Penrith High School
2010
Trial Higher School
Certificate Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question
- Staple this test to your answers

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value

Name:________________________________

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

a) Find $\int \frac{9-x^2}{9x}\,dx$. **2**

b) Use integration by parts to evaluate $\int_1^e x^2 \log_e x\,dx$. **3**

c) Find $\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$. **3**

d) By completing the square on the denominator, or otherwise, find $\int \frac{dx}{\sqrt{8x-7-x^2}}$. **3**

e) Use the substitution $t = \tan\frac{x}{2}$, to find $\int \frac{dx}{4-5\sin x}$. **4**

Question 2 on next page

Question 2

a) Evaluate i^{2010}. **1**

b)

i. Write down the expansion of $(ai+2)^3$. **2**

ii. Hence find the values of a so that $(ai+2)^3$ is real. **2**

c)

i. Write $2\sqrt{3}+2i$ in modulus-argument form. **2**

ii. Hence solve the equation $z^4 = 2\sqrt{3}+2i$, giving answers in modulus-argument form. **2**

iii. Sketch the four solutions on an Argand diagram and find the area of the square formed by the four points. **2**

d)

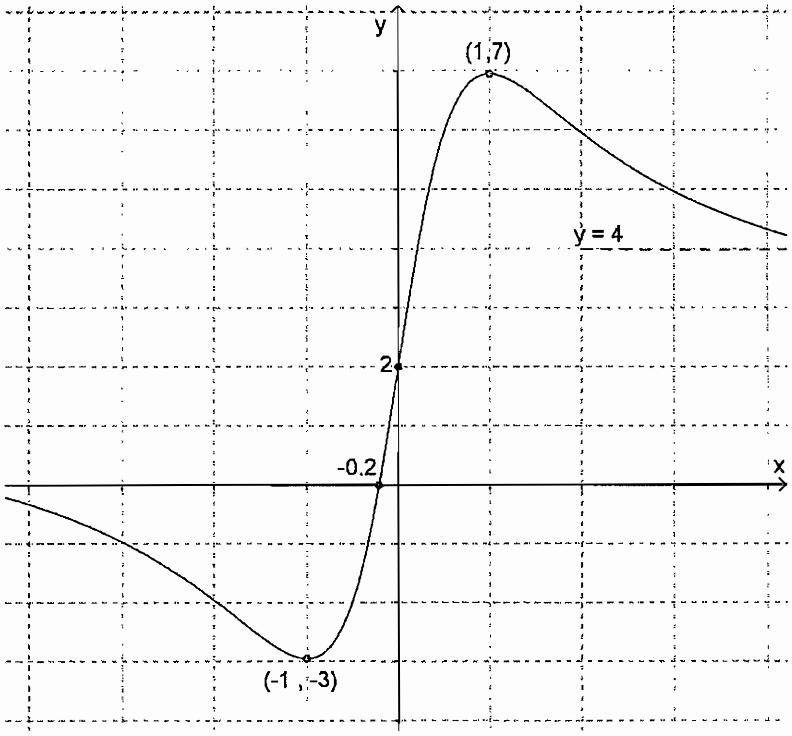
i. On an Argand diagram shade the region where both $|z-2| \le 1$ and $Re(z) \le \frac{3}{2}$ hold. **2**

ii. Find the maximum possible value for $\arg z$ for points in this region. **2**

Question 3 on next page

Question 3

a) The graph of $y = f(x)$ is shown below. The graph has two asymptotes: $y = 0$ and $y = 4$. It has turning points at $(-1, -3)$ and $(1, 7)$ as shown. The point $(0, 2)$ is a point of inflection and the intersection point with the x-axis is $(-0.2, 0)$.



An extra page with copies of this graph has been provided. Answer the following questions on the page provided and attach it to your solutions.

i. $y = |f(x)|$ **2**

ii. $y = f(|x|)$ **2**

iii. $y^2 = f(x)$ **2**

iv. $y = f'(x)$ **3**

b) $I_n = \int_0^1 \sqrt{x}(1-x)^n dx$.

i. Verify that $(1-x)^{n-1} - (1-x)^n = x(1-x)^{n-1}$. **1**

ii. Hence, by using integration by parts, show that $I_n = \frac{2n}{2n+3} I_{n-1}$. **3**

iii. Hence evaluate $\int_0^1 \sqrt{x}(1-x)^3 dx$. **2**

Question 4 on next page

Question 4

a) Find the real factors of $P(x) = 4x^3 - 5x^2 + 14x + 15$ given that one of its roots is $1 - 2i$. **3**

b) The cubic equation $4x^3 - 20x^2 + 17x - p = 0$ has p as a real number. The equation has three positive real roots, two of which are equal. Find the value p and the roots. **3**

c)

i. Use the expansion of $(\cos\theta + i\sin\theta)^3$ to show that $\cot 3\theta = \dfrac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}$. **3**

