YEAR 12 TRIAL HSC EXAMINATION
2004

MATHEMATICS

EXTENSION 2

Time Allowed: 3 hours
(plus 5 minutes reading time)

INSTRUCTIONS TO CANDIDATES:

- ALL questions should be attempted.
- All questions are of equal value (15 marks).
- Part marks are shown on the right hand side.
- Marks may not be awarded for careless or badly arranged work.
- Standard integrals are provided on the back page.
- Board-approved calculators may be used.
- If you use a second booklet for a question, place it inside the first.

ARB.LJF 19.7.04

QUESTION 1. (START A NEW BOOKLET) *Marks*

(a) Find the exact value $\int_0^{\log_e 2} \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$ **2**

(b) (i) Show that $\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$ for all real values of θ **4**

(ii) Use the above result:

(α) to find in surd form the values of $\cot\frac{\pi}{8}$ and $\cot\frac{\pi}{12}$,

(β) to show without using calculators that:

$$\operatorname{cosec}\frac{4\pi}{15} + \operatorname{cosec}\frac{8\pi}{15} + \operatorname{cosec}\frac{16\pi}{15} + \operatorname{cosec}\frac{32\pi}{15} = 0$$

(c) Find the equation of the tangent to the curve $xy(x+y)+16=0$ at the point on the curve where the gradient is -1. **4**

(d) (i) Sketch (NOT on graph paper) on the same number plane: **3**

$y = |x| - 2$ and $y = 4 + 3x - x^2$

(ii) Hence, or otherwise, solve:

$$\frac{|x|-2}{4+3x-x^2} > 0$$

(e) If $p+q=1$ and $p^2+q^2=2$, determine the values of p^3+q^3 and p^4+q^4 without finding the values of p and q. **2**

QUESTION 2. (START A NEW BOOKLET) *Marks*

Consider the function $f(x) = x - 2\sqrt{x}$

(a) Determine the domain of $f(x)$ **1**

(b) Find the x intercepts of the graph of $y = f(x)$ **1**

(c) Show that the curve $y = f(x)$ is concave upwards for all values of x from the domain. **1**

(d) Find the coordinates of the turning point and determine its nature. **1**

(e) Sketch the graph of $y = f(x)$ clearly showing all essential details. **2**

(f) Hence, sketch on separate diagrams:

(i) $y = |f(x)|$ **1**

(ii) $y = f(x-1)$ **2**

(iii) $y = f(|x|)$ **2**

(iv) $|y| = f(x)$ **2**

(v) $y = \dfrac{1}{f(x)}$ **2**

QUESTION 3. (START A NEW BOOKLET) *Marks*

(a) It is given that $|z|^2 = z + \bar{z}$. Find the locus of the point P, representing z and sketch the locus on an Argand diagram. **2**

(b) (i) Expand $z = (1 + ic)^6$ in powers of c. **6**

(ii) Hence find the five real values of c for which z is real.

(c). Solve the equation $z^5 + 16z = 0$, expressing each solution in the form $z = a + ib$ where a and b are real.

(d) Let $z = (\cos\theta + i\sin\theta),\ -\pi < \theta < \pi$. **7**

(i) Show that $$1 + z = 2\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]$$ and $$1 - z = 2\sin\frac{\theta}{2}\left[\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right]$$

(ii) Hence show that $\dfrac{z-1}{z+1} = i\tan\dfrac{\theta}{2}$ and

sketch the locus of z if $|z| = 1$ and $\left|\dfrac{z-1}{z+1}\right| \le \dfrac{1}{\sqrt{3}}$

(iii) Find z on this locus if $\text{Im}(z)$ is a maximum.

QUESTION 4. (START A NEW BOOKLET) ***Marks***

(a) (i) Show that $(x-3)$ is a factor of the polynomial $4x^3 - 15x^2 + 8x + 3$ **4**

(ii) Given that the equation $x^4 - 5x^3 + 4x^2 + 3x + 9 = 0$ has a root of multiplicity 2, solve the equation completely.

