Penrith Selective
High School

2011
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question
- All questions are of equal value
- Staple this test to your answers
- Attempt Questions 1 – 8

Total marks – 60

Question	Mark
1	/15
2	/15
3	/15
4	/15
5	/15
6	/15
7	/15
8	/15
Total	/120
Percentage	

Student's Name:______________________________

Teacher:____________________________

Question 1 (15 marks) Start a NEW Page

(a) Find: $\int cosec\, x\, dx$ by using the substitution $t = tan\frac{x}{2}$. 4

(b) Find: $\int \frac{dx}{x(1+x^2)}$. 3

(c) Find $\int \frac{dx}{x\sqrt{x^2-1}}$. 4

(d) $\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} sin\,4x\ cos\,2x\, dx$. 1

(e) Find the integral $\int \frac{e^x + e^{2x}}{1+e^{2x}} dx$. 3

Question 2 (15 marks) Start a NEW Page.

(a) Given that $z_1 = 3 - i$, $z_2 = 2 + 5i$, express in the form $a + ib$, where a, b are real,

i. $(\overline{z_1})^2$; **1**

ii. $\frac{z_1}{z_2}$; **2**

iii. $\overline{\left|\frac{z_1}{z_2}\right|}$. **2**

(b) Indicate on an Argand diagram the region in which z lies given that both $|z - (3 + i)| \leq 3$ and $\frac{\pi}{4} < arg[z - (1 + i)] \leq \frac{\pi}{2}$ are satisfied. **3**

(c) The complex numbers $z_1 = 4i$, $z_2 = 2\sqrt{3} - 2i$, $z_3 = -2\sqrt{3} - 2i$ are represented on an Argand diagram by the points A, B, C respectively.

i. Show that the triangle ABC is equilateral. **2**

ii. Show that z_1^2 and $z_2 z_3$ are represented by the same point on the Argand diagram. **2**

(d) Find the complex square roots of $7 + 6\sqrt{2i}$ giving your answers in the form $x + iy$, where x and y are real. **3**

Question 3 (15 marks) Start a NEW Page.

(a) $P(20\cos\theta, 12\sin\theta)$ is a point on the ellipse $\frac{x^2}{20^2}+\frac{y^2}{12^2}=1$. P lies in the first quadrant, and the tangent to the ellipse at P meets the directrices at Q and R where Q is nearer the focus S' and R is nearer the focus S. Q and R lie above the x-axis, and QS' meets RS in K where K lies in the third quadrant. 8

i. Sketch the ellipse showing its directrices and foci and the points P, Q, R and K.

ii. Show that the tangent at P has equation $3x\cos\theta + 5y\sin\theta = 60$.

iii. Show that K has co-ordinates $(-20\cos\theta, \frac{4(9-25\sin^2\theta)}{3\sin\theta})$.

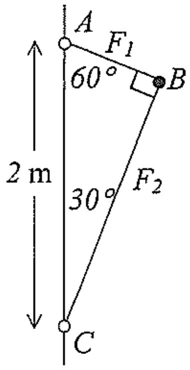
(b) (i) Show that the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a^2 > b^2)$ at the point $P(x_1, y_1)$ has equation $a^2y_1x - b^2x_1y = (a^2 - b^2)x_1y_1$. 5

(ii) This normal meets the major axis of the ellipse at G. S is one focus of the ellipse. Show that $GS = e.PS$ where e is the eccentricity of the ellipse.

(c) Show that the condition for the line $y = mx + c$ to be tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2 = a^2m^2 + b^2$. 2

Question 4 (15 marks) Start a NEW Page.

(a) A mass of 10 kg, centre B is connected by light rods to sleeves A and C which revolve freely about the vertical axis AC but do not move vertically.



i. Given $AC = 2$ metres, show that the radius of the circular path of rotation of B is $\frac{\sqrt{3}}{2}$ metres. **1**

ii. Find the tensions in the rods AB, BC when the mass makes 90 revolutions per minute about the vertical axis. **4**

(b) A railway track has been constructed around a circular curve of radius 500 metres. The distance across the track between the rails is $1{\cdot}5$ metres and the outer rail $0{\cdot}1$ metres above the inner rail.

i. If the train travels on the track at a speed v_0 which eliminates any sideways force on the wheels. **5**

  i. Draw a diagram showing all the forces acting on the train.

  ii. Show that $v_0^2 = 500g \tan \theta$, where θ is the angle the track makes with the horizontal. Taking $g = 9{\cdot}8$ ms^{-2} calculate v_0.

ii. If the train travels on the track at a speed $v > v_0$

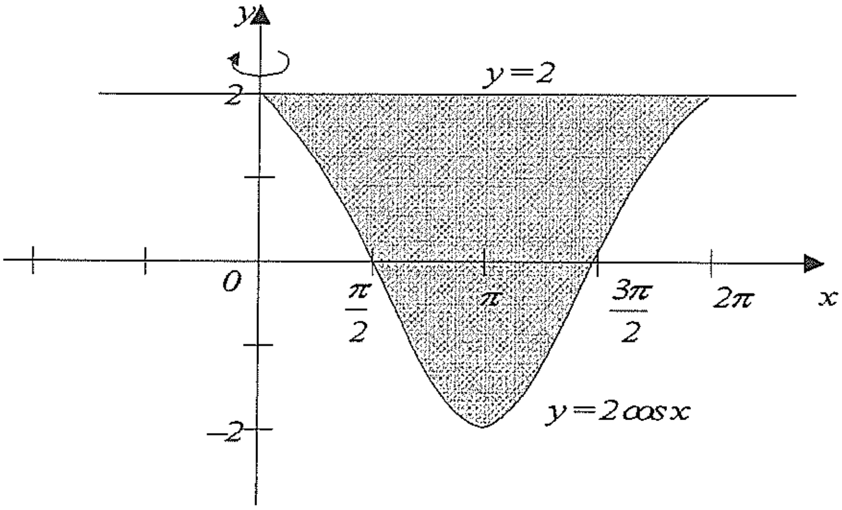
  i. State which rail exerts a lateral force on the wheel at the point of contact. **5**

