PENRITH HIGH SCHOOL

2014
HSC TRIAL EXAMINATION

Mathematics Extension 2

General Instructions:

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Answer all Questions in the booklets provided

Total marks–100

SECTION I **Pages 2–5**

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

SECTION II **Pages 6–11**

90 marks

- Attempt Questions 11–16
- Allow about 2 hours 45 minutes for this section

Student Name: ______________________ Teacher Name: ______________________

This paper MUST NOT be removed from the examination room

Assessor: Mr Ferguson

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is $-\sqrt{3}+i$ expressed in modulus-argument form?

(A) $\sqrt{2}(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$

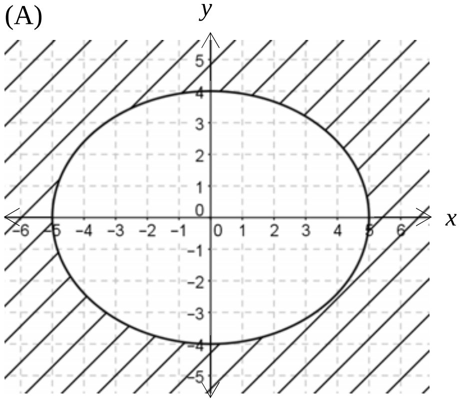
(B) $2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$

(C) $\sqrt{2}(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})$

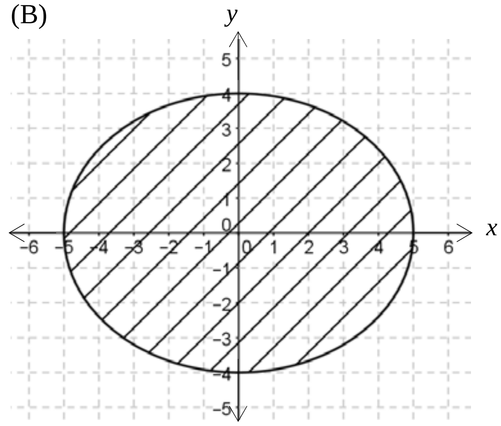
(D) $2(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})$

2 The sketch of the locus of an equation $|z-3|+|z+3|\le 10$ where $z=x+iy$ can best be represented by.

(A)



(B)



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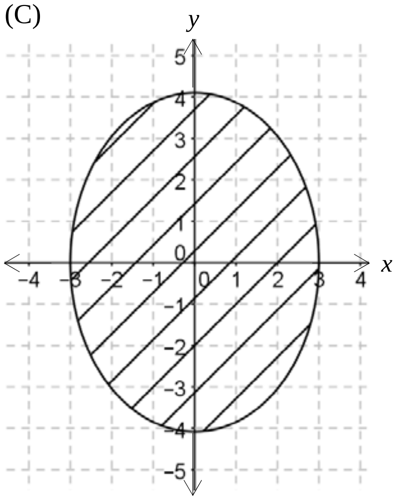
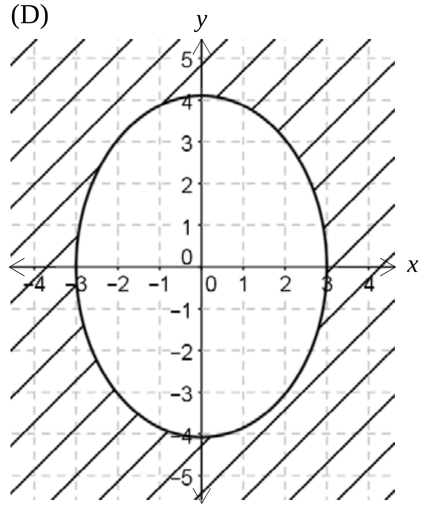


(D)



3 Which of the following expressions is equivalent to $\int_0^2 \sqrt{4-x^2}\,dx$.

(A) π

(B) 2π

(C) 4π

(D) 8π

4 Which expression is equal to $\int \frac{1}{\sqrt{4x^2-8x+5}}\,dx$?

(A) $\frac{1}{2}\sin^{-1} 2(x-3)+C$

(B) $\frac{1}{2}\cos^{-1} 2(x-3)+C$

(C) $\frac{1}{2}\ln\left(x-1+\sqrt{x^2-2x+\frac{5}{4}}\right)+C$

(D) $\frac{1}{2}\ln\left(x-1+\sqrt{x^2-2x-\frac{5}{4}}\right)+C$

5 If $a, b, c, d,$ and e are real numbers and $a \neq 0$, then the polynomial equation $ax^7+bx^5+cx^3+dx+e=0$ has.

(A) only one real root.

(B) at least one real root.

(C) an odd number of nonreal roots

(D) no real roots

6 Suppose that a function $y = f(x)$ is given with $f(x) \geq 0$ for $0 \leq x \leq 4$. If the area bounded by the curves $y = f(x)$, $y = 0$, $x = 0$, and $x = 4$ is revolved about the line $y = -1$, then the volume of the solid of revolution is given by.

(A) $\pi\int_0^4 \left[f(x-1)^2 - 1\right] dx$

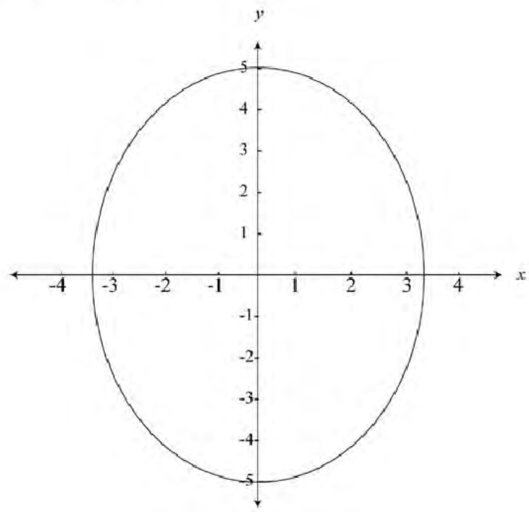
(B) $\pi\int_0^4 \left[\left(f(x)-1\right)^2 - 1\right] dx$

(C) $\pi\int_0^4 \left[f(x+1)^2 - 1\right] dx$

(D) $\pi\int_0^4 \left[\left(f(x)+1\right)^2 - 1\right] dx$

Use the following information to answer the next question.

A conic is graphed using technology, and is shown below. The distance between the x-intercepts is $2\sqrt{11}$ units, and the distance between the y-intercepts is 10 units.



