PENRITH SELECTIVE HIGH SCHOOL

2017
HSC TRIAL EXAMINATION

Mathematics Extension 2

General Instructions:

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A Reference Sheet is provided
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Answer all Questions in the writing booklets provided
- All diagrams are not drawn to scale

Total marks–100

SECTION I **Pages 3–7**

10 marks
- Attempt Questions 1–10
- Allow about 15 minutes for this section

SECTION II **Pages 8–15**

90 marks
- Attempt Questions 11–16
- Allow about 2 hours 45 minutes for this section

Student Number:________________________

This paper MUST NOT be removed from the examination room

Assessor: Mr Ferguson

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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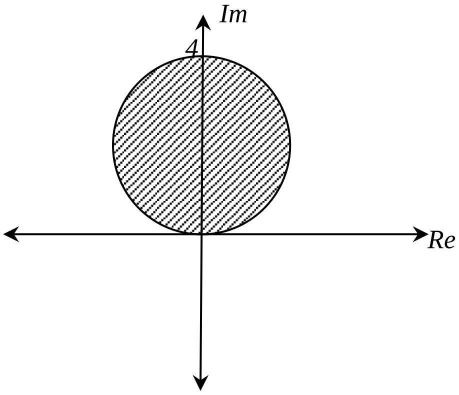
1 Which of the following inequalities is represented by the Argand diagram?



(A) $|z-2| \le 2$

(B) $|z+2| \le 2$

(C) $|z-2i| \le 2$

(D) $|z+2i| \le 2$

2 Let $u = 7\cos\frac{\pi}{4} + 7i\sin\frac{\pi}{4}$ and $v = a\cos b + ai\sin b,$ where a and b are real constants.

If $uv = 42\cos\frac{\pi}{20} + 42i\sin\frac{\pi}{20},$ then

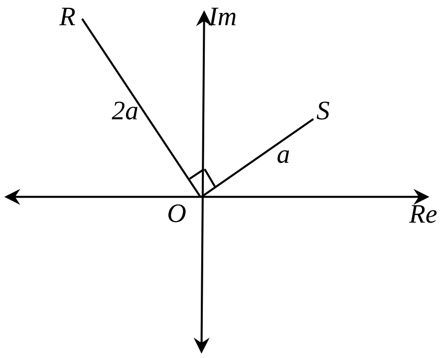
(A) $a = 35$ and $b = \frac{\pi}{5}$

(B) $a = 6$ and $b = \frac{\pi}{5}$

(C) $a = 35$ and $b = -\frac{\pi}{5}$

(D) $a = 6$ and $b = -\frac{\pi}{5}$

3 In the Argand diagram below the points R and S represent the complex numbers w and z, respectively where $\angle SOR = 90^\circ$. The distance OR is $2a$ units, and distance OS is a units. Which of the following is correct?



(A) $w = 2iz$

(B) $w = i\overline{w}$

(C) $w = \dfrac{-iz}{2}$

(D) $w = \dfrac{-z}{2i}$

4 Which of the following graphs is the locus of the point P representing the complex number z moving in an Argand diagram such that $|z-2i| = 2 + \operatorname{Im} z$?

(A) a circle

(B) a parabola

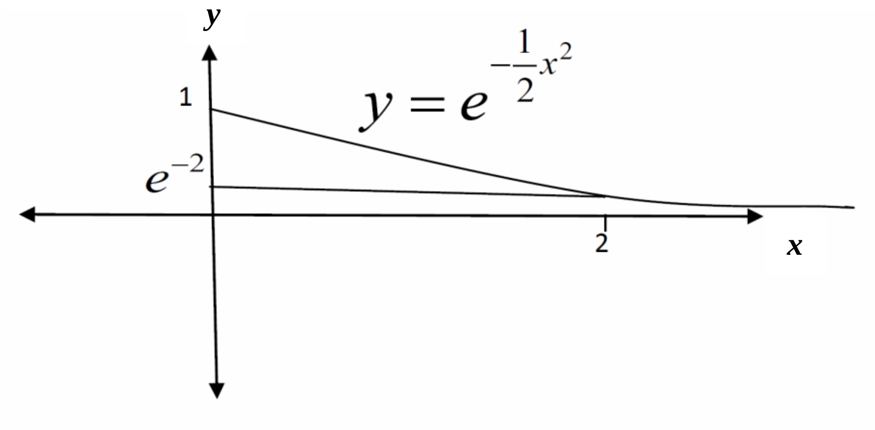
(C) a hyperbola

(D) a straight line

5 The foci of the hyperbola $xy = 8$ are

(A) $(\pm 4, \pm 4)$ (B) $(\pm 2\sqrt{2}, \pm 2\sqrt{2})$ (C) $(\pm 8\sqrt{2}, \pm 8\sqrt{2})$ (D) $(\pm 4\sqrt{2}, \pm 4\sqrt{2})$

6 The volume of the solid obtained by revolving the region bounded by $y = e^{-\frac{1}{2}x^2}$, $y = e^{-2}$ and the lines $x = 0, x = 2$ about the y – axis can be evaluated by which of the following integrals



(A) $V = 2\pi\int_{e^{-2}}^{1} x\left(e^{-\frac{1}{2}x^2} - e^{-2}\right)dx$

(B) $V = 2\pi\int_{e^{-2}}^{1} x\left(e^{-\frac{1}{2}x^2}\right)dx$

(C) $V = 2\pi\int_{0}^{2} x\left(e^{-\frac{1}{2}x^2} - e^{-2}\right)dx$

(D) $V = 2\pi\int_{0}^{2} x\left(e^{-\frac{1}{2}x^2}\right)dx$

7 Consider a polynomial $P(x)$ of degree 3.
You are given 2 numbers *a* and *b* such that

- $a < b$
- $P(a) > P(b) > 0$
- $P'(a) = P'(b) = 0$

The polynomial has

(A) 3 real zeros

(B) 1 real zero γ such that $\gamma < a$

(C) 1 real zero γ such that $a < \gamma < b$

(D) 1 real zero γ such that $\gamma > b$

8 If $\int_1^4 f(x)\,dx = 6$, what is the value of $\int_1^4 f(5-x)\,dx$?

(A) 6

(B) 3

(C) -1

(D) -6

9 Given that $x^2 + y^2 + xy = 12$, which of the following is true?

(A) $\frac{dy}{dx} = \frac{2x+y}{2y+x}$

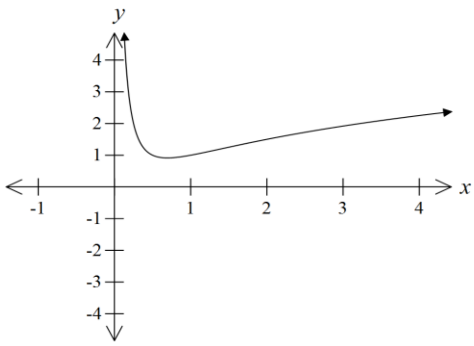
(B) $\frac{dy}{dx} = -\frac{2x+y}{2y+x}$