  ii. Draw a diagram showing all the forces on the train.

  iii. Show that the lateral force F exerted by the rail on the wheel is given by $3mg \sin\theta$, where m is the mass of the train. Deduce that F is one fifth the weight of the train when $v = 2v_0$.

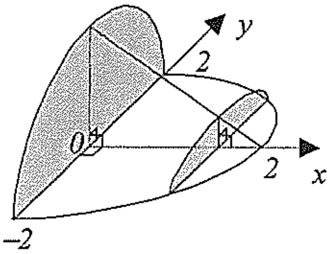
Question 5 (15 marks) Start a NEW Page.

(a)



A mathematically inclined microwave cooking enthusiast decided to design his own cake pan. The shape of the interior of the cake pan is obtained by rotating the region bounded by the curve $y = 2\cos x$, $0 \le x \le 2\pi$ and the line $y = 2$ through $360°$ about the y-axis. Use the method of cylindrical shells to show that the volume of the cake pan is given by $4\pi \int_0^{2\pi} x(1 - \cos x)\,dx$ and hence calculate this volume. 5

(b)



A solid figure has a semicircular base of radius 2. Cross sections taken at right angles to the semi-diameter of this base are semi-ellipses. The vertical cross section containing the semi-diameter of the base is a right isosceles triangle as shown. 5

i. Given that the area of an ellipse with semi-axes a and b is πab, show that the volume of the solid is given by $V = \frac{\pi}{2}\int_0^2 (2-x)\sqrt{4-x^2}\,dx$.

ii. Hence find the volume of the solid.

(c)

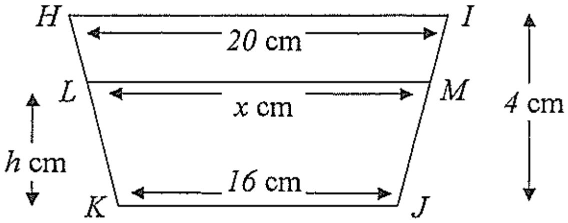


FIGURE NOT TO SCALE

i. A trapezium *HIJK* has parallel sides $KJ = 16$ cm and $HI = 20$ cm. The distance between these sides is *4* cm. *L* lies on *HK* and *M* lies on *IJ* such that *LM* is parallel to *KJ*. The shortest distance from *K* to *LM* is *h* cm and *LM* has length *x* cm. Prove that $x = 16 + h$. **2**

ii.

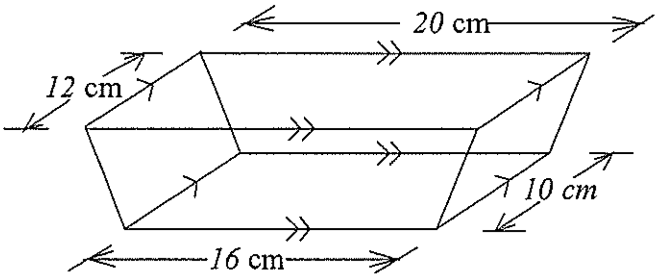


FIGURE NOT TO SCALE

The diagram above is of a cake tin with a rectangular base with sides of *16* cm and *10* cm. Its top is also rectangular with dimensions *20* cm and *12* cm. The tin has depth *4* cm and each of its four side faces is a trapezium. Find its volume. **3**

Question 6 (15 marks) Start a NEW Page.

(a) Find the integers m and n such that $(x + 1)^2$ is a factor of $x^5 + 2x^2 + mx + n$. **3**

(b) None of the roots α, β and γ of the equation $x^3 + 3px + q = 0$ is zero. **5**

i. Obtain the monic equation whose roots are $\frac{\beta\gamma}{\alpha}, \frac{\alpha\gamma}{\beta}, \frac{\alpha\beta}{\gamma}$ expressing its coefficients in terms of p and q.

ii. Deduce that $\gamma = \alpha\beta$ if and only if $(3p - q)^2 + q = 0$.

(c) i. Show that the equation $x^3 - 6x^2 + 9x - 5 = 0$ has only one real root α. **2**

ii. Determine the two consecutive integers between which α lies. **2**

iii. By considering the product of the roots of the equation express the modulus of each of the complex roots in terms of α and deduce that the value of this modulus lies between 1 and $\frac{\sqrt{5}}{2}$. **3**

Question 7 (15 marks) Start a NEW Page.

