THE PITTWATER HOUSE SCHOOLS
COLLAROY • SYDNEY • AUSTRALIA

YEAR 12 TRIAL HSC EXAMINATION
2006

MATHEMATICS

EXTENSION 2

Time Allowed: 3 hours
(plus 5 minutes reading time)

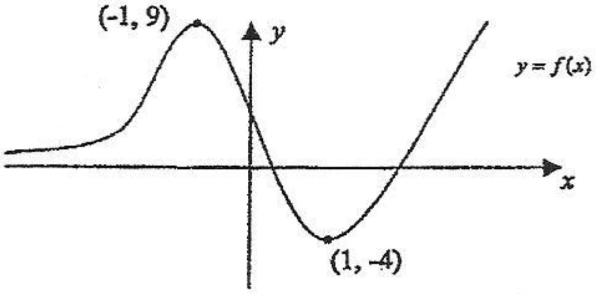
INSTRUCTIONS TO CANDIDATES:

- ALL questions should be attempted.
- All questions are of equal value (15 marks).
- Part marks are shown on the right hand side.
- Marks may not be awarded for careless or badly arranged work.
- Standard integrals are provided on the back page.
- Board-approved calculators may be used.
- If you use a second booklet for a question, place it inside the first.

ARB.LJF 25.7.06

QUESTION 1. (START A NEW BOOKLET) *Marks*

(a) If the curve below represents $y = f(x)$,



make neat sketches, on separate axes, of:

(i) $y = [f(x)]^2$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = f|x|$ **1**

(iv) $y^2 = f(x)$ **2**

(v) $y = f'(x)$ **2**

(b) (i) Draw a neat sketch of $y = \sin^{-1} x$ showing all important features. **2**

(ii) On a separate number plane draw a neat sketch of $y = x\sin^{-1}x$. **3**

(iii) Based solely on the evidence of your graph explain whether the function $y = x\sin^{-1}x$ is odd, even or neither. **1**

QUESTION 2. (START A NEW BOOKLET) *Marks*

(a) Solve equation $z^2-(1-i)z-2i=0$, writing solutions in the form $x+iy$, where x and y are real. **3**

(b) Given that $z=\dfrac{(1+\sqrt{3})+(1-\sqrt{3})i}{2(1+i)}$:

(i) Express z in mod-arg form **2**

(ii) Find all values of n for which $z^n=z$ **2**

(c) Let $A=1+2i,\ B=-3+4i$ and $z=x+iy$.

Draw clearly labelled sketches to show the loci satisfied on the Argand diagram by:

(i) $|z-A|=|B|$ **1**

(ii) $|z-A|=|z-B|$ **1**

(iii) $\arg(z-A)=\dfrac{\pi}{4}$ **1**

(d) (i) Find the five fifth roots of unity. **2**

(ii) If $w=cis\dfrac{2\pi}{5}$, show that $1+w+w^2+w^3+w^4=0$ **1**

(iii) Show that $z_1=w+w^4$ and $z_2=w^2+w^3$ are the roots of the equation $z^2+z-1=0$ **2**

QUESTION 3. (START A NEW BOOKLET) *Marks*

(a) Consider the polynomial equation $x^3 - 3x^2 + x - 5 = 0$ which has roots α, β and δ.

(i) Show that $\alpha + \beta = 3 - \delta$ **1**

(ii) Write down similar expressions for $\alpha + \delta$ and $\beta + \delta$ and hence find a polynomial equation which has the roots $\alpha + \beta$, $\alpha + \delta$ and $\beta + \delta$. **2**

(b) *P(x)* is a polynomial of degree at least 2 such that $P'(a) = 0$. Show that when *P(x)* is divided by $(x - a)^2$ the remainder is *P(a)*. **3**

(c) (i) The equation $x^3 + px^2 + qx + r = 0$ where (p, q and r are non zero) has roots α, β and δ such that $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\delta}$ are consecutive terms in an arithmetic sequence.

Show that $\beta = -\frac{3r}{q}$. **3**

(ii) The equation $x^3 - 26x^2 + 216x - 576 = 0$ has roots α, β and δ such that $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\delta}$ are consecutive terms in an arithmetic sequence. Find the values of α, β and δ. **3**

(d) Show that $(x + 1)$ is a factor of $P(x) = x^3 + 2x^2 + 2x + 1$ and hence factorise *P(x)* over the complex numbers. **3**

QUESTION 4. (START A NEW BOOKLET) ***Marks***

(a) For the ellipse $\frac{y^2}{50}+\frac{x^2}{32}=1$, find:

(i) the eccentricity, **2**

(ii) the coordinates of the foci S and S'. **1**

(b) Normals to the ellipse $4x^2+9y^2=36$ at points $P(3\cos\alpha, 2\sin\alpha)$ and $Q(3\cos\beta, 2\sin\beta)$ are at right angles to each other. Show that:

(i) the gradient of the normal at P is $\frac{3\sin\alpha}{2\cos\alpha}$ **2**

(ii) $4\cot\alpha\cot\beta=-9$ **2**

(c) $P\left(5p, \frac{5}{p}\right)$, $p>0$ and $Q\left(5q, \frac{5}{q}\right)$, $q>0$ are two points on the hyperbola, H, $xy=25$.