(C) $\frac{dy}{dx} = \frac{2x-y}{2y+x}$

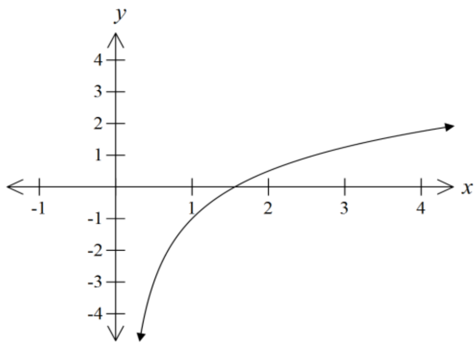
(D) $\frac{dy}{dx} = \frac{-2x+y}{2y+x}$

10 Which of the following is the sketch of $y = \log_2 x + \frac{1}{x}$?

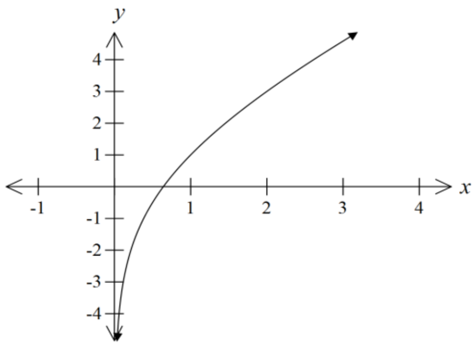
(A)



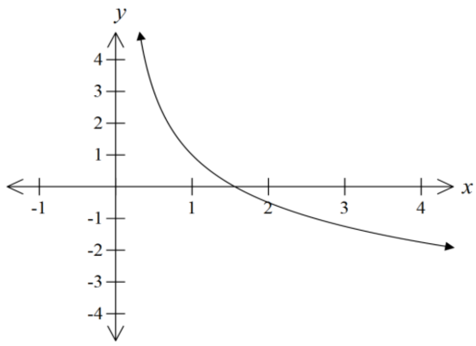
(B)



(C)



(D)



Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a new booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

a) Let $z = 3 + 2i$ and $w = -1 + i$. Express the following in the form $x + iy$, where x and y are real numbers:

i) $\dfrac{z}{iw}$ **2**

ii) $\text{Im}\left(\bar{z}w\right)$ **2**

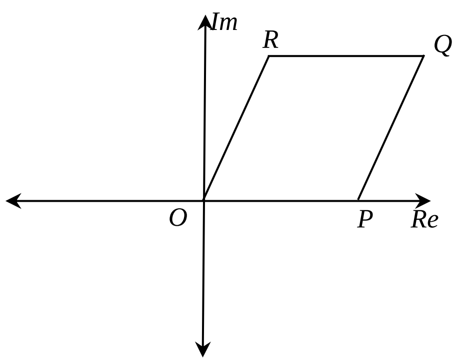
b) Let $z = -1 + i$.

i) Express z in modulus-argument form. **2**

ii) Express z^4 in modulus-argument form. **1**

iii) Hence, evaluate z^{20} **1**

c) In the diagram below $OPQR$ is a rhombus. R represents $1 + i\sqrt{3}$. Find the complex number represented by Q **2**



d) Evaluate $\displaystyle\int_0^1 \frac{2}{\sqrt{1+3x}}\,dx$ **2**

e) By using integration by parts, find $\displaystyle\int x^2 \ln 2x\ dx$ **3**

Question 12 (15 marks) Use a SEPARATE writing booklet.

a) The polynomial $z^3 - 7z^2 + 25z - 39$ has one zero equal to $2+3i$. Write down its three linear factors. **2**

b) The equation $x^4 - px^3 + qx^2 - pqx + 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Show that $(\alpha+\beta+\gamma)(\alpha+\beta+\delta)(\alpha+\gamma+\delta)(\beta+\gamma+\delta) = 1$ **4**

c) α, β, γ are the roots of $x^3 + qx + r = 0$. Find in terms of q, r the equation with roots $\dfrac{1}{\alpha^2}, \dfrac{1}{\beta^2}, \dfrac{1}{\gamma^2}$ **3**

d) Let ω be a non-real cube root of unity

i) Show that $1+\omega+\omega^2 = 0$ **1**

ii) Hence simpify $(1+\omega)^2$ **1**

iii) Show that $(1+\omega)^3 = -1$ **1**

iv) Use part iii) to simplify $(1+\omega)^{3n}$ and hence show that

$${}^{3n}C_0 - \frac{1}{2}\left({}^{3n}C_1 + {}^{3n}C_2\right) + {}^{3n}C_3 - \frac{1}{2}\left({}^{3n}C_4 + {}^{3n}C_5\right) + {}^{3n}C_6 - \ldots {}^{3n}C_{3n} = (-1)^n$$ **3**

Question 13 (15 marks) Use a SEPARATE writing booklet.

a) i) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ using the substitution $u = a - x$ **2**

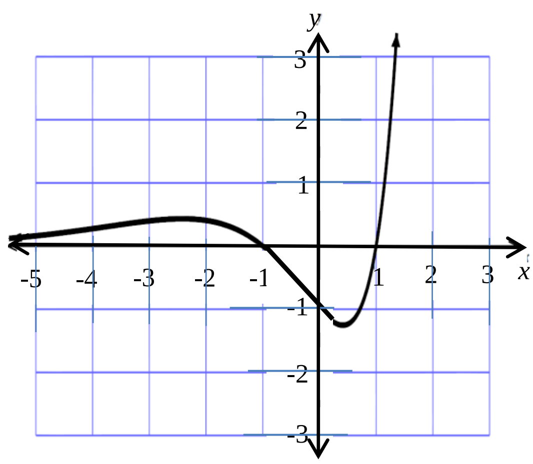
ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \sin x \cos x}\, dx$ **2**

b) i) Find real constants A, B and C such that

$$\frac{4x-2}{(x^2-1)(x-2)} \equiv \frac{Ax+B}{(x^2-1)} + \frac{C}{(x-2)}$$ **3**

ii) Hence evaluate $\int_3^6 \frac{4x-2}{(x^2-1)(x-2)} dx$ **2**

c) The diagram shows the graph of $y = f(x)$



Draw separate $\frac{1}{3}$ page sketches of the following.

i) $|y| = f(x)$ **2**

ii) $y = x.f(x)$ **2**

iii) $y = \sqrt{f(x)}$ **2**

Question 14 (15 marks) Use a SEPARATE writing sheet.

a) The equation $|z+4|+|z-4|=10$ corresponds to an ellipse in the Argand diagram.