(a) Prove that the curve: $y = x^2e^{-x}$ has a minimum turning point at $(0, 0)$ and a maximum turning point at $(2, \frac{4}{e^2})$. Sketch the curve. 3

(b) Sketch the graph of the function $y = \frac{x^2 - x + 1}{(x-1)^2}$ showing clearly the coordinates of any points of intersection with the x-axis and the y-axis, the coordinates of any turning points and the equations of any asymptotes. (There is no need to investigate points of inflection). 3

(c) i. Express $z = \sqrt{2} - i\sqrt{2}$ in modulus-argument form. 3

ii. Hence write z^{22} in the form $a + ib$, where a and b are real.

(d) $P(x)$ is a polynomial of degree 4 with real coefficients.

i. The complex number α satisfies $Im(\alpha) \neq 0$, $Re(\alpha) = a$, and $|\alpha| = r$. Show that if α is a zero of $P(x)$, then $P(x)$ has a factor $x^2 - 2ax + r^2$ over R, the field of real numbers. 3

ii. α is a non-real double zero of $P(x) = x^4 - 8x^3 + 30x^2 - 56x + 49$. Factorize $P(x)$ into irreducible factors over R, and find the four roots of $x^4 - 8x^3 + 30x^2 - 56x + 49 = 0$. 3

Question 8 (15 marks) Start a NEW Page.

(a) i. Show that $\int x\, tan^{-1} x\, dx = \frac{1}{2}(x^2 + 1)\, tan^{-1} x - \frac{1}{2}x + c$, c constant. **2**

ii. $I_n = \int_0^1 x^n\, tan^{-1} x\, dx$, $n = 0, 1, 2, \ldots$ Show that **3**

$I_0 = \frac{\pi}{4} - \frac{1}{2} ln 2$, $I_1 = \frac{\pi}{4} - \frac{1}{2}$ and

$I_n = \frac{1}{n+1} \cdot \frac{\pi}{2} - \frac{1}{n(n+1)} - \frac{n-1}{n+1} \cdot I_{n-2}$, $n = 2, 3, 4, \ldots$

(b) On the Argand diagram, let $A = 3 + 4i$, $B = 9 + 4i$.

i. Draw a clear sketch to show the important features of the curve defined by $|z - A| = 5$. Also, for z on this curve, find the maximum value of $|z|$. **2**

ii. On a separate diagram, draw a clear sketch to show the important features of the curve defined by $|z - A| + |z - B| = 12$. For z on this curve, find the greatest value of $arg\, z$. **2**

(c) A particle of mass m kg falls from rest in a medium where the resistance to motion is mkv when the particle has velocity v ms^{-1}.

i. Draw a diagram showing the forces acting on the particle. **1**

ii. Show that the equation of motion of the particle is $\ddot{x} = k(V - v)$ where V ms^{-1} is the terminal velocity of the particle in this medium, and x metres is the distance fallen in t seconds. **2**

iii. Find in terms of V and k the time T seconds taken for the particle to attain 50% of its terminal velocity, and the distance fallen in this time. **3**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

(a) $\int \operatorname{cosec} x \, dx.$

$t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}(1+t^2)$

$dx = \frac{2\,dt}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$

$\therefore \int \operatorname{cosec} x \, dx$

$= \int \frac{1+t^2}{2t} \cdot \frac{2\,dt}{1+t^2}$

$= \int \frac{1}{t}\,dt$

$= \ln|t| + c$

$= \ln\left|\tan\frac{x}{2}\right| + c.$

b) $\int \frac{dx}{x(1+x^2)}$

$= \frac{a}{x} + \frac{bx+c}{1+x^2}$

$1 \equiv a(1+x^2) + x(bx+c)$

$1 \equiv a + ax^2 + bx^2 + cx$

$= (a+b)x^2 + cx + a$

Equating coefficients

$a+b=0 \quad a=1$

$a=-b \quad \therefore b=-1$

$c=0$

$\int\left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx$

$= \ln|x| - \frac{1}{2}\ln(1+x^2) + c$

$= \ln\left|\frac{x}{\sqrt{1+x^2}}\right| + c$

c) $\int \frac{dx}{x\sqrt{x^2-1}}$

Let $u^2 = x^2 - 1$

$2u\frac{du}{dx} = 2x$

$u\frac{du}{dx} = x$

$u\,du = x\,dx$

$\int \frac{x\,dx}{x^2\sqrt{x^2-1}}$

$= \int \frac{\cancel{u}\,du}{(u^2+1)\cancel{u}}$

$= \tan^{-1}u + c$

$= \tan^{-1}\sqrt{x^2-1} + c$

d) $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin 4x \cos 2x \, dx$

$\sin 4x \cos 2x$ = even function × odd function

$\therefore$ odd function

$\therefore \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = 0.$

e) $\int \frac{e + e}{1+e^{2x}}\,dx$

Let $u = e^x$

$\frac{du}{dx} = e^x$

$du = e^x\,dx$

$\int \frac{(1+e^x)e^x\,dx}{1+(e^x)^2}$

$= \int \frac{1+u}{1+u^2}\,du$

$= \tan^{-1}u + \frac{1}{2}\ln(1+u^2) + c$

$= \tan^{-1}e^x + \frac{1}{2}\ln(1+e^{2x}) + c$

Question 2

a) $(3+i)^2 = 9+6i-1$

$= 8+6i$

b) $\frac{3-i}{2+5i} \times \frac{2-5i}{2-5i}$

$= \frac{6-15i-2i-5}{4+25}$

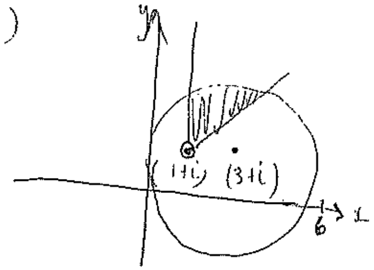
$= \frac{1}{29} - \frac{17}{29}i$

c) $\left|\overline{\frac{z_1}{z_2}}\right| = \frac{|\bar{z}_1|}{|\bar{z}_2|} = \frac{|z_1|}{|z_2|}$