(b) Given that

$$l + m + n = -3$$
$$l^2 + m^2 + n^2 = 29$$
$$lmn = -6$$

form the cubic equation whose roots are *l, m, n.* **3**

(c) Find m and n such that $(x+1)^2$ is a factor of $x^5 + 2x^2 + mx + n$ **2**

(d) Prove that if α, β are the roots of the equation $t^2 - 2t + 2 = 0$, then

$$\frac{(x+\alpha)^n - (x+\beta)^n}{(\alpha-\beta)} = \frac{\sin n\theta}{\sin^n \theta} \quad \text{where } \cot\theta = x+1$$ **6**

QUESTION 5. (START A NEW BOOKLET) *Marks*

(a) Find:

(i) $\int \frac{dx}{x^2 + 8x + 4}$ **3**

(ii) $\int \sqrt{\frac{5-x}{5+x}}\, dx$ **2**

(b) Find the exact value:

(i) $\int_0^1 x \sin^{-1} x \, dx$ **3**

(ii) $\int_1^e \ln x^3 \, dx$ **2**

(c) Given $I_{2n+1} = \int_0^1 x^{2n+1} e^{x^2} \, dx$ where n is a positive integer, show

that $I_{2n+1} = \frac{1}{2}e - nI_{2n-1}$. **5**

Hence, or otherwise, evaluate $\int_0^1 x^2 e^{x^2} \, dx$

QUESTION 6. (START A NEW BOOKLET) ***Marks***

(a) Show that the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point P($a$secθ, $b$tanθ) has equation $bx\sec\theta - ay\tan\theta = ab$, and deduce that the normal there has equation $by\sec\theta + ax\tan\theta = (a^2 + b^2)\sec\theta\tan\theta$. **6**
The tangent and the normal cut the y-axis at A and B respectively. Show that the circle on AB as diameter passes through the foci of the hyperbola. (It is enough to show that the circle passes through one focus and then to use symmetry.)

(b) Show that the condition for the line $y = mx + c$ to be tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 + b^2$. **6**
Hence or otherwise, prove that the pair of tangents from the point (3, 4) to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ are at right angles to one another.

(c) Show that the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ intersect at right angles. **3**

QUESTION 7. (START A NEW BOOKLET) ***Marks***

(a) Show by using slicing method that the volume of a right square pyramid of height H on a base of side b is given by $V = \frac{1}{3}b^2H$. **4**

(b) The region under the curve $y = \cos x,\ 0 \le x \le \frac{\pi}{2}$, is rotated about y-axis. Find the volume formed using the method of cylindrical shells. **3**

(c) The base of a particular solid is the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Find the volume of this solid if every cross section perpendicular to the major axis is an equilateral triangle with one side on the base of the solid. **4**

(d) The region between the curve $y = x^2 + 1$ and $y = 3 - x$ is rotated about the x-axis. By taking slices perpendicular to the x-axis, find the volume of the solid generated. **4**

QUESTION 8. (START A NEW BOOKLET) *Marks*

(a) A particle is projected from the origin with speed V at an angle α to the horizontal. The particle is subject to both gravity and an air resistance proportional to its velocity, so that its respective horizontal and vertical components of acceleration while it is rising are given by: **8**

$$\ddot{x} = -k\dot{x}$$
$$\ddot{y} = -g - k\dot{y}$$

(i) Show that:

(α) $\dot{x} = V\cos\alpha e^{-kt}$

(β) $\dot{y} = \left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k}$

(ii) Hence show that:

(α) $x = \frac{V\cos\alpha}{k}(1 - e^{-kt})$

(β) $y = \left(\frac{g}{k^2} + \frac{V\sin\alpha}{k}\right)(1 - e^{-kt}) - \frac{g}{k}t$

(iii) When the particle reaches its greatest height, show that it has travelled a horizontal distance of $\frac{V^2\sin 2\alpha}{2(g + Vk\sin\alpha)}$.

(b) A particle of mass m is attached to one end A of a light string, and a particle of mass M is attached to the other end B. The string passes over a smooth hook, freely pivoted at a fixed point O, so that the end B with mass M hangs freely at rest while the end A with mass m moves in a horizontal circle with constant speed v. Let OA equal l. Prove that: **7**

(i) OA is inclined at angle $\cos^{-1}\frac{m}{M}$ to the vertical.

(ii) The radius of the circle in which A moves is $\frac{\sqrt{M^2 - m^2}}{M}l$.

(iii) $v^2 = \frac{(M^2 - m^2)g}{Mm}l$

(iv) The pressure on the hook is $\sqrt{2M(M + m)}g$.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$