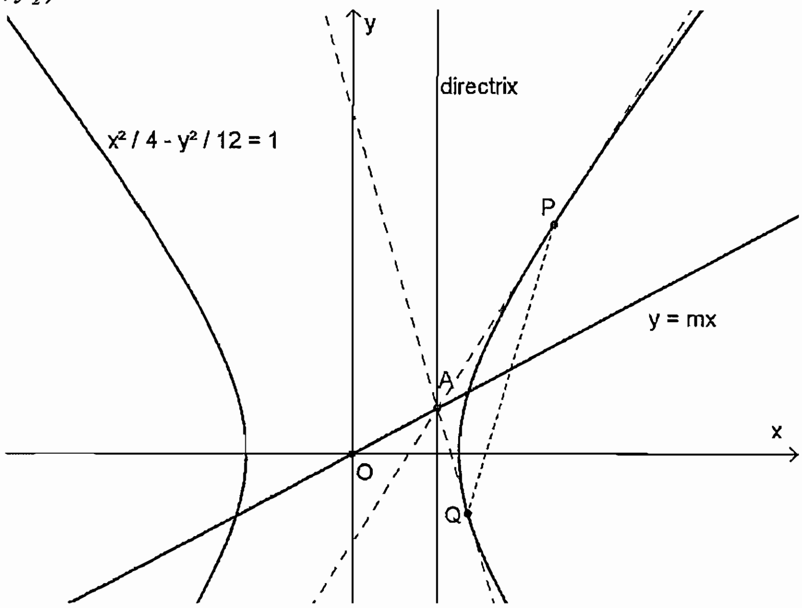
ii. Hence solve the equation $x^3 - 3x^2 - 3x + 1 = 0$. **3**

iii. Hence find the exact values of $\cot\dfrac{\pi}{12}$ and $\cot\dfrac{5\pi}{12}$. **3**

Question 5 on next page

Question 5

a) The line $y = mx$ cuts a directrix of the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$ at the point A in the first quadrant. Tangents are drawn from A to the hyperbola, touching it at points $P(x_1, y_1)$ and $Q(x_2, y_2)$.



i. Calculate the eccentricity of the hyperbola. **2**

ii. Find the co-ordinates of the foci, the equations of the directrices and the equations of the asymptotes of the hyperbola. **3**

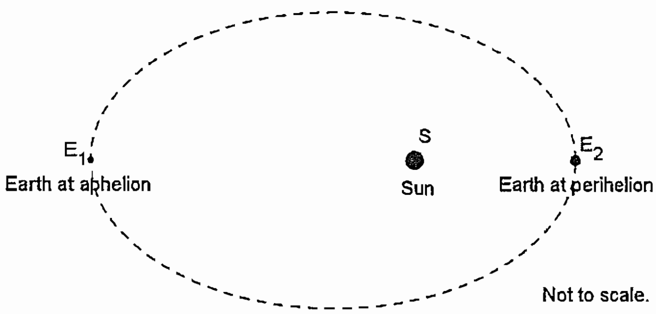
iii. Find the co-ordinates of point A. **1**

iv. Use the formula $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$ to find the equation of the chord of contact PQ. **1**

v. Show that the chord of contact PQ passes through a focus. **1**

vi. Find the midpoint M of points P and Q and show that point M lies on the line $y = mx$. **4**

b) The earth moves around the sun in an elliptical orbit.

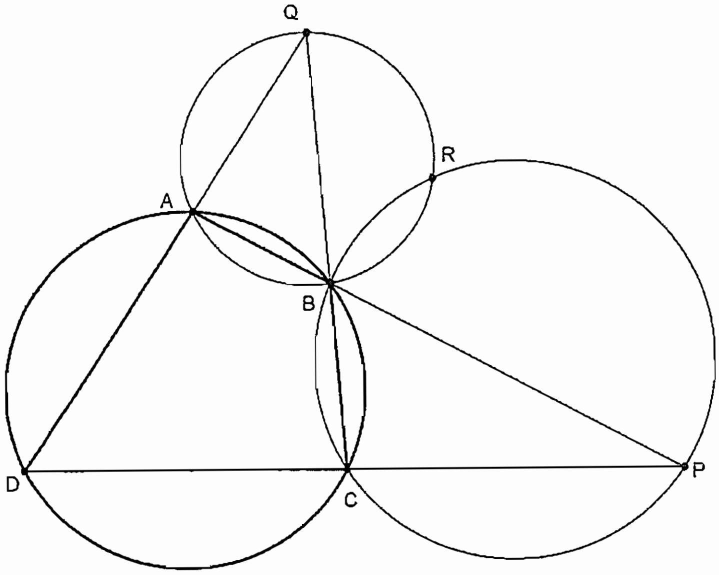


The sun lies at one of the foci of the ellipse. When the earth is furthest from the sun (called the aphelion) it is 1.521×10^8 kilometres from the sun. When the earth is closest to the sun (called the perihelion) it is 1.471×10^8 kilometres from the sun. Work out the eccentricity of the Earth's elliptical orbit to 3 significant figures. **3**

Question 6 on next page

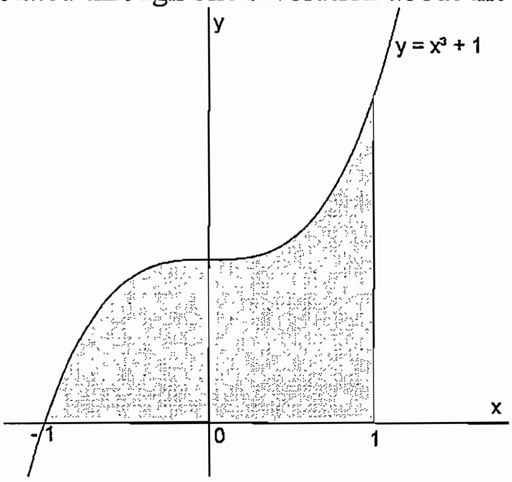
Question 6

a) *ABCD* is a cyclic quadrilateral. The opposite sides *AB* and *DC* are produced to meet at *P*, and the sides *CB* and *DA* are produced to meet at *Q*. The two circles through the vertices of the triangles *PBC* and *QAB* intersect at *R*.