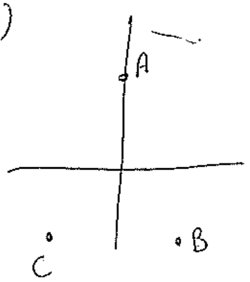
$= \frac{\sqrt{3^2+1}}{\sqrt{2^2+5^2}}$

$= \sqrt{\frac{10}{29}}$

B)



C) a)



$AB = |z_1-z_2| = |-2\sqrt{3}+6i| = \sqrt{(2\sqrt{3})^2+6^2} = 4\sqrt{\ }.$

$AC = |z_1-z_3| = |2\sqrt{3}+6i| = \sqrt{(2\sqrt{3})^2+6^2} = 4\sqrt{3}$

$BC = |z_2-z_3| = |4\sqrt{3}| = 4\sqrt{3}$

$\therefore \triangle ABC$ is equilateral.

b) $z_1^2 = (4i)^2 = -16$

$z_2z_3 = (2\sqrt{3}-2i)(-2\sqrt{3}-2i)$

$= (2i)^2-(2\sqrt{3})^2 = -4-12 = -16$

$\therefore z_1^2 = z_2z_3$

So z_1^2 and z_2z_3 represent the same point on the Argand diagram.

(D) $(x+iy)^2 = 7+6\sqrt{2}i$

$x^2-y^2=7 \quad 2xy=6\sqrt{2}$

$(x^2+y^2)^2 = (x^2-y^2)^2+(2xy)^2$

$= 7^2+(6\sqrt{2})^2.$

$=121$

$x^2+y^2=11$ — ①

$x^2-y^2=7$ — ②

$2x^2=18$

$\therefore x^2=9$

$x=\pm 3, \; y=\pm\sqrt{2}.$

so two roots are

$\pm(3+\sqrt{2}i)$

Question 3

a i) $a=20$

$b=12$

$e = \sqrt{1-\frac{b^2}{a^2}}$

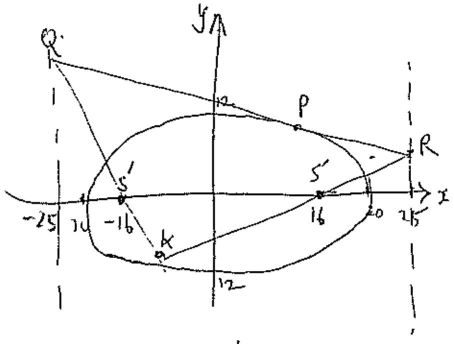
$= \sqrt{1-\frac{144}{400}}$

$= \frac{4}{5}$

$ae=16 \quad \frac{a}{e}=25$

$\therefore$ foci are $(\pm 16, 0)$

directrices are $x=\pm 25$



(ii) $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{12\cos\theta}{-20\sin\theta}$

$= \frac{3\cos\theta}{-5\sin\theta}$

Tangent at P:

$y-12\sin\theta = \frac{3\cos\theta}{-5\sin\theta}(x-20\cos\theta)$

$-5y\sin\theta+60\sin^2\theta = 3x\cos\theta-60\cos^2\theta$

$3x\cos\theta+5y\sin\theta=60$

(iii) At R, $x=25$

$75\cos\theta+5y\sin\theta=60$

$y=\frac{12-15\cos\theta}{\sin\theta}$

$\therefore R\left(25, \frac{3(4-5\cos\theta)}{\sin\theta}\right)$

At Q, $x=-25$

$-75\cos\theta+5y\sin\theta=60$

$y=\frac{12+15\cos\theta}{\sin\theta}$

$\therefore Q\left(-25, \frac{3(4+5\cos\theta)}{\sin\theta}\right)$

* Equation of RS

$\frac{y-0}{x-16} = \frac{\frac{3(4-5\cos\theta)}{\sin\theta}-0}{25-16}$

$y=\frac{4-5\cos\theta}{3\sin\theta}(x-16)$ -- ①

* Equation of QS'

$\frac{y-0}{x+16} = \frac{\frac{3(4+5\cos\theta)}{\sin\theta}-0}{-25+16}$

$y=-\frac{(4+5\cos\theta)}{3\sin\theta}(x+16)$ — ②

Find K by solving (1) and (2) simultaneously

$\frac{4-5\cos\theta}{3\sin\theta}(x-16) = \frac{-(4+5\cos\theta)}{3\sin\theta}(x+1$