7 The equation of the graph shown above is.

(A) $\frac{x^2}{25}+\frac{y^2}{44}=1$

(B) $\frac{x^2}{44}+\frac{y^2}{25}=1$

(C) $\frac{x^2}{25}+\frac{y^2}{11}=1$

(D) $\frac{x^2}{11}+\frac{y^2}{25}=1$

8 If $3x^2+2xy+y^2=2$, then the value of $\frac{dy}{dx}$ at $x=1$ is

(A) -2

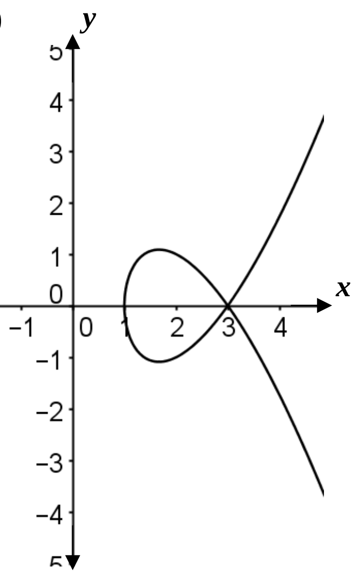
(B) 0

(C) 4

(D) not defined

9 Which diagram best represents the graph $y^2 = (x-1)(x-3)^2$?

(A)



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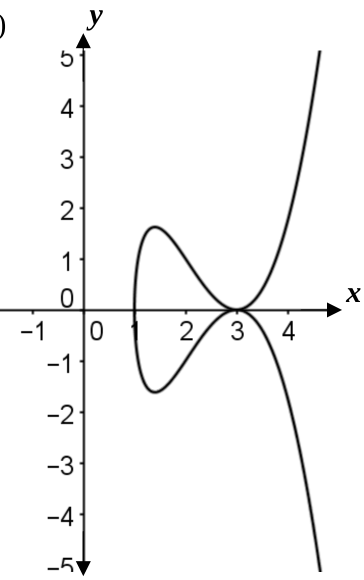
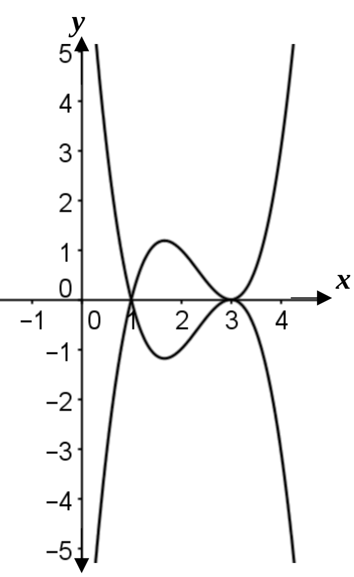
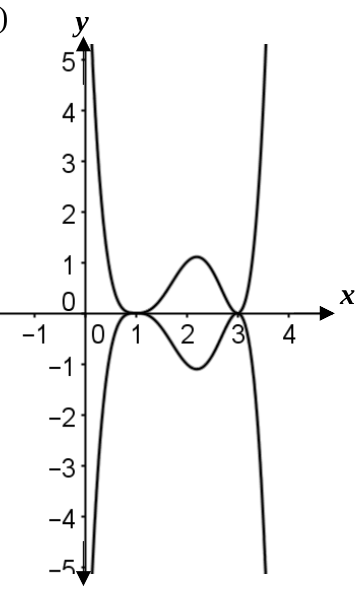


(C)



(D)



10 A person is standing on the outer edge of a circular disc that is spinning. His relative position on the disc remains unchanged. Which description below best describes the situation?

(A) The person is experiencing a force that is pushing him away from the centre of the disc.

(B) The person is experiencing a force that is pushing him towards the centre of the disc.

(C) The person is experiencing a force tangential to the edge of the disk in the direction of the motion of the disk.

(D) The person is experiencing a force tangential to the edge of the disk in the direction of the opposite direction to the motion of the disk.

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

a) If $z_1 = 3+4i$, $z_2 = 1-i$, find

(i) $\overline{z_1 z_2}$ **1**

(ii) $\left|\dfrac{z_1}{z_2}\right|$ **2**

(iii) $\sqrt{z_1}$ **3**

b) Sketch separately the following loci in an Argand plane and state the cartesian equations in each case.

(i) $\text{Re}\left(\dfrac{z-2}{2}\right) = 0$ **2**

(ii) $\arg(z+2) = -\dfrac{\pi}{6}$ **2**

c) (i) Express $\dfrac{1+2x^2}{(2+x^2)(1+x^2)}$ in the form $\dfrac{A}{2+x^2}+\dfrac{B}{1+x^2}$ **2**

(ii) Use the substitution $t = \tan x$ and your answer from part (i) to find $\displaystyle\int \frac{(1+\sin^2 x)\,dx}{1+\cos^2 x}$ **3**

(Leave your answer in term of t)

Question 12 (15 marks) Use a SEPARATE writing booklet.

a) If $\arg z_1 = \theta$ and $\arg z_2 = \phi$, show that $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ **3**

b) The equation $z^2 + (1+i)z + k = 0$ has root $1-2i$. Find the other root, and the value of k . **2**

c) Let α, β, γ be the roots (none of which is zero) of $x^3 + 3px + q = 0$

(i) Find expressions for $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \alpha\gamma$ and $\alpha\beta\gamma$ **1**

(ii) Find an expression for $\dfrac{\alpha\beta}{\gamma} + \dfrac{\beta\gamma}{\alpha} + \dfrac{\gamma\alpha}{\beta}$ **2**

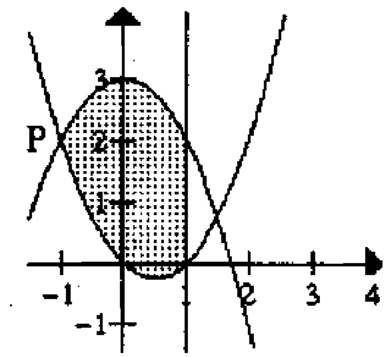
(iii) Find an expression for $\dfrac{\alpha\beta}{\gamma}.\dfrac{\beta\gamma}{\alpha} + \dfrac{\alpha\beta}{\gamma}.\dfrac{\gamma\alpha}{\beta} + \dfrac{\beta\gamma}{\alpha}.\dfrac{\gamma\alpha}{\beta}$ **2**

(iv) Hence obtain a monic equation whose roots are $\dfrac{\alpha\beta}{\gamma}, \dfrac{\beta\gamma}{\alpha}, \dfrac{\gamma\alpha}{\beta}$ **2**

d) Show that $\displaystyle\int_0^1 \frac{dx}{9-x^2} = \frac{1}{6}\ln 2$ **3**

Question 13 (15 marks) Use a SEPARATE writing booklet.

a) The shaded region bounded by $y = 3 - x^2$, $y = x^2 - x$ and $x = 1$ is rotated about the line $x = 1$. The point P is the intersection of $y = 3 - x^2$ and $y = x^2 - x$ in the second quadrant.



i) Find the x coordinate of P. **1**

ii) Use the method of cylindrical shells to express the volume of the resulting solid of revolution as an integral [DO NOT SOLVE THE INTEGRAL] **3**

b) (i) If $u_n = \dfrac{1}{n!}\displaystyle\int_0^1 x^n e^{-x}\,dx,$ where $n \geq 0$, show that $\dfrac{1}{n!} = e\left(u_{n-1} - u_n\right)$ **3**

(ii) Hence find the value of u_4 **2**

c) If a, b, c are positive real numbers;

i) Show that $a^2 + b^2 \geq 2ab$ **1**

ii) Hence prove $\left(a+b+c\right)\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right) \geq 9$ **2**

d) If z_1, z_2 are two complex numbers such that $\left|z_1 + z_2\right| = \left|z_1 - z_2\right|$, show that **3**

$$\arg z_1 - \arg z_2 = \frac{\pi}{2}$$

Question 14 (15 marks) Use a SEPARATE writing booklet.

a) $f(x)$ is defined by the equation $f(x) = x^2\left(x - \frac{3}{2}\right)$, on the domain $-2 \le x \le 2$.

Note: each sketch should take about a third of a page.