Prove that the points *P*, *R* and *Q* are collinear. **3**

b) The diagram below shows the area bounded by the curve $y = x^3 + 1$, the x –axis and the line $x = 1$. This area is rotated through one revolution about the line $x = 1$.



i. Find the volume of the generated solid using the "slice method". **5**

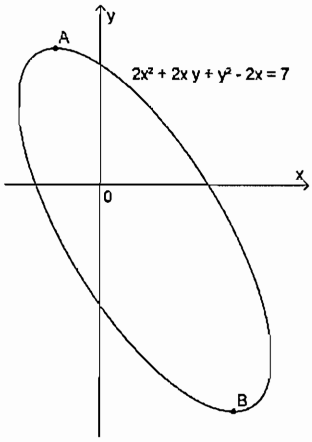
ii. Find the volume of the generated solid using cylindrical shells method. **5**

c) The equation $x^3 - 2x + 6 = 0$ has roots α , β and γ .
Find a cubic equation with roots 5α , 5β and 5γ . **2**

Question 7 on next page

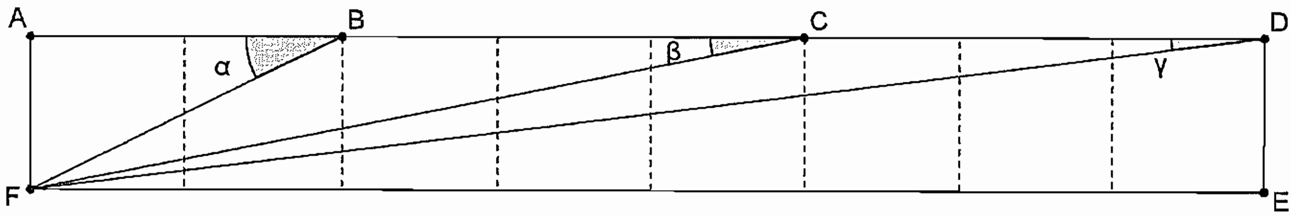
Question 7

a) The ellipse $2x^2 + 2xy + y^2 - 2x = 7$ has a maximum point at A and a minimum point at B as shown in the diagram below.



Find the coordinates of points A and B. **5**

b) The rectangle $ADEF$ has dimensions 1×8. It is divided into 8 smaller 1×1 squares as shown below.



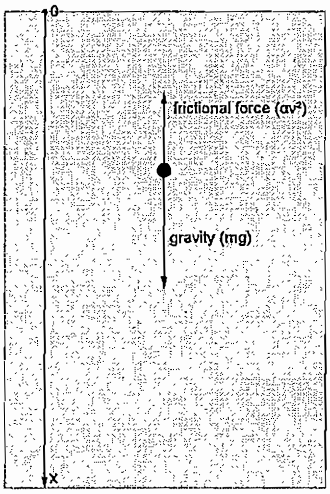
$\angle ABF = \alpha$, $\angle ACF = \beta$ and $\angle ADF = \gamma$.

Show that $\alpha + \beta + \gamma = \frac{\pi}{4}$ radians. **4**

Question 7 continued next page

Question 7 continued

c) A ball of mass m is dropped in a fluid where the resistive frictional force is proportional to the square of the velocity of the ball. Gravity is providing downward acceleration and the resistive frictional force is effectively providing upward force, since friction opposes the direction of motion.



So the forces on the object can be expressed as $ma = mg - \alpha v^2$, where a is the acceleration of the ball, v is the velocity of the ball, g is the acceleration due to gravity and α is a constant.

i. Prove that $v = \sqrt{\frac{g}{\beta}(1 - e^{-2\beta x})}$. where v is the velocity of the ball and $\beta = \frac{\alpha}{m}$. **4**

ii. If $g = 9.8\ ms^{-2}$ and $\beta = 0.2\ m^{-1}$ find how far the ball will fall before it obtains a velocity of $6\ ms^{-1}$? Give your answer to the nearest centimetre. **2**

Question 8 on next page

Question 8

a) The ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the hyperbola $\frac{x^2}{36}-\frac{y^2}{k}=1$, $(k>0)$ share the same directrices. Find the value of k. **4**

b) Prove the following for all real numbers $x \neq 0$.

i. $x^2+\frac{1}{x^2}\geq 2$. **1**

ii. $x^2+\frac{1}{x^2}\geq x+\frac{1}{x}$. **3**

c) The sequence $T_1, T_2, T_3, \ldots\ldots\ldots\ldots\ldots\ldots$ is defined so that $T_n = 4T_{n-1} - T_{n-2}$ for $n \geq 3$. It is also given that $T_1 = 2$ and $T_2 = 4$.

i. By direct calculation find the next 3 terms of the sequence and show that $T_5 = 194$. **1**

ii. Prove by induction that $T_n = (2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1}$ for $n \geq 1$. **4**

iii. Explain why, when $\left(2+\sqrt{3}\right)^{100}$ is calculated as a decimal number, it starts with more than fifty 9s in a row after the decimal point. **2**

End of test

Penrith High School
Extension Two Mathematics 2010
Solutions

QUESTION 1

(a) $\int \frac{9}{9x} - \frac{x^2}{9x}\,dx$

$= \int\left(\frac{1}{x} - \frac{x}{9}\right)dx = \ln x - \frac{x^2}{18} + C$

(b) $\int_1^e x^2 \ln x\,dx$

$= \left[\frac{1}{3}x^3 \ln x\right]_1^e - \int_1^e \frac{1}{3}x^3\left(\frac{1}{x}\right)dx$