$(4-5\cos\theta)(x-16) = -(4+5\cos\theta)(x+16)$

$4x-64-5x\cos\theta+80\cos\theta$

$=-4x-64-5x\cos\theta-80\cos\theta$

$8x = -160\cos\theta$

$x = -20\cos\theta$

sub into ①

$y = \frac{4-5\cos\theta}{3\sin\theta}(-20\cos\theta - 16)$

$= \frac{-4(4-5\cos\theta)(4+5\cos\theta)}{3\sin\theta}$

$= \frac{4(25\cos^2\theta - 16)}{3\sin\theta}$

$= \frac{4[25(1-\sin^2\theta)-16]}{3\sin\theta}$

$= \frac{4(9-\sin^2\theta)}{3\sin\theta}$

$\therefore K\left(-20\cos\theta, \frac{4(9-25\sin^2\theta)}{3\sin\theta}\right)$

b)(i) $\frac{d}{dx}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) = \frac{d}{dx}(1)$

$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$

Grad normal $\frac{a^2y_1}{b^2x_1}$

$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1)$

$a^2y_1x - b^2x_1y = (a^2-b^2)x_1y_1$

(ii) sub $y=0$

$a^2y_1x = (a^2-b^2)x_1y_1$

$x = \frac{a^2-b^2}{a^2}x_1$

$= a^2e^2\ x = e^2x_1$

$\therefore G(e^2x_1, 0)$

$S(ae, 0)$

$PS = ePM$

$= e\left(\frac{a}{e} - x_1\right)$

$= a - ex_1$

$\therefore GS = ae - e^2x_1$

$= e(a - ex_1)$

$= ePS.$

c) solve simultaneously

$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$

$(b^2+a^2m^2)x^2 + 2a^2mcx + (a^2c^2 - a^2b^2) = 0$

If the line is tangent to the ellipse then quad equation only has one solution ie $\Delta = 0$

$(2a^2mc)^2 - 4(b^2+a^2m^2)(a^2c^2-a^2b^2) = 0$

$a^2m^2c^2 = (b^2+a^2m^2)(c^2-b^2)$

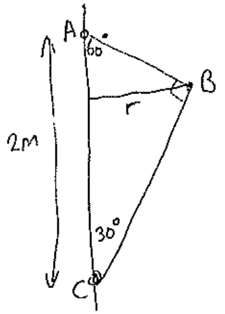
$a^2m^2c^2 = b^2c^2 - b^4 + a^2m^2c^2 - a^2b^2m^2$

$b^2c^2 = a^2b^2m^2 + b^4$

$c^2 = a^2m^2 + b^2$

Question 4

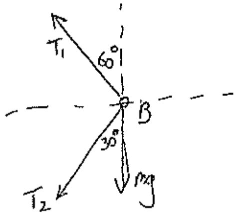
a)(i)



$\cos 60 = \frac{AB}{2}$, $AB = 1$

$\sin 60 = \frac{r}{AB}$ $r = \frac{\sqrt{3}}{2}$

(ii)



$\Sigma F_V = 0$ $T_1\cos 60 = T_2\cos 30 + mg$

$T_1\left(\frac{1}{2}\right) = T_2\left(\frac{\sqrt{3}}{2}\right) + 10g$

$T_1 = T_2\sqrt{3} + 20g$

$\Sigma F_H = mrw^2$

$T_1\sin 60 + T_2\sin 30 = mrw^2$

$T_1\left(\frac{\sqrt{3}}{2}\right) + T_2\left(\frac{1}{2}\right) = 10\left(\frac{\sqrt{3}}{2}\right)(3\pi)^2$

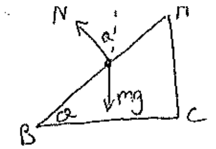
$T_1 = 90\pi^2 - \frac{T_2}{\sqrt{3}}$

$T_2\sqrt{3} + 20g = 90\pi^2 - \frac{T_2}{\sqrt{3}}$

* $T_2 = \frac{5\sqrt{3}}{2}(9\pi^2 - 2g)$

* $T_1 = \frac{5}{2}(27\pi^2 + 2g)$

(b) a(i)



(ii) $\tan\theta = \frac{AC}{BC}$

$= \frac{0 \cdot 1}{1 \cdot 5}$

$= \frac{1}{15}$

$\Sigma F_V = 0$

$N\cos\theta = mg$ —①

$\Sigma F_H = \frac{mV_0^2}{r}$

$N\sin\theta = \frac{mV_0^2}{500}$ —②

②÷① $\tan\theta = \frac{V_0^2}{500g}$

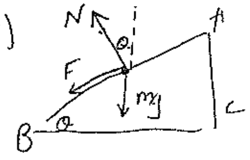
$V_0^2 = 500g\tan\theta$

$= 500(9 \cdot 8)\frac{1}{15}$

$= 18 \cdot 07$ m/s (2dec pla

(b)(i) The outer rail exerts a lateral force on the wheel at the point of contact.

(ii)



(iii) $\sum F_V = 0$

$N\cos\theta - F\sin\theta = mg$ —③

$\sum F_H = \frac{mv^2}{r}$

$N\sin\theta + F\cos\theta = \frac{mv^2}{500}$ —④

(4)× cosθ $N\sin\theta\cos\theta + F\cos^2\theta = \frac{mv^2}{500}\cos\theta$ ⑤

(3) × sinθ $N\cos\theta\sin\theta - F\sin^2\theta = mg\sin\theta$ ⑥

⑤−⑥ $F = \frac{mv^2\cos\theta}{500} - mg\sin\theta$

If $v = 2v_0$

$v^2 = 4v_0^2$

$= 2000g\tan\theta$

$F = \frac{m(2000g\tan\theta)}{500} - mg\sin\theta$

$= 4mg\sin\theta - mg\sin\theta$

$= 3mg\sin\theta$

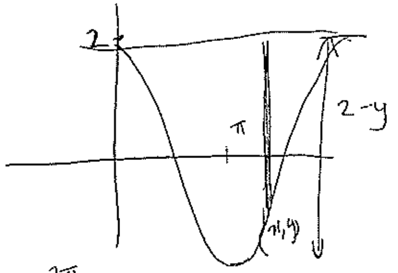
$= 3mg \times \frac{1}{15}$

$= \frac{1}{5}mg$

$= \frac{1}{5} \times$ weight of train

Question 3

a)



