THE PITTWATER HOUSE SCHOOLS
COLLAROY • SYDNEY • AUSTRALIA

YEAR 12 TRIAL HSC EXAMINATION

2005

MATHEMATICS

EXTENSION 2

Time Allowed: 3 hours
(plus 5 minutes reading time)

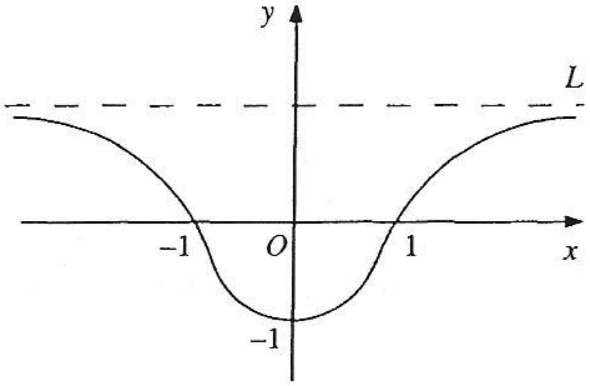
INSTRUCTIONS TO CANDIDATES:

- ALL questions should be attempted.
- All questions are of equal value (15 marks).
- Part marks are shown on the right hand side.
- Marks may not be awarded for careless or badly arranged work.
- Standard integrals are provided on the back page.
- Board-approved calculators may be used.
- If you use a second booklet for a question, place it inside the first.

ARB.LJF 25.7.05

QUESTION 1. (START A NEW BOOKLET) *Marks*

(a)



The diagram shows the graph of $y = f(x)$ where $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$.

(i) Find the equation of the asymptote L. **1**

(ii) On separate diagrams sketch the following graphs, showing any intercepts on the coordinate axes and the equations of any asymptotes: **8**

$y = \{f(x)\}^2$, $y = \sqrt{f(x)}$, $y = \dfrac{1}{f(x)}$ and $y = e^{f(x)}$

(iii) The function $f(x)$ with its domain restricted to $x \geq 0$ has an inverse $f^{-1}(x)$. Find $f^{-1}(x)$ as a function of x. **2**

(b) Consider the function $f(x) = \ln(1 + \cos x),\ -2\pi \leq x \leq 2\pi$ where $x \neq \pi,\ x \neq -\pi$.

(i) Show that the function f is even and the curve $y = f(x)$ is concave down for all values of x in its domain. **2**

(ii) Sketch the graph of the curve $y = f(x)$. **2**

QUESTION 2. (START A NEW BOOKLET) ***Marks***

(a) Find all the complex numbers $z = a + ib$, a, b real, such that $|z|^2 - iz = 16 - 2i$. **3**

(b) z and w are two complex numbers such that $|z| = 4$, $\arg z = \dfrac{5\pi}{6}$, $|w| = 2$, $\arg w = \dfrac{\pi}{3}$.

(i) Express each of z and w in the form $a + ib$, where a and b are real. **2**

(ii) In an Argand diagram the points P and Q represent the complex numbers z and w respectively. Find the distance PQ in simplest exact form. **2**

(c) (i) Express $\sqrt{3} + i$ in modulus / argument form. **1**

(ii) On an Argand diagram sketch the locus of the point P representing the complex number z such that $\left|z - \left(\sqrt{3} + i\right)\right| = 1$, and find the set of possible values of $|z|$ and $\arg z$. **3**

(d) In an Argand diagram the points P, Q and R represent the complex numbers z_1, z_2 and $z_2 + i(z_2 - z_1)$ respectively.

(i) Show that PQR is a right-angled triangle. **2**

(ii) Find in terms of z_1 and z_2 the complex number represented by the point S such that $PQRS$ is a rectangle. **2**

QUESTION 3. (START A NEW BOOKLET) ***Marks***

(a) $P(x) = 16x^4 - 32x^3 + 16x^2 + kx - 5$ where k is an integer. $P(x)$ has two rational roots which are opposites of each other, and two non-real roots.