$= \left[\frac{1}{3}x^3 \ln x\right]_1^e - \int_1^e \frac{1}{3}x^2\,dx$

$= \left[\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3\right]_1^e$

$= \left(\frac{1}{3}e^3 - \frac{1}{9}e^3\right) - \left(0 - \frac{1}{9}\right)$

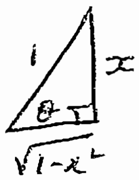
$= \frac{2}{9}e^3 + \frac{1}{9}$

(c) $\begin{cases} x = \sin\theta \\ dx = \cos\theta\,d\theta \end{cases}$

$I = \int \frac{1}{(\cos^2\theta)^{3/2}}(\cos\theta\,d\theta)$

$= \int \sec^2\theta\,d\theta$

$= \tan\theta + C$



$= \frac{x}{(1-x^2)^{1/2}} + C$

(d) $I = \int \frac{dx}{\sqrt{9-(x-4)^2}}$

$= \sin^{-1}\left(\frac{x-4}{3}\right) + C$

(e) $I = \int \frac{\frac{2\,dt}{1+t^2}}{4 - 5\left(\frac{2t}{1+t^2}\right)}$

$= \int \frac{2\,dt}{4(1+t^2) - 5(2t)}$

$= \int \frac{2\,dt}{4t^2 - 10t + 4}$

$= \int \frac{1\,dt}{(2t-1)(t-2)}$

$= \int \frac{1/3}{t-2} - \frac{2/3}{2t-1}\,dt$

$= \frac{1}{3}\ln(t-2) - \frac{1}{3}\ln(2t-1) + C$

$= \frac{1}{3}\ln\left(\frac{t-2}{2t-1}\right) + C$

$= \frac{1}{3}\ln\left[\frac{\tan\frac{\theta}{2} - 2}{2\tan\frac{\theta}{2} - 1}\right] + C$

QUESTION 2

(a) $(i^4)^{502} \cdot i^2 = (1)(-1) = -1$

(b) (i) $1(ai)^3 + 3(ai)^2(2) + 3(ai)2^2 + 2^3$

$= -a^3 i - 6a^2 + 12ai + 8$

$= (8 - 6a^2) + (12a - a^3)i$

(ii) Real if $(12a - a^3) = 0$

$a(12 - a^2) = 0$

$\therefore a = 0, \pm 2\sqrt{3}$

(c) (i) $4 \operatorname{cis} \frac{\pi}{6}$

(ii) $z^4 = 4 \operatorname{cis}\left(\frac{\pi}{6} + 2n\pi\right)$

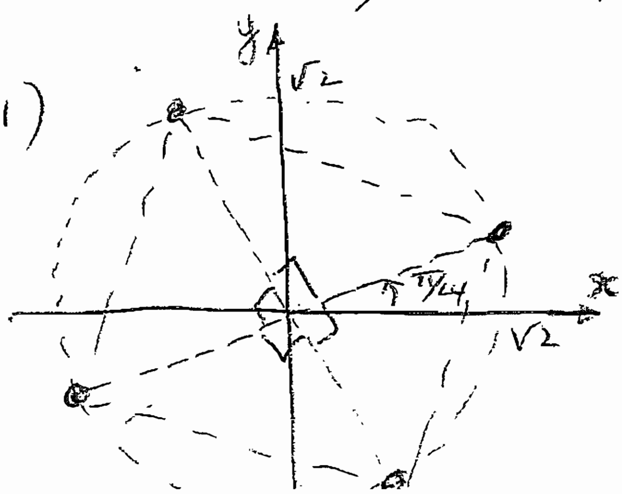
$z = 4^{1/4} \operatorname{cis}\left[\frac{1}{4}\left(\frac{\pi}{6} + 2n\pi\right)\right]$

$z = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{24} + \frac{n\pi}{2}\right)$

Taking $n = 0, 1, 2, 3$ to give 4 answers

$z = \sqrt{2} \operatorname{cis} \frac{\pi}{24}, \sqrt{2} \operatorname{cis} \frac{13\pi}{24},$
$\sqrt{2} \operatorname{cis} \frac{25\pi}{24}, \sqrt{2} \operatorname{cis} \frac{37\pi}{24}$

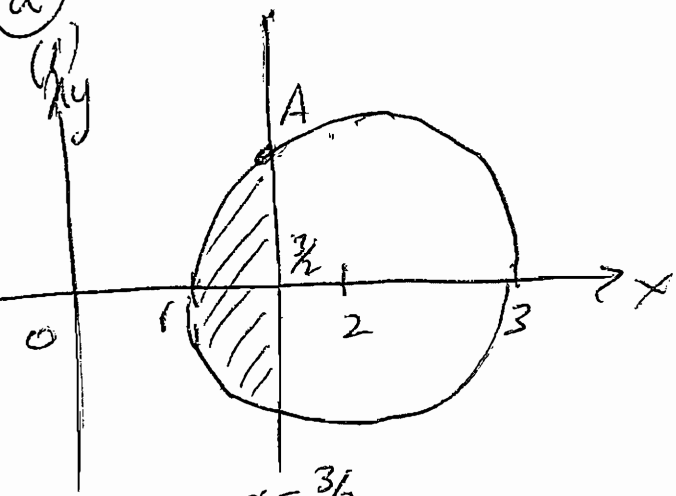
(iii)



Area $= 4 \times \left(\frac{1}{2} \times \sqrt{2} \times \sqrt{2}\right)$

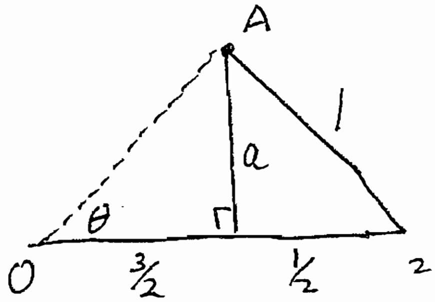
$= 4$ units^2

(d) (i)



$x = 3/2$

(ii) At point A, arg z is a maximum.