$\int_0^{2\pi} 2\pi x(2-y)\,dx$

$= \int_0^{2\pi} 2\pi x(2-2\cos x)\,dx$

* $= 4\pi\int_0^{2\pi} x(1-\cos x)\,dx$

$4\pi\left[\int_0^{2\pi} x\,dx - \int_0^{2\pi} x\cos x\,dx\right]$

$\int_0^{2\pi} x\,dx = \left[\frac{x^2}{2}\right]_0^{2\pi}$

$= 2\pi^2$

$\int_0^{2\pi} x\cos x\,dx = \int_0^{2\pi} x\,d(\sin x)$

$= \left[x\sin x\right]_0^{2\pi} - \int_0^{2\pi}\sin x\,dx$

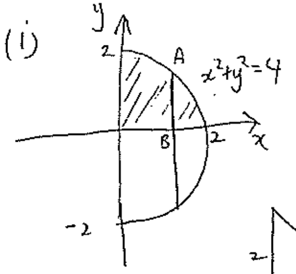
$= 0 + \left[\cos x\right]_0^{2\pi}$

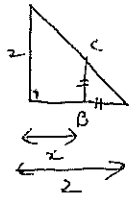
$= 0$

∴ volume $= 4\pi(2\pi^2)$

$= 8\pi^3$ units³

b) (i)





$AB = y = \sqrt{4-x^2}$

$BC = 2 - x$

Area of semi ellipse $= \frac{1}{2}\pi(AB)(BC)$

$= \frac{\pi}{2}\left(\sqrt{4-x^2}\right)(2-x)$

Volume $= \frac{\pi}{2}\int_0^2 (2-x)\sqrt{4-x^2}\,dx$

(ii) Volume $= \pi\int_0^2 \sqrt{4-x^2}\,dx$

$- \frac{\pi}{2}\int_0^2 x(4-x^2)^{\frac{1}{2}}$

$\int_0^2 \sqrt{4-x^2}\,dx =$ shaded area shown

$a = \frac{1}{4}\pi(2^2) = \pi$

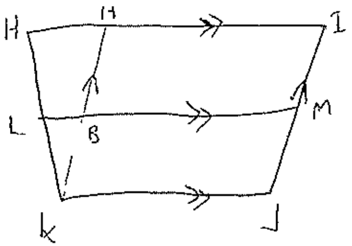
$\int_0^2 x(4-x^2)^{\frac{1}{2}}dx = -\frac{1}{3}\left[(4-x^2)^{\frac{3}{2}}\right]_0^2$

$= \frac{8}{3}$

∴ Volume $= \pi(\pi) - \frac{\pi}{2}\left(\frac{8}{3}\right)$

$= \left(\pi^2 - \frac{4\pi}{3}\right)$ units³.

(C) (i)



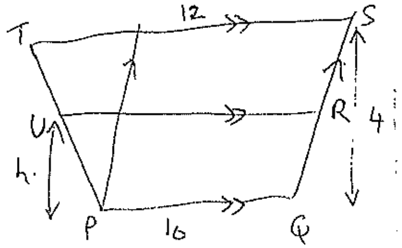
$\Delta KLB ||| \Delta KHA$

$\therefore \frac{LB}{HA} = \frac{h}{4}.$

$\frac{x-16}{4} = \frac{h}{4}$

$\therefore x = h+16.$

(ii)



Let $VR = y$

$\Delta PYU ||| \Delta PXT$

$\therefore \frac{UY}{TX} = \frac{h}{4}$

$\frac{y-10}{2} = \frac{h}{4}$

$y = \frac{1}{2}h + 10$

Volume $= \int_0^4 xy \, dh$

$= \int_0^4 (h+16)\left(\frac{1}{2}h+10\right) dh.$

$\int_0^4 \frac{1}{2}h^2 + 18h + 160) dh$

$= \left[\frac{1}{6}h^3 + 9h^2 + 160h\right]_0^4$

$= 794\frac{2}{3} cm^3$

(a) $P(x) = x^5 + 2x^2 + mx + n$

$P'(x) = 5x^4 + 4x + m$

$P(-1) = 0 \quad -1 + 2 - m + n = 0$

$\therefore m - n = -1$

$P'(-1) = 0 \quad 5 - 4 + m = 0$

$m = -1$

$\therefore n = -2$

b) (i) $\alpha\beta\gamma = -q$

so new monic roots are

$\frac{\alpha\beta\gamma}{\alpha^2}, \frac{\alpha\beta\gamma}{\beta^2}, \frac{\alpha\beta\gamma}{\gamma^2}$

$= \frac{\beta\gamma}{\alpha}, \frac{\alpha\gamma}{\beta}, \frac{\alpha\beta}{\gamma}$