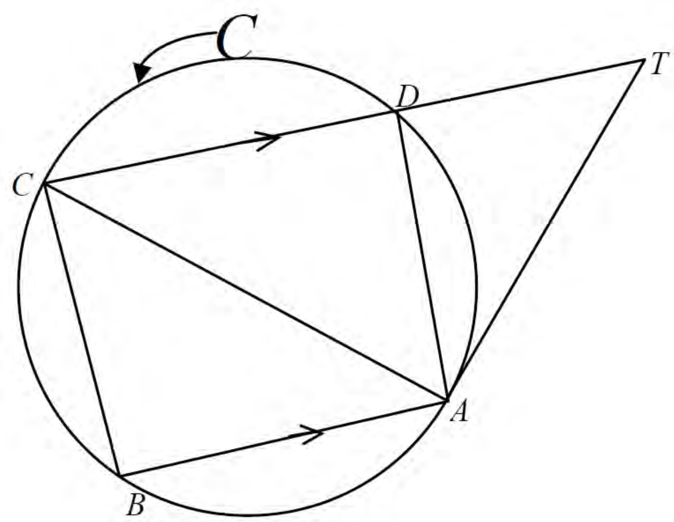
i) Draw a neat sketch of $f(x)$, labelling all intersections with coordinate axes and turning points **2**

ii) Sketch $y = \dfrac{1}{f(x)}$ **2**

iii) Sketch $y = \sqrt{f(x)}$ **2**

iv) Sketch $y = \ln\left(f(|x|)\right)$ **2**

b) The points *A*, *B*, *C* and *D* lie on the circle *C*. From the exterior point *T*, a tangent is drawn to point *A* on *C*. The line *CT* passes through *D* and *TC* is parallel to *AB*.



i) Copy or trace the diagram onto your page.

ii) Prove that ΔADT is similar to ΔABC. **3**

The line *BA* is produced through *A* to point *M*, which lies on a second circle.
The points *A*, *D*, *T* also lie on this second circle and the line *DM* crosses *AT* at *O*.

iii) Show that ΔOMA is isosceles. **2**

iv) Show that $TM = BC$. **2**

Question 15 (15 marks) Use a SEPARATE writing booklet.

a) Given that $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$, find $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ **2**

b) The hyperbola H has an equation $xy = 9$. $P\left(3p, \frac{3}{p}\right)$, where $p > 0$, and $Q\left(3q, \frac{3}{q}\right)$, where $q > 0$, are two distinct arbitrary points on H.

(i) Prove that the equation of the tangent at P is $x + p^2 y = 6p$ **2**

(ii) The tangents at P and Q intersect at T. Find the coordinates of T. **3**

(iii) The chord PQ produced passes through the point $(0,6)$. Given that the equation of this chord is $x + pqy = 3(p+q)$ find;

(a) Find the equation of the locus of T **3**

(b) Give a geometrical description of this locus **1**

c) A light inextensible string of length *3L* is threaded through a smooth vertical ring which is free to turn. The string carries a particle at each end. One particle *A* of mass *m* is at rest at a distance *L* below the ring. The other particle *B* of mass *M* is rotating in a horizontal circle whose centre is *A*.

(i) Find *m* in terms of *M*. **2**

(ii) Find the angular velocity of *B* in terms of g and L **2**

Question 16 (15 marks) Use a SEPARATE writing booklet.

a) Use mathematical induction to prove that for all n where n can be any positive integer that $(a-b)$ is a factor of $a^n - b^n$ **3**

b) A car travels around a banked circular track of radius 90 metres at 54 km/h.

(i) Draw a diagram showing all the forces acting on the car **1**

(ii) Show that the car will have tendency to slip sideways if the angle at which the banked track is banked is $\tan^{-1}\left(\frac{1}{4}\right)$. **3**

(iii) A second car of mass 1.2 tonnes travels around the same bend at 72 km/h. Find the sideways frictional force exerted by the road on the wheels of the car in Newtons. You may assume gravity = 10 m/s^2. (Answer correct to 1decimal place) **3**

c) (i) Using $\tan(2\theta+\theta)=\tan 3\theta$, show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$ **2**

(ii) Find the value of x for which $3\tan^{-1}x = \dfrac{\pi}{2} - \tan^{-1}3x$, **3**

where $\tan^{-1}x$ and $\tan^{-1}3x$ both lie between 0 and $\dfrac{\pi}{2}$

End of Exam

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \ n \neq -1; \ x \neq 0, \ if \ n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \ x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \ a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \ a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \ a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \ a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \ a \neq 0, \ -a < x < a$$

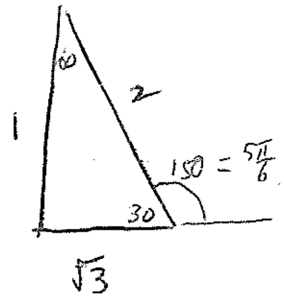
$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2} \right), \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, \ x > 0$

Multiple Choice

1 $-\sqrt{3}+i$



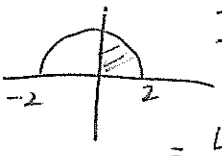
$2\,\text{cis}\,\frac{5\pi}{6}$

$= D.$

2. sum of focal lengths is a constant $\Rightarrow 2a$

$\therefore a = 5.$

B

3. 

$\frac{\pi r^2}{4}$

$= \frac{4\pi}{4}$

$= \pi.$

A.

4. $\int \frac{1\,dx}{\sqrt{4(x^2-2x+\frac{5}{2})}}$

$= \frac{1}{2}\int \frac{1}{\sqrt{x^2-2x+\frac{5}{2}}}\,dx$

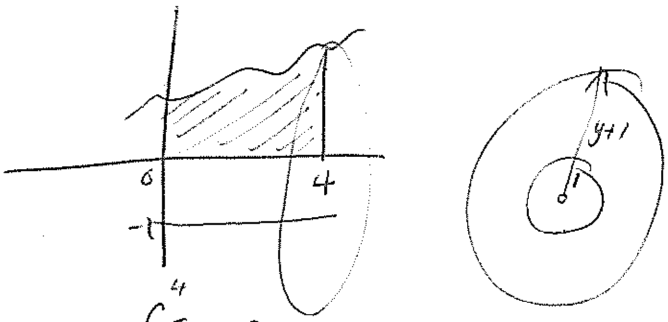
$= \frac{1}{2}\int \frac{1}{\sqrt{(x-1)^2+(\sqrt{\frac{3}{2}})^2}}$

$= \frac{1}{2}\ln\left[(x-1)+\sqrt{x^2-2x+\frac{5}{2}}\right. + c$

$= C$

5. B

unreal occur in conjugate pairs since real coefficient so since the degree of polynomial is odd there must be at least 1 real root.

6. 

$\pi \int_0^4 [(y+1)^2 - 1^2]\,dx$

$= D.$

7. D

8. when $x = 1$.