$a = \sqrt{1^2 - \frac{1}{2}^2} = \frac{\sqrt{3}}{2}$

$\arg z = \theta$

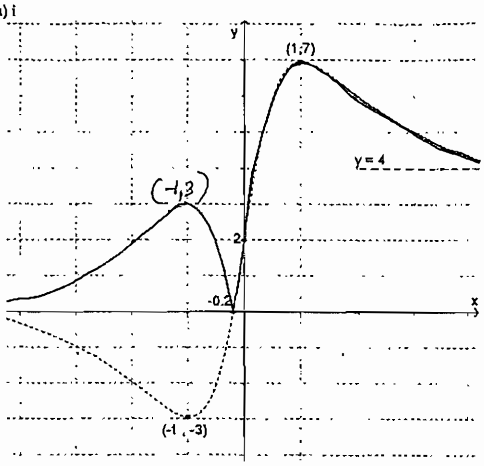
$= \tan^{-1} \frac{\sqrt{3}/2}{3/2}$

$= \pi/6$

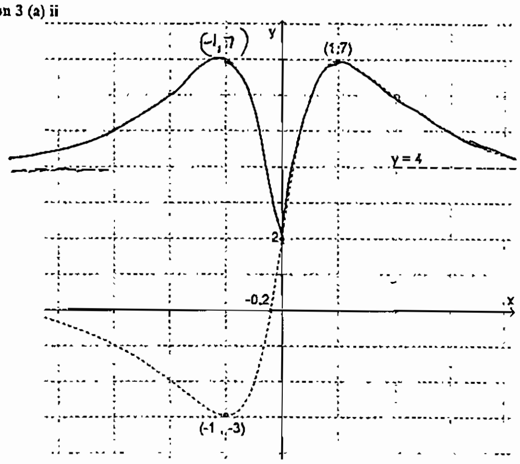
NB: OA is tangent to the circle.

QUESTION 3

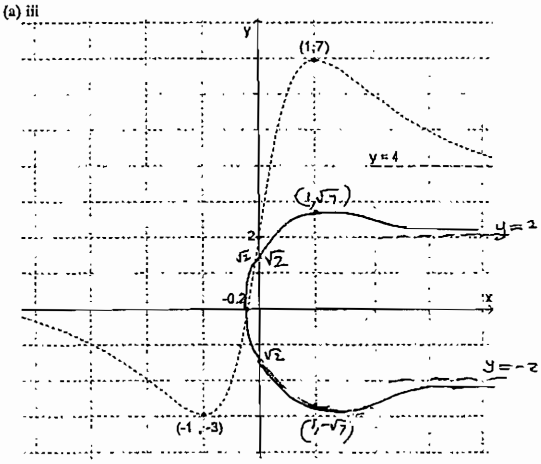
Question 3 (a) i



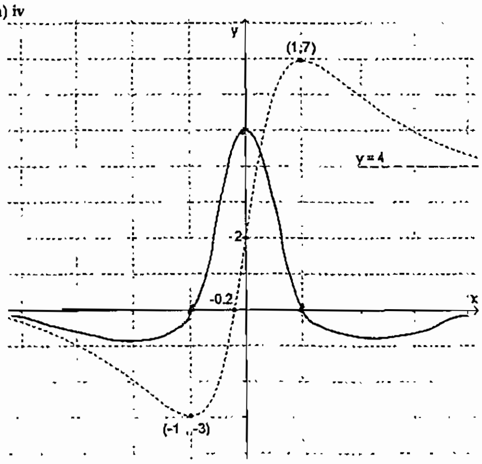
Question 3 (a) ii



Question 3 (a) iii



Question 3 (a) iv



(b) (i) $(1-x)^{n-1} - (1-x)^n = (1-x)^{n-1}[1-(1-x)]$

$= (1-x)^{n-1}(1-1+x)$

$= x(1-x)^{n-1}$

(ii)

$u = (1-x)^n$	$v' = \sqrt{x}$
$u' = -n(1-x)^{n-1}$	$v = \frac{2}{3}x\sqrt{x}$

$I_n = \int_0^1 \sqrt{x}(1-x)^n dx$

$= \left[\frac{2}{3}x\sqrt{x}(1-x)^n\right]_0^1 + \frac{2}{3}n\int_0^1 x\sqrt{x}(1-x)^{n-1} dx$

$= 0 + \frac{2}{3}n\int_0^1 \sqrt{x}\left[(1-x)^{n-1} - (1-x)^n\right] dx$

$= \frac{2n}{3}(I_{n-1} - I_n)$

$3I_n = 2nI_{n-1} - 2nI_n$

$(3+2n)I_n = 2nI_{n-1}$

$\therefore \quad I_n = \frac{2n}{2n+3} I_{n-1}$

(iii) $I_0 = \int_0^1 \sqrt{x}\,dx = \left[\frac{2}{3}x\sqrt{x}\right]_0^1 = \frac{2}{3}$

$I_1 = \frac{2}{5}I_0 = \frac{2}{5} \times \frac{2}{3} = \frac{4}{15}$

$I_2 = \frac{4}{7}I_1 = \frac{4}{7} \times \frac{4}{15} = \frac{16}{105}$

$I_3 = \frac{6}{9}I_2 = \frac{6}{9} \times \frac{16}{105} = \frac{32}{315}$

QUESTION 4

(a) $\overline{1-2i} = 1+2i$ is also a root since the coefficients are real.