$= \frac{-q}{\alpha^2}, \frac{-q}{\beta^2}, \frac{-q}{\gamma^2}$

Let $y = \frac{-q}{x^2}$ ie $x = \sqrt{\frac{-q}{y}}$

$\left(\frac{-q}{y}\right)^{\frac{3}{2}} + 3p\left(\frac{-q}{y}\right)^{\frac{1}{2}} + q = 0$

$\left(\frac{-q}{y}\right)^{\frac{1}{2}}\left[\left(-\frac{q}{y}\right) + 3p\right] + q = 0$

$\left(-\frac{q}{y}\right)^{\frac{1}{2}} = \frac{-qy}{3py-q}$

$-\frac{q}{y} = \left(\frac{-qy}{3py-q}\right)^2$

$-(3py-q)^2 = qy^3$

$-9p^2y^2 + 6pqy - q^2 = qy^3$

one of the roots of the new equation would be $y=1$.

$1 + \frac{9p^2}{q} - 6p + q = 0$

$q + 9p^2 - 6pq + q^2 = 0$

$q + (3p-q)^2 = 0$

c i) If $f(x) = x^3 - 6x^2 + 9x$

$= x(x-3)^2$

$f(x)$ cuts x axis at $x=0$

and touches the x axis at $x=3$

$\therefore y = f(x) - 5$ crosses the x-axis once at $x = \alpha$.

(ii) $4^3 - 6(4^2) + 9(4) - 5 = -1 \quad < 0$

$5^3 - 6(5^2) + 9(5) - 5 = 15 \quad > 0$

$\therefore 4 < \alpha < 5$

(iii) Real coefficients $\therefore$ complex root has conjugate root.

$\alpha(a+ib)(a-ib) = 5$

$a^2 + b^2 = \frac{5}{\alpha}$

$\sqrt{a^2+b^2} = \sqrt{\frac{5}{\alpha}}$ since $4 < \alpha < 5$

$\sqrt{\frac{5}{5}} < |a \pm ib| < \sqrt{\frac{5}{4}}$

ie $1 < |a \pm ib| < \sqrt{5}$

Question 7

a) $y = \overset{u}{x^2}\overset{v}{e^{-x}}$

$\frac{dy}{dx} = x^2 \cdot -e^{-x} + e^{-x} \cdot 2x$

$= e^{-x}(2x - x^2)$

Let $\frac{dy}{dx} = 0$

$2x - x^2 = 0$

$x(2-x) = 0$

$x = 0$ or $x = 2$

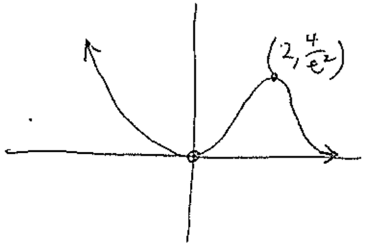
$y = 0 \qquad y = \frac{4}{e^2}$

$y' = \overset{u}{e^{-x}}\overset{v}{(2x - x^2)}$

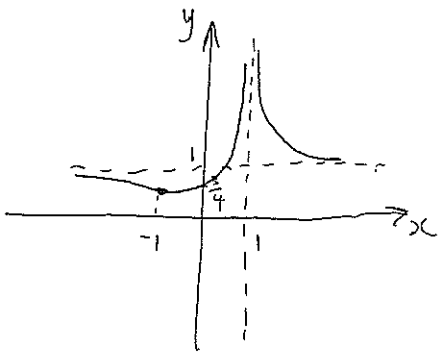
$y'' = -(2x - x^2)e^{-x} + e^{-x}(2 - 2x)$

$f''(0) = +2$ +ve $\therefore$ min

$f''(2) =$ −ve $\therefore$ max.



b)



c)(i) $2\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

or $2\operatorname{cis}\left(-\frac{\pi}{4}\right)$

(ii) $Z^{22} = 2^{22}\left[\cos\left(-\frac{22\pi}{4}\right) + i\sin\left(\frac{-22}{4}\right.\right.$

$= 2^{22}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= 2^{22}i$

d(i) If $\alpha = r(\cos\theta + i\sin\theta)$

$\bar{\alpha} = r(\cos\theta - i\sin\theta)$

$\alpha + \bar{\alpha} = 2r\cos\theta$

$= 2a$

$\alpha\bar{\alpha} = r^2$

Since α and $\bar{\alpha}$ are zeros of $P(x)$ then a factor of $P(x)$ is:

$(x - \alpha)(x - \bar{\alpha}) = x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha}$

$= x^2 - 2ax + r^2$

(ii) $P(x) = (x^2 - 2ax + r^2)^2$

$P(0) = 49 = r^4$

$\therefore r^2 = 7.$

$P(1) = 16 = (8 - 2a)^2$

$\pm 4 = 8 - 2a$

$2a = 4$ or $2a = 7$.

$\therefore 2a = 4$ and $r^2 = 7$.