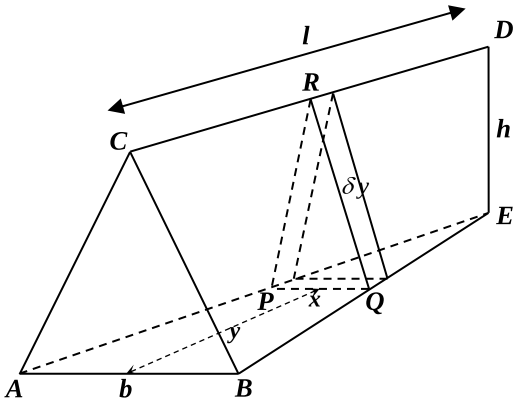
Sketch the ellipse, and state the lengths of the major and minor axes **2**

b) Let $I_n = \int_0^{\pi} x^n \sin x \, dx$, where $n = 0, 1, 2, \ldots$

i) Show that $I_n = \pi^n - n(n-1)I_{n-2}$ for $n = 2, 3, 4, \ldots$ **3**

ii) Hence, evaluate $\int_0^{\pi} x^4 \sin x \, dx$ **2**

c) *ABC* is an isosceles triangle with *AC*=*BC* and *AB*=*b*. *ABCDE* is a wedge shape with height *DE*=*h* and length *CD*=*l*. Triangle *ABC* and line *DE* are perpendicular to the plane of *ABE* as shown in the diagram.



Consider a slice of the wedge height *h* and depth δy as in the diagram. The slice is parallel to the plane *ABC* at *PQR*.

i) Show that the area of the triangle *PQR* can be expressed as $\frac{h}{2}\left(b - \frac{by}{l}\right)$. **2**

ii) Hence calculate the volume of the wedge **3**

d) Find $\int \frac{dx}{1+\sin x}$ **3**

Question 15 (15 marks) Use a SEPARATE writing booklet.

a) Draw a large neat sketch of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ labelling clearly the,
the foci, the directices **3**

b) Find all the possible values of k if $\frac{x^2}{12-k}+\frac{y^2}{k+4}=1$ represents an ellipse **2**

c) $P\left(a\sec\theta, b\tan\theta\right)$ is any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Given that the equation of the normal at P is given by $ax\tan\theta+by\sec\theta=\left(a^2+b^2\right)\sec\theta\tan\theta$.(DO NOT PROVE THIS)
A line through P parallel to the y axis meets an asymptote at Q and the x axis at N.
The normal at P meets the x axis at R.

i) Find the coordinates of Q, N, R. **3**

ii) Show that QR is perpendicular to the asymptote **2**

iii) Show that $OR=e^2ON$ where e is the eccentricity. **2**

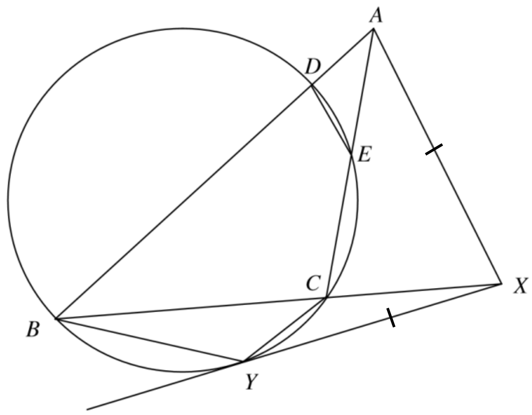
d) In a state swimming championships, 12 swimmers (including the Jones twins) are chosen to represent their club and are divided into three teams of four swimmers to form 3 relay teams.

i) In how may ways can the 3 teams be formed? **2**

ii) Find the number of ways this can be done if the Jones twins (Angela and Bethany) are not to swim in the same relay team. **1**

Question 16 (15 marks) Use a SEPARATE writing booklet.

a) In the diagram XY is a tangent to the circle and $XY = XA$.



i) Show that $\Delta XCY \;|||\; \Delta XBY$ **2**

ii) Hence explain why $\dfrac{XY}{BX} = \dfrac{CX}{XY}$ **1**

iii) Show that $\Delta AXC \;|||\; \Delta AXB$ **3**

iv) Prove that $DE \parallel AX$ **2**

b) Given $a_1, a_2, a_3 \; .... \; a_n$ and $b_1, b_2, b_3 \; .... \; b_n$ are positive real numbers. Where $A_n = a_1 + a_2 + a_3 + ...+ a_n$ and $B_n = b_1 + b_2 + b_3 + ...+ b_n$, are such that $a_1, a_2, a_3 \; .... \; a_n > 0$, $b_1, b_2, b_3 \; .... \; b_n > 0$ and $A_r \le B_r$, for $r = 1, 2, 3, \ldots, n$.

(i) Prove, by mathematical induction for $n = 1, 2, 3, \ldots$, that: **3**

$$\frac{1}{\sqrt{b_n}} B_n + \left(\frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}} \right) B_{n-1} + \left(\frac{1}{\sqrt{b_{n-2}}} - \frac{1}{\sqrt{b_{n-1}}} \right) B_{n-2} + \ldots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}} \right) B_1$$
$$= \sqrt{b_1} + \sqrt{b_2} + \sqrt{b_3} + \ldots + \sqrt{b_n}$$

(ii) Hence given

$$\frac{a_1}{\sqrt{b_1}} + \frac{a_2}{\sqrt{b_2}} + \frac{a_3}{\sqrt{b_3}} + \ldots + \frac{a_n}{\sqrt{b_n}} =$$
$$\frac{1}{\sqrt{b_n}} A_n + \left(\frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}} \right) A_{n-1} + \left(\frac{1}{\sqrt{b_{n-2}}} - \frac{1}{\sqrt{b_{n-1}}} \right) A_{n-2} + \ldots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}} \right) A_1$$

Show that $\displaystyle\sum_{r=1}^{n} \frac{a_r}{\sqrt{b_r}} \le \sum_{r=1}^{n} \sqrt{b_r}$ **1**

c) ΔABC has sides a, b, c. If $a^2 + b^2 + c^2 = ab + bc + ca$, show that ΔABC is equilateral. **3**

End of Exam

Multiple choice.

Q1 C

2 D

3 A

4 $|x+iy-2i| = 2+y$

$|x+(y-2)i| = 2+y$

$x^2+(y-2)^2 = (2+y)^2$

$x^2+y^2-4y+4 = 4+4y+y^2$

$x^2 = 8y-4$

$x^2 = 4(2y-1)$

$\therefore$ B (parabola).