$3 + 2y + y^2 = 2$

$y^2 + 2y + 1 = 0$

$(y+1)^2 = 0$

$y = -1.$

D

$\frac{d}{dx} \Rightarrow 6x + y.2 + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$2x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2y - 6x$

$\frac{dy}{dx}(2x+2y) = -2y - 6x$

$\frac{dy}{dx} = \frac{-2y-6x}{2x+2y}$

when $x = +1$

$y = -1$

$2x + 2y = 0$

$\therefore$ not defined

9. A

10. B.

Question 11

a(i) $\overline{z_1 z_2}$

$z_1 z_2 = (3+4i)(1-i)$

$= 3-3i+4i-4i^2$

$= 7+i$

$\therefore \overline{z_1 z_2} = 7-i$

(ii) $\left|\frac{z_1}{z_2}\right|$

$\frac{z_1}{z_2} = \frac{3+4i}{1-i} \times \frac{1+i}{1+i}$

$= \frac{3+3i+4i+4i^2}{1+1}$

$= \frac{-1+7i}{2}$

$= -\frac{1}{2}+\frac{7}{2}i$

$\left|\frac{z_1}{z_2}\right| = \left|-\frac{1}{2}+\frac{7}{2}i\right|$

$= \sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{7}{2}\right)^2}$

$= \sqrt{\frac{1}{4}+\frac{49}{4}}$

$= \sqrt{\frac{50}{4}}$

$= \frac{5\sqrt{2}}{2}$

OR $\frac{|z_1|}{|z_2|} = \frac{\sqrt{9+16}}{\sqrt{1+1}}$

$= \frac{5}{\sqrt{2}}$

$= \frac{5\sqrt{2}}{2}$

(iii) $(x+iy)^2 = 3+4i$

$x^2+2xyi-y^2 = 3+4i$

$x^2-y^2=3 \quad 2xy=4$

$\therefore xy=2$

$y=\frac{2}{x}$

$x^2-\left(\frac{2}{x}\right)^2=3$

$x^4-3x^2-4=0$

$(x^2+1)(x^2-4)=0$

$x^2=4$

$x=\pm 2$

$\therefore x=2 \quad y=1$

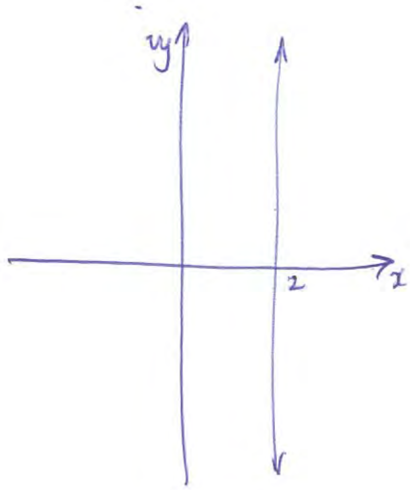
$x=-2 \quad y=-1$

$\therefore \sqrt{z_1} = \pm(2+1i)$

b

(i)

$$Re\left(\frac{z-2}{2}\right)=0$$



$$\frac{z-2}{2} = \frac{x+iy-2}{2}$$

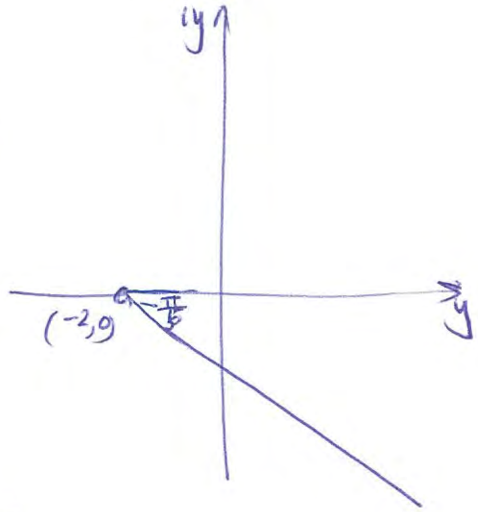
$$= \frac{x-2}{2} + \frac{iy}{2}$$

$$Re\left(\frac{z-2}{2}\right)=0 \Rightarrow \frac{x-2}{2}=0$$

$$\therefore x-2=0$$

$$\text{or } \underline{x=2}$$

(ii) $\arg(z+2) = -\frac{\pi}{6}$



$$m = \tan\frac{\cancel{5}\pi}{6} = -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

For y intercept $\Rightarrow \tan\frac{\pi}{6} = \frac{y}{2}$

$$y = 2\tan\frac{\pi}{6}$$

$$= \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{3}$$

$\therefore$ equation is $y = -\frac{\sqrt{3}}{3}x - \frac{2\sqrt{3}}{3}$

for $x>-2$,

C(i) $1+2x^2 \equiv A(1+x^2) + B(2+x^2)$

$\equiv A + Ax^2 + 2B + Bx^2$

$= A+2B+(A+B)x^2$

$A+2B=1$ — ①

$A+B=2$ — ②

①−② $B=-1$

sub into ①

$A-2=1$

$A=3$.

$$\frac{3}{2+x^2} + \frac{-1}{1+x^2}$$

(ii) $$\int \frac{(1+\sin^2 x)\,dx}{1+\cos^2 x} = \int \frac{1+\frac{1-\cos 2x}{2}}{1+\frac{1+\cos 2x}{2}}\,dx$$

$$= \int \frac{3-\cos 2x}{3+\cos 2x}\,dx$$

$$= \int \frac{3-\frac{1-t^2}{1+t^2}}{3+\frac{1-t^2}{1+t^2}} \cdot \frac{1}{1+t^2}\,dt$$

$$= \int \frac{3+3t^2-1+t^2}{(3+3t^2+1-t^2)} \frac{dt}{(1+t^2)}$$

$$= \int \frac{1+2t^2}{(2+t^2)(1+t^2)}\,dt$$

Let $t=\tan x$

$\frac{dt}{dx} = \sec^2 x$

$dt = \sec^2 x\,dx$

$dt = (1+t^2)\,dx$

$dx = \frac{1}{1+t^2}\,dt$

from part (i)

$$= \int \frac{3}{2+t^2}\,dt - \int \frac{1}{1+t^2}\,dt$$

$$= \frac{3}{\sqrt{2}} \tan^{-1}\frac{t}{\sqrt{2}} - \tan^{-1} t + C.$$

$$= \frac{3}{\sqrt{2}} \tan^{-1}\frac{\tan x}{\sqrt{2}} - x + c$$

Question 12

a) Let $z_1 = r_1(\cos\theta + i\sin\theta)$ and $z_2 = r_2(\cos\phi + i\sin\phi)$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta + i\sin\theta)}{r_2(\cos\phi + i\sin\phi)}$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta + i\sin\theta)}{r_2(\cos\phi + i\sin\phi)} \times \frac{(\cos\phi - i\sin\phi)}{\cos\phi - i\sin\phi}$$

$$= \frac{r_1(\cos\theta\cos\phi - i\sin\phi\cos\theta + i\sin\theta\cos\phi - i^2\sin\theta\sin\phi)}{\cos^2\phi - i^2\sin^2\phi}$$

$$= \frac{r_1\left[(\cos\theta\cos\phi + \sin\theta\sin\phi) - i(\sin\phi\cos\theta - \sin\theta\cos\phi)\right]}{r_2 \cdot (\cos^2\phi + \sin^2\phi)}$$

$$= \frac{r_1\left[\cos(\theta - \phi) - i\sin(\phi - \theta)\right]}{r_2}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta - \phi) + i\sin(\theta - \phi)\right]$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta - \phi$$

$$= \arg z_1 - \arg z_2.$$

b) Let α, β be the roots

$\alpha = 1 - 2i$

$\alpha + \beta = -\frac{b}{a}$.

$1 - 2i + \beta = -1 - i$

$\beta = -2 + i$

$\alpha\beta = \frac{c}{a}$

$= k$.