Sum $1-2i+1+2i+\alpha = -b/a$

$2+\alpha = 5/4$

$\therefore$ other root, $\alpha = -3/4$

$\therefore$ Factors are

$(4x+3)(x-(1-2i))(x-(1+2i))$

Real factors are

$(4x+3)(x^2-2x+5)$

(b) $\frac{d}{dx}(4x^3-20x^2+17x)=0$ has that double root once

$12x^2-40x+17=0$

$(2x-1)(6x-17)=0$

$x=\frac{1}{2}$ or $x=\frac{17}{6}$

Sum of roots $\frac{1}{2}+\frac{1}{2}+\alpha=\frac{20}{4}$

$\alpha = 4$

or $\frac{17}{6}+\frac{17}{6}+\alpha=\frac{20}{4}$

$\alpha = -2/3$

But roots are all positive.

So roots are $1/2, 1/2, 4$.

Product of roots: $\frac{1}{2}\times\frac{1}{2}\times 4=\frac{P}{4}$

$\therefore P=4$

(c) $(\cos\theta)^3 = \cos 3\theta$

(i) Also

$(\cos\theta+i\sin\theta)^3 = (\cos\theta)^3 + 3\cos^2\theta(i\sin\theta)+3\cos\theta(i\sin\theta)^2+(i\sin\theta)^3$

$=(\cos^3\theta-3\cos\theta\sin^2\theta)+i(3\cos^2\theta\sin\theta-\sin^3\theta)$

Equating real and imag. parts

$\frac{\cos 3\theta}{\sin 3\theta} = \frac{\cos^3\theta-3\cos\theta\sin^2\theta}{3\cos^2\theta\sin\theta-\sin^3\theta}\begin{matrix}\div\sin^3\theta\\ \div\sin^3\theta\end{matrix}$

$\cot 3\theta = \frac{\cot^3\theta-3\cot\theta}{3\cot^2\theta-1}$

(ii) Rearrange to get

$\frac{x^3-3x}{3x^2-1}=1$

Let $x=\cot\theta$

$\frac{\cot^3\theta-3\cot\theta}{3\cot^2\theta-1}=1$

$\cot 3\theta = 1$

$3\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}\ldots$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}\ldots$

$\therefore x = \cot\frac{\pi}{12}, \cot\frac{5\pi}{12}, -1$

(iii) Since $x=-1$ is a solution, $(x+1)$ is a factor

$x^3-3x^2-3x+1=(x+1)(x^2-4x+1)$ by long division.

$\therefore \cot\frac{\pi}{12}, \cot\frac{5\pi}{12}$ are roots of $x^2-4x+1=0$

$(x-2)^2=3$

$x=2\pm\sqrt{3}$

Note $\cot\frac{\pi}{12} > \cot\frac{5\pi}{12}$

$\therefore \cot\frac{\pi}{12} = 2+\sqrt{3}$

$\cot\frac{5\pi}{12} = 2-\sqrt{3}$

QUESTION 5

(a) (i) $b^2 = a^2(e^2-1)$ $\qquad a = 2$

$12 = 4(e^2-1)$ $\qquad b = 2\sqrt{3}$

$e = 2$

(ii) Foci: $(\pm ae, 0) = (\pm 4, 0)$

Directrices: $x = \pm \frac{a}{e} = \pm 1$

Asymptotes $y = \pm \frac{b}{a}x = \pm\sqrt{3}\,x$

(iii) $A(1, m)$

(iv) $\frac{x(1)}{4} - \frac{y(m)}{12} = 1$

$y = \frac{3}{m}(x-4)$ (*)

(v) When $x = 4$

$y = \frac{3}{m}(4-4) = 0$

$\therefore$ PQ passes through focus $(4, 0)$

(vi) Sub (*) into $\frac{x^2}{4} - \frac{y^2}{12} = 1$

$$\frac{x^2}{4} - \frac{\left[\frac{3}{m}(x-4)\right]^2}{12} = 1$$

$$\frac{x^2}{4} - \frac{3(x-4)^2}{4m^2} = 1$$

$$m^2x^2 - 3(x-4)^2 = 4m^2$$

$$m^2x^2 - 3x^2 + 24x - 48 = 4m^2$$

$$(m^2-3)x^2 + 24x^2 - (48+4m^2) = 0$$

This equation has roots x_1 and x_2

$$x_1 + x_2 = \frac{-24}{m^2-3}$$

$\therefore$ x value of M

$$= \frac{x_1+x_2}{2} = \frac{-12}{m^2-3} = \frac{12}{3-m^2}$$

Sub into *

$$y = \frac{3}{m}\left(\frac{12}{3-m^2} - 4\right)$$

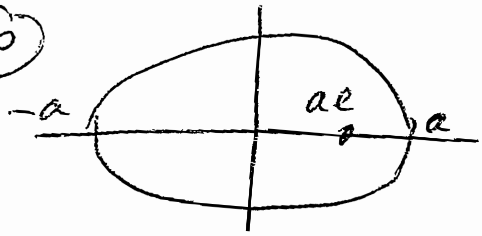
$$y = \frac{3}{m}\left(\frac{4m^2}{3-m^2}\right) = \frac{12m}{3-m^2}$$

$\therefore M\left(\frac{12}{3-m^2}, \frac{12m}{3-m^2}\right)$

which satisfies $y = mx$

i.e M lies of $y = mx$

(b)



$a - ae = 1.471 \times 10^8$ (1)

$a + ae = 1.521 \times 10^8$ (2)