5 $c = 2\sqrt{2}$

foci $(\pm 2\sqrt{2}\times\sqrt{2}, \pm 2\sqrt{2}\times\sqrt{2})$

$= (\pm 4, \pm 4)$

$=$ A.

6. C

7. B

8. A.

9. $2x+2y\frac{dy}{dx}+y+x\frac{dy}{dx}=0$

$(2y+x)\frac{dy}{dx} = -(2x+y)$

B $\quad \frac{dy}{dx} = -\frac{(2x+y)}{2y+x}$

10 A.

Exam Ext 2 Maths	MATHEMATICS	: Question 11

Suggested Solutions	Marker's Comments
Q11 a) $z = 3+2i \qquad w = -1+i$	Most students have answered part a) correctly.

i) $\dfrac{z}{iw}$

$= \dfrac{3+2i}{i(-1+i)}$

$= \dfrac{3+2i}{-i-1} \times \dfrac{-1+i}{-1-i}$

$= \dfrac{-3+3i-2i-2}{(-1)^2 - i^2}$

$= \dfrac{-5+i}{1+1}$

$= -\frac{5}{2} + \frac{1}{2}i$

ii) $\text{Im}(\bar{z}w)$

$= \text{Im}\left[(3-2i) \times (-1+i)\right]$

$= \text{Im}(-3+3i+2i+2)$

$= \text{Im}(-1+5i)$

$= 5$

Suggested Solutions	Marker's Comments
Q11 b) $z = -1 + i$	
i) $\lvert z\rvert = \sqrt{(-1)^2 + 1^2}$ $\arg(z) = \tan^{-1}\left(\frac{1}{-1}\right)$	Some students obtained the wrong $\arg(z)$.
$\lvert z\rvert = \sqrt{2}$ $= \frac{3\pi}{4}$	
$z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$	
ii) $z^4 = \left[\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^4$	
$= (\sqrt{2})^4\left(\cos\frac{3\pi}{4}\times 4 + i\sin\frac{3\pi}{4}\times 4\right)$	
$= 4(\cos 3\pi + i\sin 3\pi)$	
$= 4(\cos\pi + i\sin\pi)$	
iii) $z^{20} = (z^4)^5$	some students didn't go on and evaluate z^{20}.
$= [4(\cos\pi + i\sin\pi)]^5$	
$= 4^5(\cos 5\pi + i\sin 5\pi)$	
$= 1024(\cos\pi + i\sin\pi)$	
$= 1024 \times (-1)$	
$= -1024$	

Exam Ext 2 Maths	MATHEMATICS	: Question 11

Suggested Solutions	Marker's Comments

Q11

c) R $1+i\sqrt{3}$

$|OR| = \sqrt{1^2+3^2}$

$= 2$

$|OP| = |OR|$ (sides of a rhombus)

$\overrightarrow{OQ} = \overrightarrow{OR} + \overrightarrow{OP}$

$= 1+i\sqrt{3}+2$

$= 3+i\sqrt{3}$

$\therefore Q(3+i\sqrt{3})$

Students who understood vector representation did this question successfully.

d) $\int_0^1 \frac{2}{\sqrt{1+3x}}\,dx$

let $u = 1+3x$

$du = 3dx$

$x=1 \quad u=4$

$x=0 \quad u=1$

$= \frac{2}{3}\int_1^4 \frac{du}{\sqrt{u}}$

$= \frac{2}{3}\left[\frac{u^{1/2}}{1/2}\right]_1^4$

$= \frac{4}{3}(\sqrt{4}-\sqrt{1})$

$= \frac{4}{3}$

e) $\int x^2 \ln 2x\,dx$

$u = \ln 2x \quad v' = x^2$

$u' = \frac{1}{x} \quad v = \frac{x^3}{3}$

$= \frac{x^3}{3}\ln 2x - \int \frac{1}{x}\times\frac{x^3}{3}\,dx$

$= \frac{x^3}{3}\ln 2x - \frac{1}{3}\left[\frac{x^3}{3}\right]+C$

$= \frac{x^3}{3}\ln 2x - \frac{x^3}{9}+C$

Exam Ext 2 Maths	MATHEMATICS	: Question 12

Suggested Solutions	Marker's Comments
Q12. a) $P(z) = z^3 - 7z^2 - 25z - 39$	

$P(2+3i) = 0$

Since $2+3i$ is a root, then $2-3i$ is also a root for $P(z)$ (all real coefficients)

Let the roots of $P(z)$ be α, β, γ

$\alpha + \beta + \gamma = 7$

$2+3i + 2-3i + \gamma = 7$

$\gamma = 3$

$\therefore P(z) = (z-(2+3i))(z-(2-3i))(z-3)$

b) $x^4 - px^3 + qx^2 - pqx + 1 = 0$

$\alpha + \beta + \gamma + \delta = p$

$\alpha + \beta + \gamma = p - \delta$

$\alpha + \beta + \delta = p - \gamma$

$\alpha + \gamma + \delta = p - \beta$

$\beta + \gamma + \delta = p - \alpha$

most students could identify roots in terms of p, however there were struggles with expansion after.

$(p-\alpha)(p-\beta)(p-\gamma)(p-\delta)$

$= p^4 - (\Sigma\alpha)p^3 + (\Sigma\alpha\beta)p^2 - (\Sigma\alpha\beta\gamma)p + (\alpha\beta\gamma\delta)$

$= p^4 - p \times p^3 + q \times p^2 - pq \times p + 1$

$= p^4 - p^4 + p^2q - p^2q + 1$

$= 1$

Exam Ext 2 Maths MATHEMATICS : Question 12

Suggested Solutions	Marker's Comments
Q12	students forgot to square to remove surds.

c) $x^3 + qx + r = 0$

let $x = \frac{1}{\alpha^2}$

$\alpha^2 = \frac{1}{x}$

$\alpha = \pm \frac{1}{\sqrt{x}}$

$\left(\pm \frac{1}{\sqrt{x}}\right)^3 + q\left(\pm \frac{1}{\sqrt{x}}\right) + r = 0$