$(-2+i)(1-2i) = k$

$k = -2 + 4i + i - 2i^2$

$= 5i$

c) $x^3+3px+q=0$

(i) $\alpha+\beta+\gamma=0$

$\alpha\beta+\beta\gamma+\alpha\gamma=3p$

$\alpha\beta q=-q$

(ii) $$\frac{\alpha\beta}{\gamma}+\frac{\beta\gamma}{\alpha}+\frac{\gamma\alpha}{\beta}=\frac{(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2}{\alpha\beta\gamma}$$

$$=\frac{(\alpha\beta+\beta\gamma+\gamma\alpha)^2-2(\alpha\beta.\beta\gamma+\alpha\beta.\gamma\alpha+\beta\gamma.\gamma\alpha)}{\alpha\beta\gamma}$$

$$=\frac{(3p)^2-2[(\alpha\gamma\beta^2+\alpha^2\beta\gamma+\gamma^2\beta\alpha)]}{\alpha\beta\gamma}$$

$$=\frac{(3p)^2-2[\alpha\beta\gamma(\alpha+\beta+\gamma)]}{\alpha\beta\gamma}$$

$$=\frac{9p^2-0}{-q}$$

$$=\frac{9p^2}{q}$$

(iii) $$\frac{\alpha\beta^3\gamma+\alpha^3\beta\gamma+\alpha\beta\gamma^3}{\alpha\beta\gamma}$$

$$=\frac{\alpha\beta\gamma(\alpha^2+\beta^2+\gamma^2)}{\alpha\beta\gamma}$$

$$=\alpha^2+\beta^2+\gamma^2$$

$$=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$$

$$=0-2(3p)$$

$$=-6p$$

(iv) (ii) $\Rightarrow$ sum of roots $-\frac{3p^2}{q}$

(iii) $\Rightarrow$ sum of root 2 at a time $-6p$

product of roots

$$\frac{\alpha\beta}{\gamma}\times\frac{\beta\gamma}{\alpha}\times\frac{\gamma\alpha}{\beta}=\alpha\beta\gamma$$

$$=-q$$

$\therefore\ x^3-\frac{9p^2}{q}x^2-6px+q=0$

(d) $\displaystyle\int_0^1 \frac{dx}{9-x^2} = \frac{1}{6}\ln 2.$

$$\int_0^1 \frac{1}{(3-x)(3+x)}\,dx.$$

$$\int_0^1 \frac{1}{(3-x)(3+x)} = \int\left[\frac{a}{(3-x)} + \frac{b}{(3+x)}\right]dx$$

$$1 \equiv a(3+x) + b(3-x)$$

Let $x = 3$ $\quad 1 = 6a$

Let $x = -3$ $\quad 1 = 6b$

$$a = \frac{1}{6} \quad b = \frac{1}{6}$$

$$\int_0^1 \frac{1}{(3-x)(3+x)} = \frac{-1}{6}\int_0^1 \frac{-1}{3-x} + \frac{1}{6}\int_0^1 \frac{1}{3+x}\,dx.$$

$$= \left[-\frac{1}{6}\ln(3-x)\right]_0^1 + \left[\frac{1}{6}\ln(3+x)\right]_0^1$$

$$= \left[-\frac{1}{6}\ln 2 + \frac{1}{6}\ln 3\right] + \left[\frac{1}{6}\ln 4 - \frac{1}{6}\ln 3\right]$$

$$= -\frac{1}{6}\ln 6$$

$$= \frac{1}{6}\left[\cancel{\ln 3} - \ln 2 + \ln 4 - \cancel{\ln 3}\right]$$

$$= \frac{1}{6}\left[\ln 4 - \ln 2\right]$$

$$= \frac{1}{6}\ln\frac{4}{2}$$

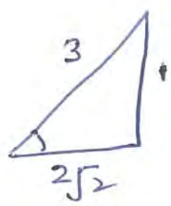
$$= \frac{1}{6}\ln 2.$$

Alternative approach: Let $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta.$

$$\int_0^1 \frac{dx}{9-x^2} = \int_0^{\sin^{-1}\left(\frac{1}{3}\right)} \frac{1}{3}\sec\theta\, d\theta$$

$$= \frac{1}{3}\int \frac{\sec\theta(\tan\theta + \sec\theta)}{\tan\theta + \sec\theta}\,d\theta$$



$$= \frac{1}{3}\Big[\ln(\tan\theta + \sec\theta)\Big]_0^{\sin^{-1}\left(\frac{1}{3}\right)}$$

$$= \frac{1}{3}\left[\ln\left(\frac{1}{2\sqrt{2}} + \frac{3}{2\sqrt{3}}\right)\right] = \frac{1}{3}\left[\ln\frac{4}{2\sqrt{2}}\right] = \frac{1}{3}\ln\sqrt{2} = \frac{1}{6}\ln 2$$

Question 13

a (i) $P: \; 3 - x^2 = x^2 - x$

$2x^2 - x - 3 = 0$

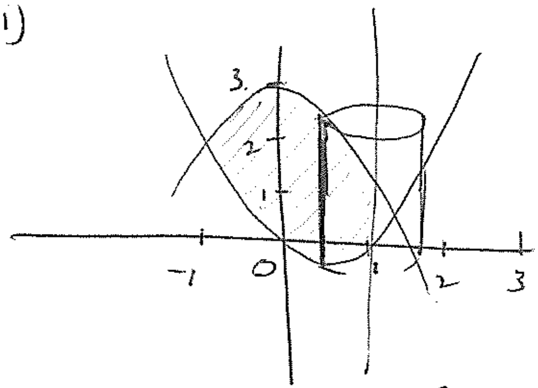
$(x+1)(2x-3) = 0$

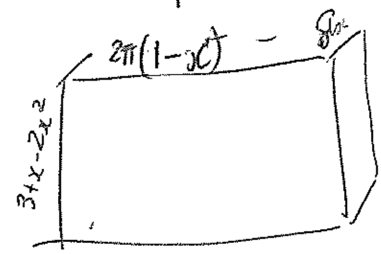
$\therefore x = -1, \frac{3}{2}$

$\therefore$ x coord of P is -1

(as P is in 2nd quadrant)

(ii)





$h = (3 - x^2) - (x^2 - x)$

$= 3 + x - 2x^2$

Volume $\delta V = \delta A h.$

$\delta V = 2\pi(1-x)(3+x-2x^2)\,\delta x$

$V = \lim_{\delta x \to 0} \sum_{x=-1}^{1} 2\pi(3 - 2x - 3x^2 + 2x^3)\,\delta x$

$= 2\pi \int_{-1}^{1} 3 - 2x - 3x^2 + 2x^3 \, dx$

b(i) $U_n = \frac{1}{n!}\int_0^1 \overset{u}{x^n}\, \overset{\frac{dv}{dx}}{e^{-x}}\, dx$ $\qquad u = x^n \qquad \frac{du}{dx} = nx^{n-1}$