(1)+(2) $2a = 2.992 \times 10^8$

$a = 1.496 \times 10^8$

From (2)

$$1 + e = \frac{1.521 \times 10^8}{a}$$

$1 + e = 1.0167123$

$\therefore e \doteqdot 0.0167$

QUESTION 6

(a) Join PR, BR, PR

$\angle QRB = \angle DAB$ (ext $\angle$ of cyc. quad = opp int. angle)

$\angle PRB = \angle BCD$ (")

But $\angle DAB + \angle BCD = 180$ (opp. $\angle$s of a cycl. quad)

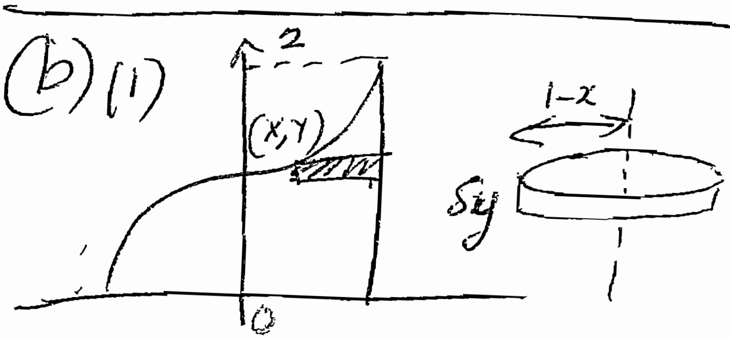
$\therefore \angle QRB + \angle PRB = 180°$

$= \angle DAB + \angle BCD = 180°$

$\therefore \angle QRB$ is straight

ie Q, R, P are collinear.

(b) (i)



$$V = \int_0^2 \pi (1-x)^2 \, dy$$

$$V = \int_0^2 \pi \left[1-(y-1)^{1/3}\right]^2 dy$$

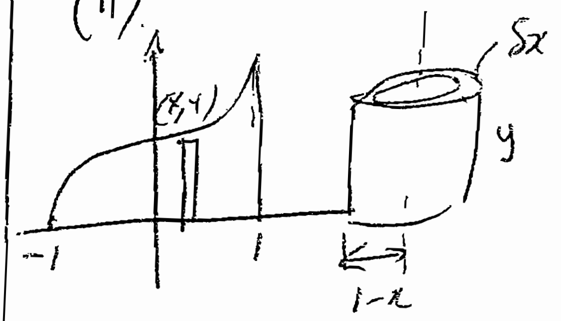
$$V = \pi \int_0^2 1 - 2(y-1)^{1/3} + (y-1)^{2/3} \, dy$$

$$V = \pi \left[y - \tfrac{3}{2}(y-1)^{4/3} + \tfrac{3}{5}(y-1)^{5/3}\right]_0^2$$

$$V = \pi \left[\left(2 - \tfrac{3}{2} + \tfrac{3}{5}\right) - \left(0 - \tfrac{3}{2} - \tfrac{3}{5}\right)\right]$$

$$V = \frac{16\pi}{5} \text{ units}^3$$

(ii)



$$V = \int_{-1}^{1} 2\pi (1-x) y \, dx$$

$$V = 2\pi \int_{-1}^{1} (1-x)(x^3+1) \, dx$$

$$V = 2\pi \int_{-1}^{1} (1 - x + x^3 - x^4) \, dx$$

$$V = 2\pi \left[x - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^5}{5}\right]_{-1}^{1}$$

$$V = 2\pi \left[\frac{11}{20} - \left(-1\frac{1}{20}\right)\right] = \frac{16\pi}{5} \text{ units}^3$$

c Let $y = 5x$

$$\left(\frac{y}{5}\right)^3 - 2\left(\frac{y}{5}\right) + 6 = 0$$

$$y^3 - 50y + 750 = 0$$

QUESTION 7

(a) Differenciate both sides

$4x + 2xy' + 2y + 2yy' - 2 = 0$

At A, B $y' = 0$

$4x + 0 + 2y + 0 - 2 = 0$

$\therefore y = 1 - 2x$

sub into ellipse equation

$2x^2 + 2x(1-2x) + (1-2x)^2 - 2x = 7$

$2x^2 + 2x - 4x^2 + 1 - 4x + 4x^2 - 2x = 7$

$2x^2 - 4x - 6 = 0$

$x^2 - 2x - 3 = 0$

$(x+1)(x-3) = 0$

$x = -1 \qquad x = 3$

From $y = 1-2x$: $y = 3 \qquad y = -5$

ie $A(-1, 3) \qquad B(3, -5)$

(b) $\tan\alpha = \frac{1}{2}, \tan\beta = \frac{1}{5}, \tan\gamma = \frac{1}{8}$

$\tan[\alpha+\beta+\gamma] = \dfrac{\tan(\alpha+\beta) + \tan\gamma}{1 - \tan(\alpha+\beta)\tan\gamma}$

Now $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \dfrac{\frac{1}{2} + \frac{1}{5}}{1 - \frac{1}{2} \times \frac{1}{5}} = \dfrac{7}{9}$

$\therefore \tan(\alpha+\beta+\gamma) = \dfrac{\frac{7}{9} + \frac{1}{8}}{1 - \frac{7}{9} \times \frac{1}{8}} = \dfrac{65/72}{65/72}$