$\pm \frac{1}{x\sqrt{x}} \pm \frac{q}{\sqrt{x}} + r = 0$

$\pm \frac{1}{\sqrt{x}}\left(\frac{1}{x} + q\right) = -r$

$\frac{1}{x}\left(\frac{1}{x} + q\right)^2 = (-r)^2$

$\frac{1}{x}\left(\frac{1+qx}{x}\right)^2 = r^2$

$\frac{(1+qx)^2}{x^3} = r^2$

$1 + 2qx + q^2x^2 = r^2x^3$

$\therefore r^2x^3 - q^2x^2 - 2qx - 1 = 0$

Exam Ext 2 Maths MATHEMATICS : Question 12

Suggested Solutions	Marker's Comments
Q12	
d) i) If ω is a root of $z^3=1$	Some students didn't explain why $\omega - 1 \neq 0$.
$\omega^3 = 1$	
$\omega^3 - 1 = 0$	
$(\omega-1)(\omega^2+\omega+1)=0$	
since ω is a non-real root, $\omega \neq 1$.	
$\therefore \omega^2+\omega+1=0$	
ii) $(1+\omega)^2 = (-\omega^2)^2 = \omega^4 = \omega \times \omega^3 = \omega$	
iii) $(1+\omega)^3 = (-\omega^2)^3 = -\omega^6 = -(\omega^3)^2 = -1^2 = -1$	
iv) $(1+\omega)^{3n} = [(1+\omega)^3]^n = (-1)^n$ from part iii)	students were able to get $(1+\omega)^{3n} = (-1)^n$, however they struggled with the rest of this question.
$(1+\omega)^{3n} = {}^{3n}C_0 + {}^{3n}C_1\omega + {}^{3n}C_2\omega^2 + {}^{3n}C_3\omega^3 + \dots + {}^{3n}C_{3n}\omega^{3n}$	
$(-1)^n = {}^{3n}C_0 + {}^{3n}C_1\omega + {}^{3n}C_2\omega^2 + {}^{3n}C_3 + \dots + {}^{3n}C_{3n}$	
$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$	
$\omega^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$	
$(-1)^n = {}^{3n}C_0 + {}^{3n}C_1\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) + {}^{3n}C_2\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) + {}^{3n}C_3 + \dots + {}^{3n}C_{3n}$	

Exam Ext 2 Maths. MATHEMATICS	: Question 12
Suggested Solutions	Marker's Comments

$$(-1)^n = {}^{3n}C_0 + \left(-\tfrac{1}{2}\right)\left({}^{3n}C_1 + {}^{3n}C_2\right) + \tfrac{\sqrt{3}}{2}i\left({}^{3n}C_1 - {}^{3n}C_2\right)$$

$$+ {}^{3n}C_3 + \dots + {}^{3n}C_{3n}$$

Equate real and imaginary part.

$$* \quad \tfrac{\sqrt{3}}{2}i\left({}^{3n}C_1 - {}^{3n}C_2\right) + \tfrac{\sqrt{3}}{2}i\left({}^{3n}C_4 - {}^{3n}C_5\right) + \dots = 0$$

$$* \quad {}^{3n}C_0 - \tfrac{1}{2}\left({}^{3n}C_1 + {}^{3n}C_2\right) + {}^{3n}C_3 - \tfrac{1}{2}\left({}^{3n}C_4 + {}^{3n}C_5\right)$$

$$+ {}^{3n}C_6 - \dots + {}^{3n}C_{3n} = (-1)^n.$$

Suggested Solutions	Marker's Comments

Q13. a)

i) $\int_0^a f(x)\,dx$ let $u = a - x$

$du = -dx$

$= \int_a^0 f(u) \times (-du)$ $x = a$ $u = 0$

$x = 0$ $u = a$

$= \int_0^a f(u)\,du$

$= \int_0^a f(x)\,dx$

(ii) $I = \int_0^{\frac{\pi}{2}} \dfrac{\cos x - \sin x}{1 + \sin x \cos x}\,dx$

$I = \int_0^{\frac{\pi}{2}} \dfrac{\cos(\frac{\pi}{2} - x) - \sin(\frac{\pi}{2} - x)}{1 + \sin(\frac{\pi}{2} - x)\cos(\frac{\pi}{2} - x)}\,dx$

$= \int_0^{\frac{\pi}{2}} \dfrac{\sin x - \cos x}{1 + \cos x \sin x}\,dx$

$2I = 0$

$I = 0$

Suggested Solutions	Marker's Comments

b) i) $\dfrac{4x-2}{(x^2-1)(x-2)} = \dfrac{Ax+B}{x^2-1} + \dfrac{C}{x-2}$

$(Ax+B)(x-2) + C(x^2-1) = 4x-2$

let $x=2$ $\quad 3C = 6$

$C = 2$

let $x=0$ $\quad -2B - C = -2$

$-2B = 0$

$B = 0$

let $x=1$ $\quad (A+B)\times(-1) + 0 = 2$

$A = 2$

$\dfrac{4x-2}{(x^2-1)(x-2)} = \dfrac{-2x}{x^2-1} + \dfrac{2}{x-2}$

(ii) $\displaystyle\int_3^6 \frac{4x-2}{(x^2-1)(x-2)}\,dx$

$= \displaystyle\int_3^6 \left(\frac{-2x}{x^2-1} + \frac{2}{x-2}\right) dx$

$= \Big[-\ln|x^2-1| + 2\ln|x-2|\Big]_3^6$

$= \left[-\ln 35 - (-\ln 8) + 2\ln 4 - 2\ln 1\right]$

$= -\ln 35 + \ln 8 + \ln 16$

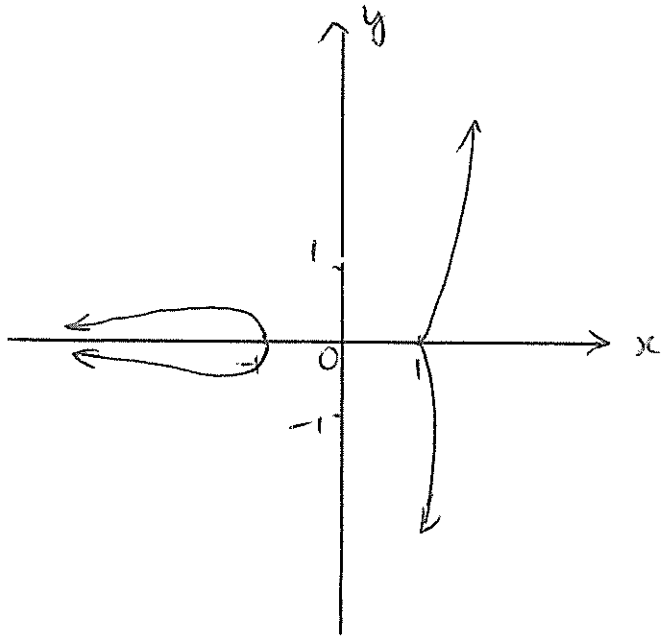
$= \ln\left(\dfrac{16\times 8}{35}\right)$