$\qquad\qquad\qquad\qquad\qquad\qquad \frac{dv}{dx} = e^{-x} \qquad v = -e^{-x}$

$$U_n = \frac{1}{n!}\left\{\left[-e^{-x}x^n\right]_0^1 - \int_0^1 -e^{-x}nx^{n-1}\,dx\right\}$$

$$= \frac{1}{n!}\left\{-e^{-1} + n\int_0^1 e^{-x}x^{n-1}\,dx\right\}$$

$$= \frac{-1}{en!} + \frac{1}{(n-1)!}\int_0^1 e^{-x}x^{n-1}\,dx$$

$$U_n = -\frac{1}{en!} + U_{n-1}$$

$$\therefore -\frac{1}{en!} = U_{n-1} - U_n.$$

$$\frac{1}{n!} = e(U_{n-1} - U_n)$$

(ii) $U_n = U_{n-1} - \frac{1}{en!}$

$$U_4 = U_3 - \frac{1}{4!e}$$

$$= U_2 - \frac{1}{3!e} - \frac{1}{4!e}$$

$$\vdots$$

$$= U_0 - \frac{1}{1!e} - \frac{1}{2!e} - \frac{1}{3!e} - \frac{1}{4!e}$$

$$= \frac{1}{0!}\int_0^1 x^0 e^{-x}\,dx - \frac{1}{e} - \frac{1}{2e} - \frac{1}{6e} - \frac{1}{24e}$$

$$= \int_0^1 e^{-x}\,dx - \frac{41}{24e}$$

$$= -e^{-1} - (-e^0) - \frac{41}{24e}$$

$$= -\frac{1}{e} + 1 - \frac{41}{24e}$$

$$= -\frac{65}{24e} + 1$$

c) (i) $(a-b)^2 \geq 0$

$a^2-2ab+b^2 \geq 0$

$a^2+b^2 \geq 2ab$

(ii) $a\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+b\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+c\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9.$

L.H.S. $1+\frac{a}{b}+\frac{a}{c}+1+\frac{b}{a}+\frac{b}{c}+1+\frac{c}{a}+\frac{c}{b}$

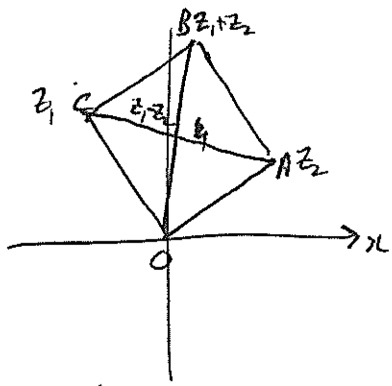
$3+\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}$

$3+\frac{a^2+b^2}{ab}+\frac{a^2+c^2}{ac}+\frac{b^2+c^2}{cb}$

$\geq 3+\frac{2\cancel{ab}}{\cancel{ab}}+\frac{2\cancel{ac}}{\cancel{ac}}+\frac{2\cancel{bc}}{\cancel{cb}}$

$\geq 9.$

d)



since $|z_1+z_2|=|z_1-z_2|$

$|AC|=|OB|$

OABC is a parallelogram with equal diagonals

$\therefore$ it is a rectangle

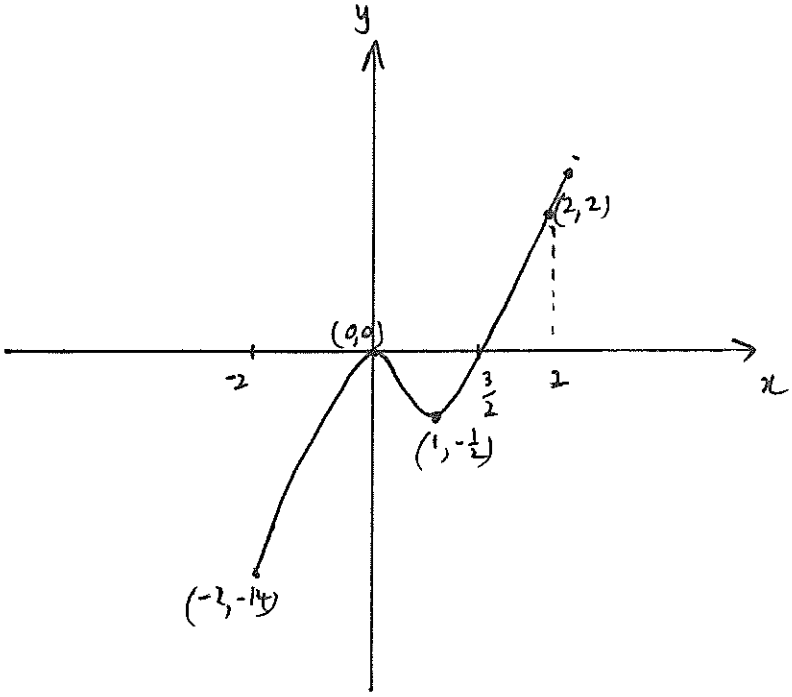
$\therefore \angle AOC$ is a right angle

$\angle xOC - \angle xOA = 90°$

or $\arg z_1 - \arg z_2 = \frac{\pi}{2}.$

Question 14

(i)