$\therefore \tan(\alpha+\beta+\gamma) = 1$

$\alpha + \beta + \gamma = \pi/4$

(c) (i) $\ddot{x} = g - \beta v^2$

$v\dfrac{dv}{dx} = g - \beta v^2$

$\dfrac{dx}{dv} = \dfrac{v}{g - \beta v^2}$

$\therefore x = \dfrac{-1}{2\beta}\log_e(g - \beta v^2) + c$

When $x = 0$ $v = 0$ $\therefore c = \frac{1}{2\beta}\log_e g$

$\therefore -2\beta x = \log_e(g - \beta v^2) - \log_e g$

$-2\beta x = \log_e\left(\dfrac{g - \beta v^2}{g}\right)$

$e^{-2\beta x} = 1 - \dfrac{\beta}{g}v^2$

$\dfrac{\beta}{g}v^2 = 1 - e^{-2\beta x}$

$v^2 = \dfrac{g}{\beta}(1 - e^{-2\beta x})$

$v = \sqrt{\dfrac{g}{\beta}(1 - e^{-2\beta x})}$

since $v > 0$

(ii) $x = \dfrac{-1}{2\beta}\log_e\left(1 - \dfrac{\beta}{g}v^2\right)$

$x = \dfrac{-1}{2(0.2)}\log_e\left(1 - \dfrac{0.2}{9.8}(6^2)\right)$

$x = -\dfrac{\log_e(0.2653)}{0.4}$

$x = 3.32$ metres

So the ball will have fallen 332 centimetres

QUESTION 8

(a) **Ellipse** $b^2 = a^2(1-e^2)$

$4 = 9(1-e^2)$

$\therefore e = \sqrt{5}/3$

Directrices: $x = \frac{\pm a}{e} = \frac{\pm 9}{\sqrt{5}}$

Hyperbola $B^2 = A^2(E^2-1)$

$K = 36(E^2-1)$

$E = \sqrt{1+\frac{K}{36}}$

Directrices: $x = \frac{\pm A}{E} = \frac{\pm 6}{\sqrt{1+\frac{K}{36}}}$

$\therefore \frac{9}{\sqrt{5}} = \frac{6}{\sqrt{1+\frac{K}{36}}}$

$\sqrt{1+\frac{K}{36}} = \frac{6\sqrt{5}}{9} = \frac{2\sqrt{5}}{3}$

$1 + \frac{K}{36} = \frac{20}{9}$

$K = 44$

(b) (a) $x^2 + \frac{1}{x^2} - 2 = \left(x - \frac{1}{x}\right)^2 \geq 0$

$\therefore x^2 + \frac{1}{x^2} \geq 2 \qquad x \neq 0$

EQUALITY $x=1$

(b) $x^2 + \frac{1}{x^2} - x - \frac{1}{x}$

$= \frac{x^4 - x^3 - x + 1}{x^2}$

$= \frac{x^3(x-1) - 1(x-1)}{x^2}$

$= \frac{(x^3-1)(x-1)}{x^2} \geq 0$

Since $x^2 \geq 0$ and

If $x > 1$ $\quad x^3 - 1 > 0,\ x-1 > 0$

If $x < 1$ $\quad x^3 - 1 < 0,\ x-1 < 0$

If $x = 1$ $\quad x^3 - 1 = 0 \quad x - 1 = 0$

$\therefore x^2 + \frac{1}{x^2} \geq x + \frac{1}{x}$

(c) (i) $T_3 = 4T_2 - T_1 = 4\times4 - 2 = 14$

$T_4 = 4T_3 - T_2 = 4\times14 - 4 = 52$

$T_5 = 4T_4 - T_3 = 4\times52 - 14 = 194$

(ii) Check $n=1$ $(2+\sqrt{3})^0 + (2-\sqrt{3})^0 = 1+1 = 2$

$\therefore$ True if $n = 1, 2$

Assume true for $n = k-2$ & $k-1$

ie $T_{k-2} = (2+\sqrt{3})^{k-3} + (2-\sqrt{3})^{k-3}$

$T_{k-1} = (2+\sqrt{3})^{k-2} + (2-\sqrt{3})^{k-2}$

Now $T_k = 4T_{k-1} - T_{k-2}$

$T_k = 4\left[(2+\sqrt{3})^{k-2} + (2-\sqrt{3})^{k-2}\right] - \left[(2+\sqrt{3})^{k-3} + (2-\sqrt{3})^{k-3}\right]$

$= (2+\sqrt{3})^{k-3}(4(2+\sqrt{3})-1)$

$\quad (2-\sqrt{3})^{k-3}(4(2-\sqrt{3})-1)$

$= (2+\sqrt{3})^{k-3}(7+4\sqrt{3}) + (2-\sqrt{3})^{k-3}(7-4\sqrt{3})$

$= (2+\sqrt{3})^{k-3}(2+\sqrt{3})^2 + (2-\sqrt{3})^{k-3}(2-\sqrt{3})^2$

$= (2+\sqrt{3})^{k-1} + (2-\sqrt{3})^{k-1}$

$\therefore$ If true for $n = k-2$ & $n = k-1$, it's true for $n = k$

Since true for $n=1, n=2$ it's true for $n=3$

Since true for $n=2, n=3$ it's true for $n=4$

and so on for all $n \geq 1$

(iii) Since $T_1 = 2$, $T_2 = 4$ and $T_n = 4T_{n-1} - T_{n-2}$

all T_n are integers.

$\therefore T_{101} = (2+\sqrt{3})^{100} + (2-\sqrt{3})^{100}$ is an integer.

$(2-\sqrt{3})^{100} \doteqdot 6.4 \times 10^{-58}$ by calculator

$\therefore (2+\sqrt{3})^{100} = $ an integer $- 6.4 \times 10^{-58}$

$\therefore 2+\sqrt{3}$ starts with 57 nines!