$= \ln\left(\dfrac{128}{35}\right)$

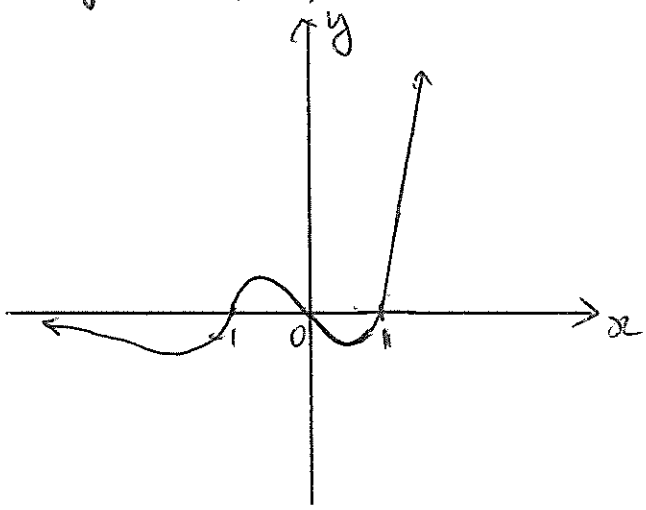
Exam Ext 2 Maths	MATHEMATICS	: Question 13

Suggested Solutions	Marker's Comments

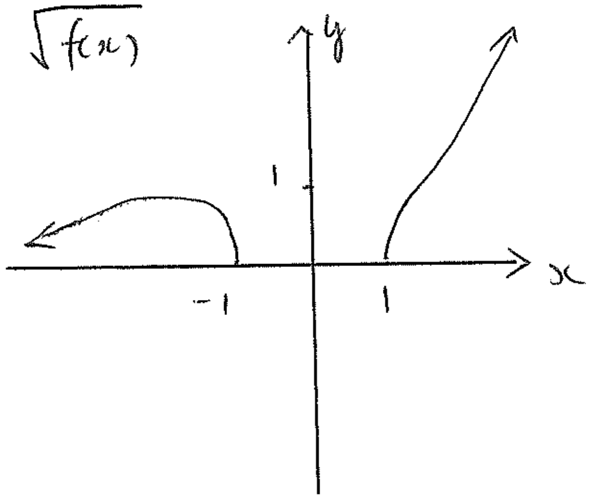
c) i) $|y| = f(x)$



ii) $y = x \cdot f(x)$



iii) $y = \sqrt{f(x)}$

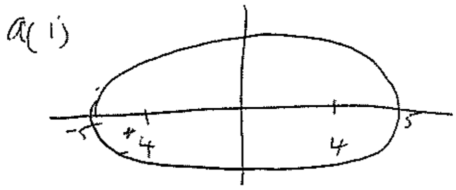


Exam MATHEMATICS : Question 14

Suggested Solutions	Marker's Comments

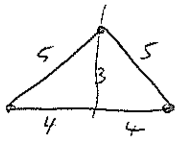
Question 14:

a (i)



$PS + PS' = 2a$.

$\therefore$ Major axis is 10



$\therefore b = 3$.

$\therefore$ minor axis is 6.

Marker's comments: well done. some students gave a semi axis measurement

b. (i) $\int_0^{\pi} x^n \sin x \, dx$.

$u = x^n \quad v' = \sin x$

$u' = nx^{n-1} \quad v = -\cos x$

$I_n = -x^n \cos x + n\int_0^{\pi} x^{n-1} \cos x \, dx$

$u = x^{n-1} \quad v' = \cos x$

$u' = (n-1)x^{n-2} \quad v = \sin x$

$I_n = \left[-x^n \cos x\right]_0^{\pi} + n\left[x^{n-1}\sin x\right]_0^{\pi} - n\int_0^{\pi}(n-1)x^{n-2}\sin x \, dx$

$I_n = \left(-\pi^n \cos\pi - 0\right) + n\left(\pi^{n-1}\sin\pi - 0\right) - n(n-1)I_{n-2}$

$I_n = -\pi^n \times (-1) - n(n-1)I_{n-2}$

$\therefore I_n = \pi^n - n(n-1)I_{n-2}$.

Marker's comments: well done.

(ii) $\int_0^{\pi} x^4 \sin x \, dx$

$I_4 = \pi^4 - 4\times 3\times I_2$

$I_2 = \pi^2 - 2\times 1\times I_0$

$= \pi^2 - 2\times 1\times 2$

$= \pi^2 - 4$

$I_4 = \pi^4 - 12\times(\pi^2 - 4)$

$\boxed{= \pi^4 - 12\pi^2 + 48}$

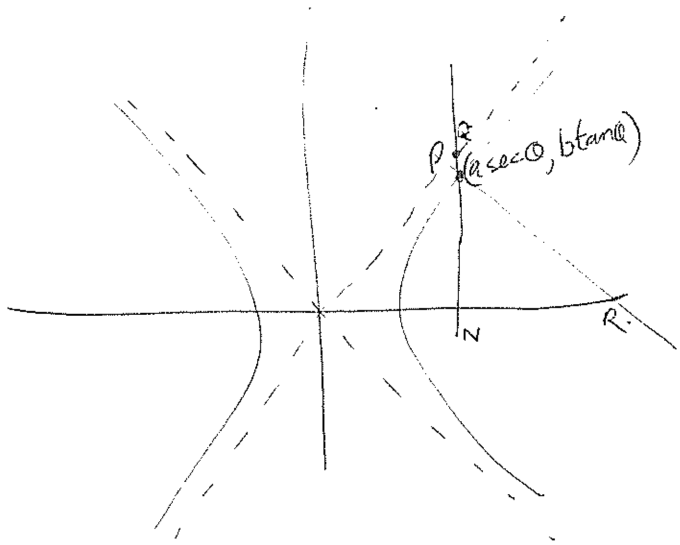
$\boxed{I_0 = \int_0^{\pi} x^0 \sin x \, dx = \left(-\cos x\right]_0^{\pi} = 2.}$

Marker's comments: well done

Exam MATHEMATICS : Question 14

Suggested Solutions | Marker's Comments

(c)