(i) If α is a non-real root of $P(x)$, show that $\text{Re}(\alpha) = 1$ and $|\alpha| > 1$. **2**

(ii) If the rational roots are $\pm\beta$, deduce that $\beta^2 < \frac{5}{16}$. **1**

(iii) Find the rational roots and the value of k. **3**

(iv) Factorise $P(x)$ completely into irreducible factors over real numbers. **2**

(b) The equation $x^3 + px - 1 = 0$ has three real, non-zero roots α, β, γ.

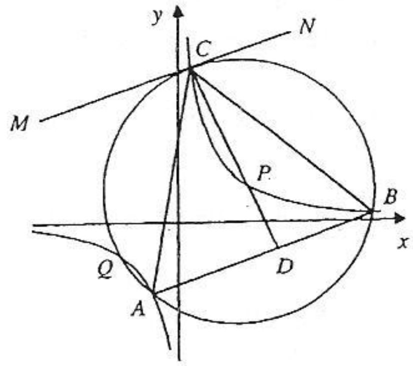
(i) Find the values of $\alpha^2 + \beta^2 + \gamma^2$ and $\alpha^4 + \beta^4 + \gamma^4$ in terms of p, and hence show that p must be strictly negative. **3**

(ii) Find the monic equation, with coefficients in terms of p, whose roots are $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}$. **2**

(c) The equation $x^3 - x^2 - 3x + 2 = 0$ has roots α, β, γ.
Use the value of $\alpha + \beta + \gamma$ to find the monic polynomial equation with roots $2\alpha + \beta + \gamma$, $\alpha + 2\beta + \gamma$, $\alpha + \beta + 2\gamma$. **2**

QUESTION 4. (START A NEW BOOKLET) *Marks*

(a) $P(c\theta, \frac{c}{\theta})$ and $Q(-c\theta, -\frac{c}{\theta})$ where $\theta>0$ and $c > 0$, are two points on the rectangular hyperbola $xy = c^2$. The circle with centre P and radius PQ cuts the hyperbola again at points $A(c\alpha, \frac{c}{\alpha})$, $B(c\beta, \frac{c}{\beta})$ and $C(c\gamma, \frac{c}{\gamma})$. CP produced meets AB at D. MCN is tangent to the circle at C.

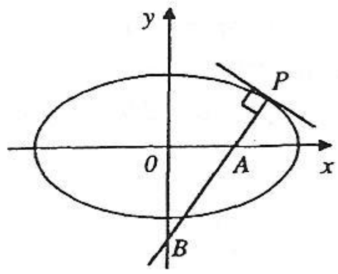


(i) Show that the circle cuts the hyperbola at points $(ct, \frac{c}{t})$ where t satisfies the equation $t^4 - 2t^3\theta - 3t^2\left(\theta^2 + \frac{1}{\theta^2}\right) - \frac{2}{\theta}t + 1 = 0$. Hence deduce that $\alpha\beta\gamma\theta = -1$. **4**

(ii) Show that $CPD \perp AB$. Hence show that $MCN \| AB$. **2**

(iii) Show that $CA = CB$. **3**

(b) $P(a\cos\theta, b\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b > 0$. The normal to the ellipse at P has equation $ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$. This normal cuts the x axis at A and the y axis at B.



(i) Show that ΔOAB has area $\frac{(a^2 - b^2)^2}{2ab}\sin\theta\cos\theta$. **3**

(ii) Find the maximum area of ΔOAB and the coordinates of P when this maximum occurs. **3**

QUESTION 5. (START A NEW BOOKLET) *Marks*

(a) (i) Find $\int \frac{\cos^2 x}{1-\sin x}\,dx$ **1**

(ii) Find $\int \frac{1}{x(x^2+1)}\,dx$ **3**

(b) (i) Use the substitution $u = e^x$ to find $\int \frac{e^x}{\sqrt{e^{2x}+1}}\,dx$. **2**

(ii) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1-\cos x}\,dx$. **3**

(c) (i) If $I_n = \int_0^t \frac{1}{(1+x^2)^n}\,dx$, $n = 1, 2, 3, \ldots$, show that **4**

$$2nI_{n+1} = (2n-1)I_n + \frac{t}{(1+t^2)^n} \quad \text{for } n = 1, 2, 3, \ldots$$

(ii) Hence find the value of I_3 in terms of t. **2**

QUESTION 6. (START A NEW BOOKLET) *Marks*

(a) The circle $x^2 + (y-3)^2 = 1$ is rotated about the line $x = 5$.

