-2}\theta+\dots+(-1)^n$$

Question 16

c) (ii) Using (i) $\binom{2n+1}{1}\cot^{2n}\theta - \binom{2n+1}{3}\cot^{2n-2}\theta + \dots + (-1)^n = \frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta}$

Let $x = \cot^2\theta$

$$\binom{2n+1}{1}x^n - \binom{2n+1}{3}x^{n-1} + \dots + (-1)^n = \frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta}$$

Solutions of $\binom{2n+1}{1}x^n - \binom{2n+1}{3}x^{n-1} + \dots + (-1)^n = 0$ ①

correspond to solns of $\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta} = 0$ where $x = \cot^2\theta$

$\therefore \sin(2n+1)\theta = 0$

$$(2n+1)\theta = m\pi \qquad m \in \mathbb{Z}$$

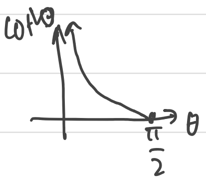
$$\theta = \frac{m\pi}{2n+1}$$

$\therefore$ solutions to ① are $x = \cot^2\left(\frac{m\pi}{2n+1}\right)$

As the eqn is a degree n polynomial, there can be up to n solutions. Note that $m \neq 0$, $\because \cot 0$ is undefined

Now $0 < \frac{m\pi}{2n+1} < \frac{\pi}{2}$ for $m = 1$ to n

and $\cot\theta$ is one-to-one in this interval, and yields n distinct values.



Thus, the solutions are $x = \cot^2\left(\frac{m\pi}{2n+1}\right)$ for $m = 1, 2, \dots n$

Question 16

c) (iii) $\sum_{m=1}^{n} \cot^2\frac{m\pi}{2n+1} = -\frac{a_{n-1}}{a_n} = \binom{2n+1}{3} \div \binom{2n+1}{1}$ sum of roots

$$= \frac{\cancel{(2n+1)}(\cancel{2}n)(2n-1)}{3\times\cancel{2}\times 1} \times \frac{1}{\cancel{2n+1}}$$

$$= \frac{n(2n-1)}{3}$$

set $n = 8$ $\quad 2n+1 = 17$

$$\cot^2\frac{\pi}{17} + \cot^2\frac{2\pi}{17} + \dots + \cot^2\frac{8\pi}{17} = \frac{8\times\cancel{15}\,5}{\cancel{3}}$$

$$= 40$$

Question 16 Markers Feedback

a) Most students could not negate this statement correctly. This led to further errors when writing the proof.

Statement	Negation
$\mathbb{R}, P(x)$	$\mathbb{R}: \ \sim P(x)$

Some students incorrectly wrote the negation as $\forall x \in \mathbb{R}, \sim P$. They then found a counterexample that "contradicted" this statement. This could not be awarded any further marks because all they have done is find <u>one</u> example where the statement given in the question is true.

When completing a proof by contradiction you are trying to disprove the opposite rather than prove the original. The technique involves assuming the negation and then often working with that until you arrive at an absurdity. Instead, many students just ended up trying to complete a direct proof of the original statement in the question. This could not be awarded any further marks because they were not demonstrating the proof technique required.

b) (i) Many students took the long road and proved this result using Mathematical Induction. Generally, if a question requires Induction it will be stated in the question. Applying a partial fractions method to $\dfrac{1}{k(k+1)}$ was a much more efficient way to show the result.

(ii) Not well done. Often students tried to introduce an inequality but had the sign the wrong way around which meant they could not get to the result.

c) (i) Time seemed to be a factor for most students by this point in the paper as many responses appeared rushed. Make sure that you are showing enough steps to justify the awarding of two marks. The question says 'use De Moivre's theorem' so this suggests you need first to get the LHS in terms of $cis\theta$.

(ii) Not well done at all. The word 'deduce' suggests that the use of the identity in part (i) will be required. Some students realised that the binomial coefficients implied a binomial expansion but then did not make much progress. While the solution explains in detail why there are n distinct solutions, this was not required to earn the marks, although students needed to use justify why $m \neq 0$.

(iii) This question could be done without successfully completing part (ii) and it was pleasing to see a number of students persevere and get this part out successfully.