(i) N is $(a\sec\theta, 0)$

asymptote $y = \pm\frac{b}{a}x$

$Q\left(a\sec\theta, \frac{b}{a}a\sec\theta\right)$

$\therefore Q(a\sec\theta, \pm b\sec\theta)$

for R $\quad ax\tan\theta + by\sec\theta = (a^2+b^2)\sec\theta\tan\theta$

when $y=0$. $\quad ax\tan\theta = (a^2+b^2)\sec\theta\tan\theta$

$$x = \frac{(a^2+b^2)\sec\theta}{a}$$

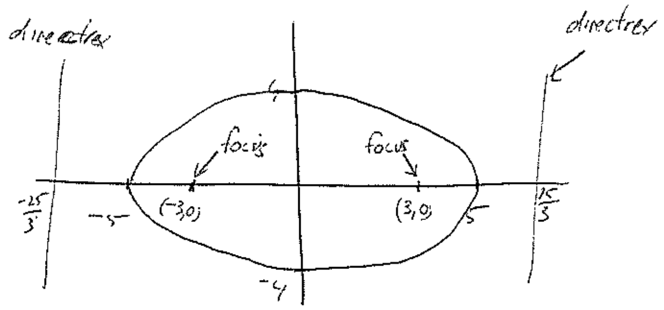
$$R\left(\frac{(a^2+b^2)\sec\theta}{a}, 0\right)$$

students forgot the $\pm$ case. This wasn't penalised

Exam	MATHEMATICS : Question 14

Suggested Solutions	Marker's Comments
(ii) asymptote $y = \pm\frac{b}{a}x$ $m = \pm\frac{b}{a}$. $M_{QR} = \dfrac{b\sec\theta - 0}{a\sec\theta - \frac{(a^2+b^2)\sec\theta}{a}}$ $= \dfrac{ab\sec\theta}{a^2\sec\theta - a^2\sec\theta - b^2\sec\theta}$ $= -\frac{a}{b}$. $M_{QR} = \dfrac{-b\sec\theta}{a\sec\theta - \frac{(a^2+b^2)\sec\theta}{c}}$ $= \frac{a}{b}$ $-\frac{a}{b} \times \frac{b}{a} = -1$ $\quad\quad$ $\frac{a}{b} \times -\frac{b}{a} = -1$ $\therefore$ $QR \perp$ asymptote	$m_1 m_2 = -1$ concept needed to be used at some stage. some students made algebraic errors.
(iii) $OR = \dfrac{(a^2+b^2)\sec\theta}{a}$ $ON = a\sec\theta$. $b^2 = a^2(e^2-1)$ $\frac{b^2}{a^2} + 1 = e^2$ $\frac{b^2+a^2}{a^2} = e^2$ $e^2 ON = \frac{b^2+a^2}{a^2} \times a\sec\theta$ $= \frac{(b^2+a^2)\sec\theta}{a}$ $= OR$	well done.
d) (i) $\dfrac{{}^{12}C_4 \cdot {}^{8}C_4 \cdot {}^{4}C_4}{3!}$ (ii) ${}^{10}C_3 \cdot {}^{7}C_3 \cdot {}^{4}C_4$ $= 4200$.	(ii) some students used correct alternative method from part (i). working was essential

Exam	MATHEMATICS	: Question 15

Suggested Solutions	Marker's Comments
Question 15 a) $\frac{x^2}{25}+\frac{y^2}{16}=1$ foci $(\pm ae, 0)$ Directrices $x=\pm\frac{a}{e}$. $b^2=a^2(1-e^2)$ $16=25(1-e^2)$ $\frac{16}{25}=(1-e^2)$ $e^2=\frac{9}{25}$ $e=\frac{3}{5}$ foci $(\pm 3, 0)$ Directrices $x=\pm\frac{25}{3}$  	well done
b) $12-k>0$ and $k+4>0$ $k<12$ $\quad$ $k>-4$ $-4<k<12$	students used more complicated approaches and forgot $a^2=(12-k)$ etc and ended up with complicated algebraic expressions

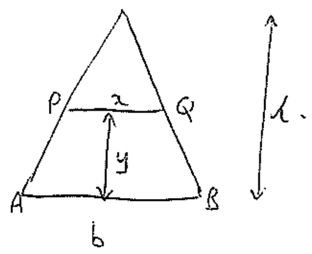
Exam	MATHEMATICS : Question 15	
	Suggested Solutions	Marker's Comments

c (i)

$$A_{\Delta PQR} = \tfrac{1}{2} \times PQ \times h$$

$$= \tfrac{1}{2} x h.$$

$$= \tfrac{1}{2}\left(b - \tfrac{b}{l}y\right)h.$$



$$\frac{x}{b} = \frac{l-y}{l}.$$

$$x = \frac{b(l-y)}{l}.$$

$$x = b - \frac{by}{l}$$

(ii) $V = \int_0^l \frac{1}{2}\left(b - \frac{b}{l}y\right)h \times dy$

$$= \int_0^l \left(\frac{1}{2}bh - \frac{1}{2}\frac{bh}{l}y\right) dy$$

$$= \left[\frac{1}{2}bhy - \frac{bh}{l}\cdot\frac{y^2}{4}\right]_0^l$$

$$= \frac{1}{2}bh \times l - \frac{bh}{4l} \times l^2$$

$$= \frac{1}{2}bhl - \frac{bhl}{4}$$

$$= \frac{1}{4}bhl.$$

Marker's comment: some students lacked understanding and $\int_0^h$ etc. this was penalised

d) $\int \frac{dx}{1+\sin x}\, dx$

Let $t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$2dt = (1+t^2)dx$

$\frac{2dt}{(1+t^2)} = dx$

$$= \int \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2\,dt}{1+t^2}$$

$$= \int \frac{1+t^2}{1+t^2+2t} \times \frac{2\,dt}{1+t^2}$$

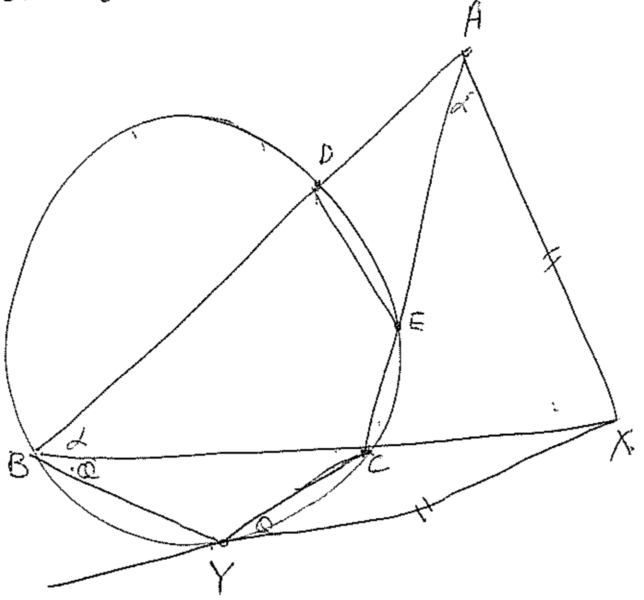
$$= 2\int \frac{dt}{t^2+2t+1}$$

$$= 2\int \frac{dt}{(t+1)^2} = 2\frac{(t+1)^{-1}}{-1} + c = \frac{-2}{t+1} + c = \frac{-2}{\tan\frac{x}{2}+1} + c$$