$y = x^3 - \frac{3x^2}{2}$

$y' = 3x^2 - 3x$

Let $y' = 0$

$0 = 3x^2 - 3x$

$0 = 3x(x-1)$

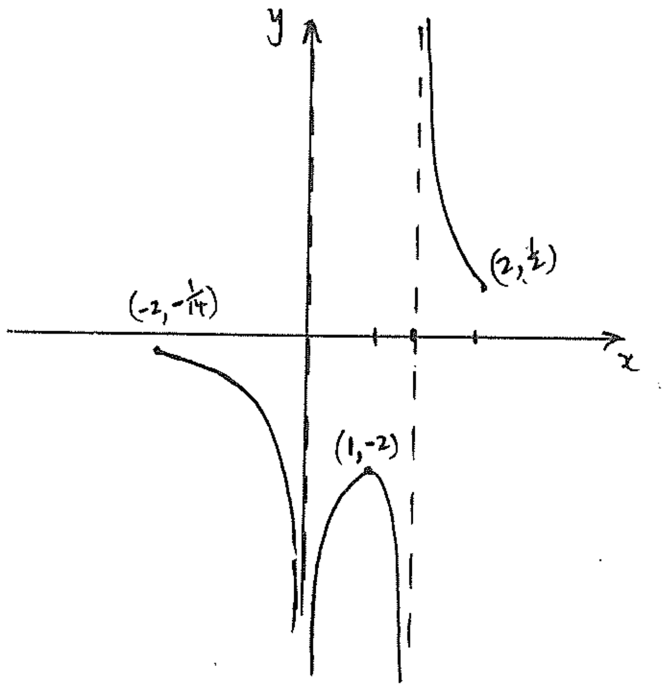
$x = 0$ or $x = 1$

$y = 0 \quad y = 1\left(-\frac{1}{2}\right)$

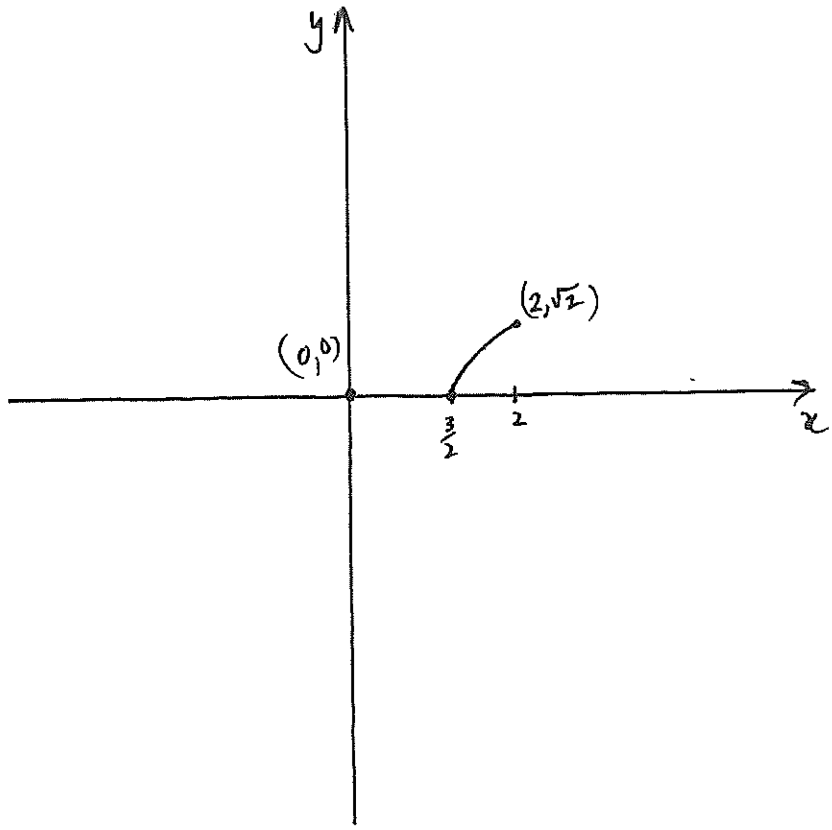
$= -\frac{1}{2}$

$(0,0) \quad \left(1, -\frac{1}{2}\right)$

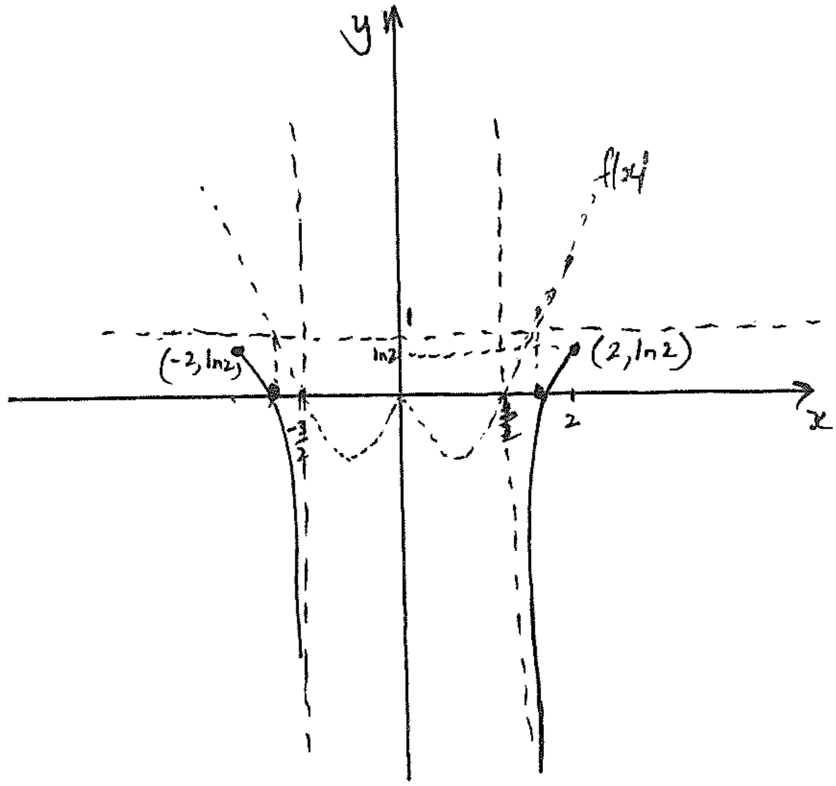
(ii)



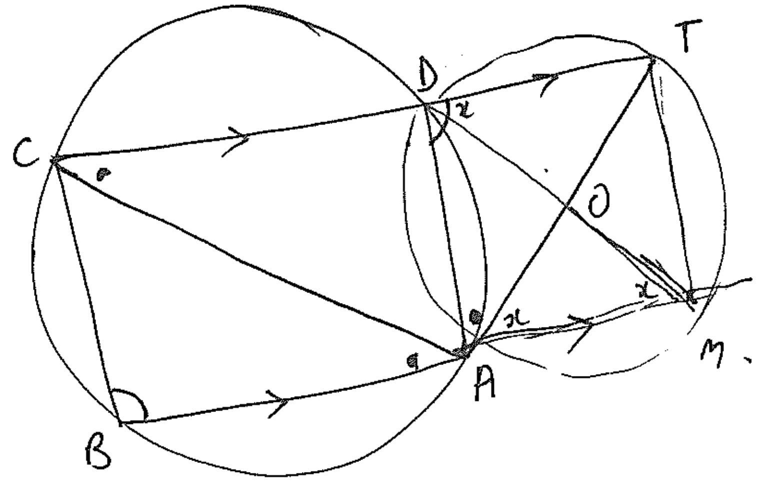
(iii)



(iv)



b(i)



(ii) In ΔADT and ΔABC

$\angle TAD = \angle DCA$ (angle in the alternate segment are equal.)

$\angle DCA = \angle CAB$ (alternate angles on parallel lines DC and AB are equal)

$\therefore \angle TAD = \angle CAB$. (angle)

$\angle TDA = \angle CBA$ (exterior angle of a cylic quadrilateral is equal to the interior opposite angle)

$\therefore \Delta ADT ||| \Delta ABC$ (Two angles in one triangle are equal to two angles in the other.

(iii) Let $\angle TDM = x$

$\angle TAM = x$ (angle in the same segment are equal)

$\angle OMA = x$ (alternate angles on parallel lines TD and AM are equal)

$\therefore \angle OAM = \angle OMA$

$\therefore \Delta OMA$ is isosceles.

(iv) $\angle TMA = \angle CDA$ (~~opposite~~ exterior angles of a cyclic quadrilateral ~~are equal~~ is equal to interior opposite)

$\angle CDA = 180 - \angle CBA$ (opposite angles of a cyclic quadrilateral are supplementary

$\therefore \angle CBA = 180 - \angle TMA$ $\therefore CB \parallel TM$ since cointerior angles are supplementary

$\therefore$ TMBC is a parallelogram $\therefore TM = CB$ (opposite sides of parallelogram

Question 15

(a) $$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx$$

$$= \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}$$ since $\sin(\pi-x) = \sin x$.

and $\cos^2(\pi-x) = (-\cos x)^2 = \cos^2 x$

$$\int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}\,dx - \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$$

$$\therefore 2\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}\,dx$$

$$\therefore \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$$

$$= -\frac{\pi}{2}\left[\tan^{-1}(\cos x)\right]_0^\pi$$ since $\frac{d}{dx}(\cos x) = -\sin x$.