(i) Use the method of cylindrical shells to show that the volume generated is given by $4\pi \int_{-1}^{1} (5-x)\sqrt{1-x^2}\ dx$. **3**

(ii) Hence find the volume. **3**

(b) The base of a particular solid is the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Find the volume of this solid if every cross section perpendicular to the major axis is an equilateral triangle with one side on the base of the solid. **5**

(c) The region between the curve $y = x^2 + 1$ and $y = 3 - x$ is rotated about the x axis. By taking slices perpendicular to the x axis, find the volume of the solid generated. **4**

QUESTION 7. (START A NEW BOOKLET) *Marks*

(a) A particle of mass m kilograms is dropped from rest in a medium where the resistance to motion has magnitude $\frac{1}{10}mv^2$ Newtons when the speed of the particle is $v\text{ms}^{-1}$. After t seconds, the particle has fallen x metres, and has velocity $v\text{ms}^{-1}$ and acceleration $a\text{ms}^{-2}$. The particle hits the ground $\ln(1+\sqrt{2})$ seconds after it is dropped. Take $g = 10\text{ms}^{-2}$.

(i) Draw a diagram showing the forces acting on the particle. Deduce that $a = \frac{1}{10}(100 - v^2)$. **2**

(ii) Express v as a function of t. Hence find the speed with which the particle hits the ground, giving the answer in simplest exact form. **4**

(iii) Find in simplest exact form the distance fallen by the particle before it hits the ground. **3**

(b) A particle is projected from a point O with speed V and angle of elevation α. At a certain point P on its trajectory, the direction of motion of the particle and the line OP are inclined (in opposite senses) at equal angles β to the horizontal. Show that:

(i) the time taken to reach P from O is $\frac{4V\sin\alpha}{3g}$ **3**

(ii) $3\tan\beta = \tan\alpha$ **3**

QUESTION 8. (START A NEW BOOKLET) *Marks*

(a) Solve the inequality

$$|2x-3| \le \frac{1}{2}x + 1$$ **3**

(b) Prove by mathematical induction

$12^n > 7^n + 5^n$ for $n \ge 2$ **3**

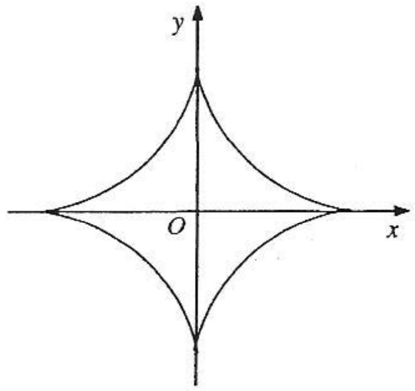
(c) Consider the curve defined by the parametric equations $\left.\begin{array}{l} x = t^2 + t - 1 \\ y = te^{2t} \end{array}\right\}$.

(i) Show that $\dfrac{dy}{dx} = e^{2t}$ **2**

(ii) Hence show that the tangent to the curve at the point on the curve where $t = -1$ passes through the origin. **2**

(d) Use De Moivre's Theorem to show that $(\cot\theta + i)^n + (\cot\theta - i)^n = \dfrac{2\cos n\theta}{\sin^n\theta}$. **2**

(e)



The diagram shows the graph of the relation $|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} = L^{\frac{1}{2}}$ for some $L > 0$.

Show that the area of the region enclosed by the curve is $\dfrac{2}{3}L^2$. **3**