Marker's comment: well done. students need to remember to convert back to $\tan\frac{x}{2}$

Question 16

a)



(i) In ΔXCY and ΔXBY

$\angle BXY = \angle CXY$ (common)

$\angle CYX = \angle XBY$ (alternate segment theorem)

$\therefore \Delta XCY \parallel\!| \Delta XBY$ (equiangular)

Marker's comments: well done

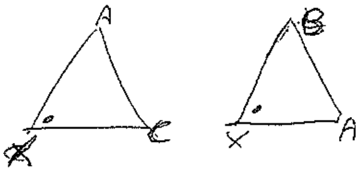
(ii) $\dfrac{XY}{CX} = \dfrac{BY}{CY} = \dfrac{BX}{XY}$ (corresponding sides of similar triangles are in proportion)

$\therefore \dfrac{XY}{BX} = \dfrac{CX}{XY}$

Marker's comments: well done

(iii) In ΔAXC and ΔAXB

$\angle AXC = \angle AXC$ (common)



$XC \cdot XB = XY^2$ (tangent and secant)

$\therefore XC \cdot XB = AX^2$ ($XY = AX$ given)

$\therefore \dfrac{AX}{XB} = \dfrac{XC}{AX}$

$\therefore \Delta AXC \parallel\!| \Delta AXB$ (included angle equal with corresponding sides in equal ratio)

Marker's comments: Some students used different methods but most were successful.

Exam	MATHEMATICS : Question 16	
	Suggested Solutions	Marker's Comments

(IV) DE // AX.

Let $\angle ABX = \alpha$.

$\angle XAC = \angle XBA$ (from part iii – corresponding angles are equal)

$\therefore \angle XAC = \alpha$.

$\angle CED = 180 - \alpha$ (opposite angles in a cyclic quadrilateral are supplementary)

$\therefore \angle AED = \alpha$ (straight angle)

So $\angle XAC = \angle AED$ and since the angles are alternate and equal then DE//AX.

Marker's comments: well done.

b(i) Test $n=1$ $A_1 = a_1$

$B_1 = b_1$

$\frac{1}{\sqrt{b_1}} B_1 = \frac{1}{\sqrt{b_1}} \times b_1$

$= \sqrt{b_1}$

$= R.HS$ $\therefore$ proven true for $n=1$

Assume true for $n=k$

$$\frac{1}{\sqrt{b_k}} B_k + \left(\frac{1}{\sqrt{b_{k-1}}} - \frac{1}{\sqrt{b_k}}\right) B_{k-1} + \dots + \frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}} B_1 = \sqrt{b_1} + \sqrt{b_2} + \dots + \sqrt{b_k}$$

prove true for $n=k+1$

RTP $$\frac{1}{\sqrt{b_{k+1}}} B_{k+1} + \left(\frac{1}{\sqrt{b_k}} - \frac{1}{\sqrt{b_{k+1}}}\right) B_k + \left(\frac{1}{\sqrt{b_{k-1}}} - \frac{1}{\sqrt{b_k}}\right) B_{k-1} + \dots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) B_1 = \sqrt{b_1} + \sqrt{b_2} + \dots \sqrt{b_k} + \sqrt{b_{k+1}}$$

$$\frac{1}{\sqrt{b_{k+1}}} B_{k+1} - \frac{1}{\sqrt{b_{k+1}}} B_k + \overbrace{\frac{1}{\sqrt{b_k}} B_k + \left(\frac{1}{\sqrt{b_{k-1}}} - \frac{1}{\sqrt{b_k}}\right) B_{k-1} + \dots \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) B_1}^{S_k}$$

$\frac{1}{\sqrt{b_{k+1}}} \left[B_{k+1} - B_k\right] + \sqrt{b_1} + \sqrt{b_2} + \dots + \sqrt{b_k}$ (from assumption) now

$B_{k+1} - B_k = (b_1 + b_2 + \dots + b_{k+1}) - (b_1 + b_2 + \dots b_k) = b_{k+1}$

so $\frac{1}{\sqrt{b_{k+1}}} \cdot b_{k+1} + \sqrt{b_1} + \sqrt{b_2} + \dots \sqrt{b_k} = \sqrt{b_1} + \sqrt{b_2} + \dots \sqrt{b_k} + \sqrt{b_{k+1}}$

Marker's comments: students generally tested for $n=1$ and assumed for $n=k$ correctly. for $n=k+1$ unless students saw $B_{k+1} - B_k = b_{k+1}$, they were unsuccessful.

Exam	MATHEMATICS : Question 16	
	Suggested Solutions	Marker's Comments

(ii) $A_r \leq B_r$

$$\frac{1}{\sqrt{b_n}}A_n + \left(\frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}}\right)A_{n-1} + \dots \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}A_1\right)$$

$$< \frac{1}{\sqrt{b_n}}B_n + \left(\frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}}\right)B_{n-1} + \dots \left(\frac{1}{\sqrt{b_1}} + \frac{1}{\sqrt{b_2}}\right)B_1$$

since $A_r < B_r$ for $r = 1, 2, 3, \dots n$

$$\therefore \frac{a_1}{\sqrt{b_1}} + \frac{a_2}{\sqrt{b_2}} + \frac{a_3}{\sqrt{b_3}} + \dots \frac{a_n}{\sqrt{b_n}} \leq \sqrt{b_1} + \sqrt{b_2} + \sqrt{b_3} + \dots \sqrt{b_n}$$

or $\displaystyle\sum_{r=1}^{n} \frac{a_r}{\sqrt{b_r}} \leq \sum_{r=1}^{n} \sqrt{b_r}$

well done

(iii) $a^2 + b^2 - 2ab = (a-b)^2$

$b^2 + c^2 - 2bc = (b-c)^2$

$c^2 + a^2 - 2ca = (c-a)^2$

$\therefore 2\left(a^2+b^2+c^2 - (ab+bc+ca)\right) = (a-b)^2 + (b-c)^2 + (c-a)^2$

$(a-b), (b-c), (c-a)$ are real numbers since a, b, c positive real number

$\therefore (a-b)^2 \geq 0$ with equality if and only if $a = b$.

similarly for $(b-c)^2, (c-a)^2$

Hence if $a^2 + b^2 + c^2 = ab + bc + ca$

then $(a-b)^2 + (b-c)^2 + (c-a)^2 = 0$

$\therefore (a-b)^2 = (b-c)^2 = (c-a)^2 = 0$

$\therefore a = b \quad b = c \quad c = a$

Hence $a = b = c$ and ΔABC is equilateral

students must use $a^2 + b^2 + c^2 = ab + bc + ca$ if ⇒ then and not assume equality and see if it works.