$$= -\frac{\pi}{2}\left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$$

$$= \frac{\pi^2}{4}.$$

(b) (i) $x = 3p$ $\quad \frac{dx}{dp} = 3 \quad \frac{dy}{dp} = -\frac{3}{p^2}$

$y = \frac{3}{p}$

$$\frac{dy}{dx} = \frac{dy}{dp}\times\frac{dp}{dx}$$

$$= -\frac{3}{p^2}\times\frac{1}{3}$$

$$= -\frac{1}{p^2}$$

$$y - \frac{3}{p} = -\frac{1}{p^2}(x - 3p)$$

$$p^2 y - 3p = -x + 3p$$

$$x + p^2 y = 6p$$

(ii) $x + p^2 y = 6p$ — ①

$x + q^2 y = 6q$ — ②

① − ② $(p^2 - q^2)y = 6(p - q)$

$$\frac{(\cancel{p-q})(p+q)}{(\cancel{p-q})} y = 6$$

$$(p+q)y = 6$$

$$y = \frac{6}{p+q}$$

sub into ①

$$x + \frac{6p^2}{p+q} = 6p$$

$$x = 6p - \frac{6p^2}{p+q}$$

$$(p+q)x = 6p(p+q) - 6p^2$$

$$(p+q)x = \cancel{6p^2} + 6pq - \cancel{6p^2}$$

$$x = \frac{6pq}{p+q}$$

$$\therefore T\left(\frac{6pq}{p+q}, \frac{6}{p+q}\right)$$

(iii) $x + pqy = 3(p+q)$

sub $(0,6)$

$$6pq = 3(p+q)$$

$$\frac{pq}{p+q} = \frac{1}{2}$$

$$x = \frac{6pq}{p+q} \quad y = \frac{6}{p+q}$$

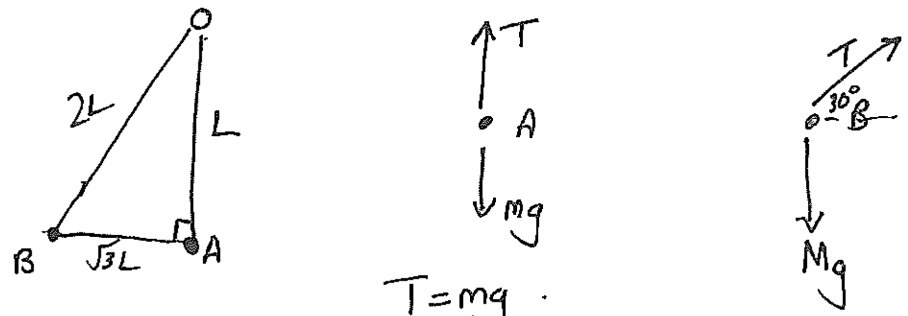
$$x = 3 \quad y = \frac{6}{p+q}.$$

$$T\left(3, \frac{6}{p+q}\right)$$

$\therefore x = 3$.

vertical line passing through $x=3$.

c,



$T = mg$.

vertically $T\sin 30 = Mg$

$\frac{1}{2}T = Mg$

or $\frac{1}{2}mg = Mg$

$m = 2M$.

horizontally $T\cos 30 = Mr\omega^2$

$\frac{\sqrt{3}}{2}mg = M(\sqrt{3}L)\omega^2$

$\frac{\sqrt{3}}{2}mg = \left(\frac{m}{2}\right)(\sqrt{3}L)\omega^2$

$\omega = \sqrt{\frac{g}{L}}$

Question 16

(a)

Test for $n=1$

$a-b$ is a factor of a^1-b^1

Assume true for $n=k$

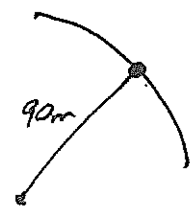
$a^k-b^k=(a-b)F$ where F is an integer

Test for $n=k+1$

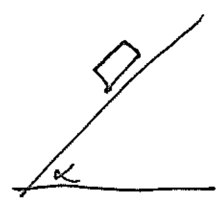
$$a^{k+1}-b^{k+1} = a^{k+1}-a^kb+a^kb-b^{k+1}$$
$$= a^k(a-b)+b(a^k-b^k)$$
$$= a^k(a-b)+b(a-b)F \quad \text{(from assumption)}$$
$$= (a-b)(a^k+bF)$$

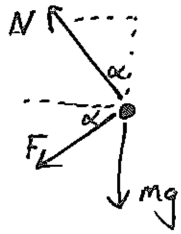
since $[a^m+bF]$ is another polynomial in a and b, we have show what we set out to prove.

b)



(i)





N - Normal reaction

F - frictional force

mg - weight

(ii) Vertically $N\cos\alpha = F\sin\alpha + mg$

$N\cos\alpha - F\sin\alpha = mg$ — (1)

Horizontally $N\sin\alpha + F\cos\alpha = \frac{mv^2}{r}$ — (2)

From (1) $N\cos\alpha\sin\alpha - F\sin^2\alpha = mg\sin\alpha$ (3)

and (2) $N\sin\alpha\cos\alpha + F\cos^2\alpha = \frac{mv^2}{r}\cos\alpha$ (4)

(4)-(3) $F(\cos^2\alpha+\sin^2\alpha) = \frac{mv^2}{r}\cos\alpha - mg\sin\alpha$

$$F = \frac{mv^2}{r}\cos\alpha - mg\sin\alpha, \quad *$$

no sideway slip when $F=0$

$$\therefore \quad mg\sin\alpha = \frac{mv^2}{r}\cos\alpha$$

$$\tan\alpha = \frac{v^2}{rg}$$

$$v = \frac{54\times1000}{60\times60}\ ms^{-1} \quad r=90.$$

$$\tan\alpha = \frac{15\times15}{90\times10}$$

$$= \frac{225}{900}$$

$$= \frac{1}{4}$$

$$\alpha = \tan^{-1}\left(\frac{1}{4}\right)$$

(iii) now $F = m\cos\alpha\left(\frac{v^2}{r} - g\tan\alpha\right)$ *

$$F = 1200\cdot\frac{4}{\sqrt{17}}\left(\left(\frac{72\times1000}{60\times60}\right)^2\cdot\frac{1}{90} - \frac{10}{4}\right) \qquad \cos\alpha = \frac{4}{\sqrt{17}}$$

$$= \frac{4800}{\sqrt{17}}\left(\frac{400}{90} - \frac{10}{4}\right)$$

$$= \frac{4800}{\sqrt{17}}\left(\frac{800-450}{180}\right)$$

$$= \frac{4800}{\sqrt{17}} \times \frac{350}{180}$$

$$\doteqdot 2263.665\ldots$$

$$= 2263.7\ N$$

(c).

(i) $\tan(2\theta+\theta) = \dfrac{\tan 2\theta + \tan\theta}{1-\tan 2\theta \tan\theta}$

$$= \left(\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta\right) \div \left(1 - \frac{2\tan^2\theta}{1-\tan^2\theta}\right)$$

$$= \left(\frac{2\tan\theta + \tan\theta - \tan^3\theta}{1-\tan^2\theta}\right) \div \left(\frac{1-\tan^2\theta - 2\tan^2\theta}{1-\tan^2\theta}\right)$$

$$= \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

(ii) Let $\theta = \tan^{-1}x$

$$\therefore 3\theta = 3\tan^{-1}x$$

$$= \frac{\pi}{2} - \tan^{-1}3x$$

$$\tan^{-1}3x = \frac{\pi}{2} - 3\theta$$

$$\tan(\tan^{-1}3x) = \tan\left(\frac{\pi}{2} - 3\theta\right)$$

$$3x = \cot 3\theta$$

$$3x = \frac{1-3\tan^2\theta}{3\tan\theta - \tan^3\theta} \quad (\text{from (i)})$$

$$3x = \frac{1-3x^2}{3x-x^3} \quad (\theta = \tan^{-1}x \Rightarrow x = \tan\theta$$

$$3x(3x-x^3) = 1-3x^2$$

$$9x^2 - 3x^4 = 1-3x^2$$

$$3x^4 - 12x^2 + 1 = 0$$

$$\therefore x^2 = \frac{6 \pm \sqrt{33}}{3}$$

$$\therefore x = \sqrt{\frac{6\pm\sqrt{33}}{3}}$$