$P(x) = (x^2 - 4x + 7)^2$

$(x^2 - 4x + 7)^2 = 0.$

$x^2 - 4x = -7$

$(x-2)^2 = -3$

$x = 2 \pm \sqrt{3}i$

Both roots occur twice

Question 0

(a) (i) $\int \overset{\frac{dv}{dx}}{x} \overset{u}{\tan^{-1}x}\ dx$

$u = \tan^{-1}x \quad \frac{du}{dx} = \frac{1}{1+x^2}$

$\frac{dv}{dx} = x \qquad v = \frac{x^2}{2}$

$\frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{2} \cdot \frac{1}{1+x^2}\, dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[\int\frac{x^2+1}{1+x^2} - \int\frac{1}{1+x^2}\right]$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C$

$= \frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x + C$

(ii) $I_0 = \int_0^1 \overset{\frac{dv}{dx}}{1} \cdot \overset{u}{\tan^{-1}x}\, dx$

$u = \tan^{-1}x \quad \frac{du}{dx} = \frac{1}{1+x^2}$

$\frac{dv}{dx} = 1 \qquad v = x$

$= \cancel{\frac{\pi}{4}}$

$= \left[x\tan^{-1}x\right]_0^1 - \int_0^1 x \cdot \frac{1}{1+x^2}$

$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\, dx$

$= \frac{\pi}{4} - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

$I_1 = \int_0^1 x\tan^{-1}x\, dx$

$= \left[\frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x\right]_0^1$

$= \frac{\pi}{4} - \frac{1}{2}$

$I_n + \frac{n-1}{n+1} I_{n-2}$

$= \int_0^1 \overset{\frac{dv}{dx}}{\left(x^n + \frac{n-1}{n+2}x^{n-2}\right)} \overset{u}{\tan^{-1}x}\, dx$

$u = \tan^{-1}x \qquad \frac{du}{dx} = \frac{1}{1+x^2}$

$\frac{dv}{dx} = x^n + \frac{n-1}{n+2}x^{n-2} \quad v = \frac{x^{n+1}+x^{n-1}}{n+1}$

$= \left[\frac{x^{n+1}+x^{n-1}}{n+1}\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x^{n+1}+x^{n-1}}{n+1} \cdot \frac{1}{1+x^2}$

$= \frac{1}{n+1}\left(\frac{\pi}{2}\right) - \int_0^1 \frac{x^{n-1}(x^2+1)}{n+1} \cdot \frac{dx}{1+x^2}$

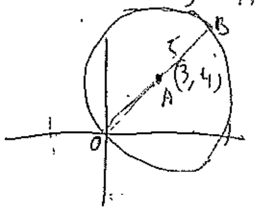
$= \frac{1}{n+1}\left(\frac{\pi}{2}\right) - \int_0^1 \frac{x^{n-1}}{n+1}\, dx$

$= \frac{1}{n+1}\left(\frac{\pi}{2}\right) - \left[\frac{x^n}{n(n+1)}\right]_0^1$

$= \frac{1}{(n+1)}\frac{\pi}{2} - \frac{1}{n(n+1)}$

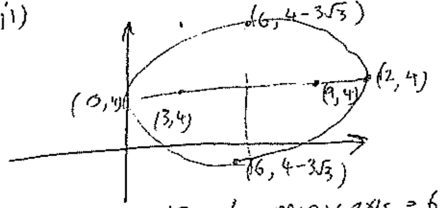
$\therefore I_n = \frac{1}{n+1}\left(\frac{\pi}{2}\right) - \frac{1}{n(n+1)} - \frac{n-1}{n+1} I_{n-2}.$

(b) (i)



At B $|z|$ has max value of 10

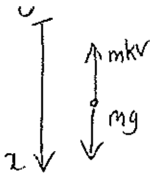
(ii)



Major axis = 12 units, minor axis = $6\sqrt{3}$

At (0, 4) ie $4i$, arg z has max value $\frac{\pi}{2}$

(c) (i)



(ii) $m\ddot{x} = mg - mkv$

$\ddot{x} = g - kv$

Terminal velocity V occurs when $\ddot{x} = 0$

$0 = g - kV$

$V = \frac{g}{k}$

$\therefore \ddot{x} = Vk - kv$

$\ddot{x} = k(V-v)$

(iii) $\frac{dv}{dt} = k(V-v)$

$\frac{dt}{dv} = \frac{1}{k}\left(\frac{1}{V-v}\right)$

$t = -\frac{1}{k}\ln(V-v) + C_1$

when $t=0 \quad v=0$

$C_1 = \frac{1}{k}\ln V$

$t = \frac{1}{k}\ln\left(\frac{V}{V-v}\right)$

When $v = \frac{1}{2}V \quad t = T$

$T = \frac{1}{k}\ln\left(\frac{V}{V-\frac{1}{2}V}\right)$

$= \frac{1}{k}\ln 2$

$\ddot{x} = k(V-v)$

$v\frac{dv}{dx} = k(V-v)$

$\frac{dx}{dv} = \frac{1}{k}\left(\frac{v}{V-v}\right)$

$\frac{dx}{dv} = -\frac{1}{k}\left[\frac{(V-v)-V}{V-v}\right]$

$\frac{dx}{dv} = -\frac{1}{k}\left(1 - \frac{V}{V-v}\right)$

$x = -\frac{1}{k}\left[v + V\ln(V-v)\right] + C_2$

when $x=0,\ v=0$

$C_2 = \frac{V}{k}\ln V$

$x = \frac{V}{k}\ln\left(\frac{V}{V-v}\right) - \frac{v}{k}$

when $v = \frac{1}{2}V$

$x = \frac{V}{k}\ln\left(\frac{V}{V-\frac{1}{2}V}\right) - \frac{\frac{1}{2}V}{k}$

$x = \frac{V}{2k}(2\ln 2 - 1)$