(i) Derive the equation of the chord PQ **2**

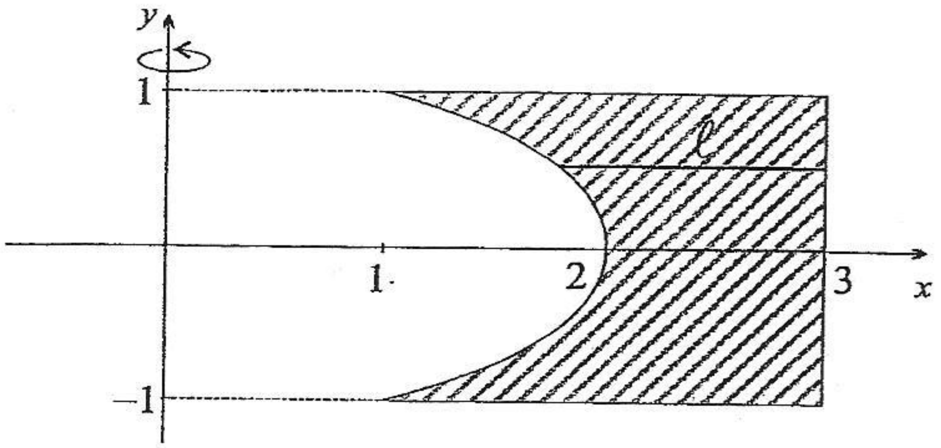
(ii) State the equations of the tangents at P and Q **2**

(iii) If the tangents at P and Q intersect at R, find the co-ordinates of R. **2**

(iv) If the secant PQ passes through the point $S(15, 0)$, find the locus of R. **2**

QUESTION 5. (START A NEW BOOKLET) ***Marks***

(a)



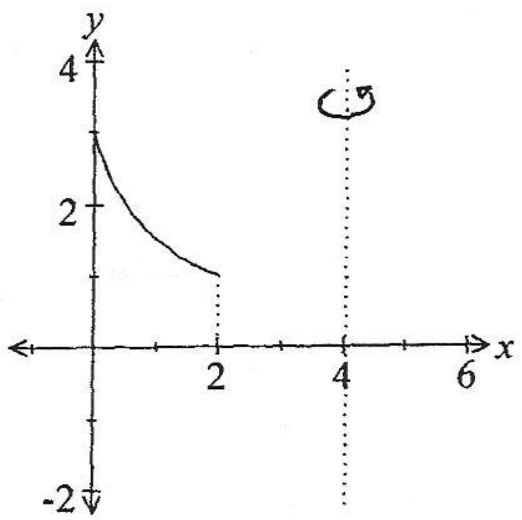
The diagram above shows the region bounded by the curve $x = 2 - y^2$ and the lines $x = 3$, $y = 1$ and $y = -1$. This region is rotated about the y-axis to form a solid. The interval l at height y sweeps out an annulus.

(i) Show that the annulus at height y has area equal to

$\pi\left(5 + 4y^2 - y^4\right)$ **2**

(ii) Find the volume of the solid. **2**

(b) The region bounded by the curve $y = \dfrac{3}{x+1}$, the x-axis and $x = 0$ and $x = 2$ is rotated about the line $x = 4$. Use the method of cylindrical shells to find the volume of the resulting solid. **4**



QUESTION 5. (Continued) ***Marks***

(c) (i) Prove that the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab. **3**

(ii) Find the volume of an elliptical cone of height H, standing on an elliptical base of equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **4**

QUESTION 6. (START A NEW BOOKLET) *Marks*

(a) Find:

(i) $\int x\sec^2(5-3x^2)\ dx$ **2**

(ii) $\int \frac{3x^2+2x+5}{x-2}\ dx$ **2**

(b) Evaluate:

(i) $\int_0^{\frac{1}{\sqrt{2}}} \frac{x^2}{\sqrt{1-x^2}}\ dx$ **3**

(ii) $\int_0^{\ln 2} xe^{-2x}\ dx$ **2**

(c) $I_n = \frac{1}{n!}\int_0^1 x^n e^{-x} dx$ where n non-negative integer **5**

(i) Prove that:

$$I_n = I_{n-1} - \frac{1}{en!}$$

(ii) Hence, evaluate I_4

QUESTION 7. (START A NEW BOOKLET) *Marks*

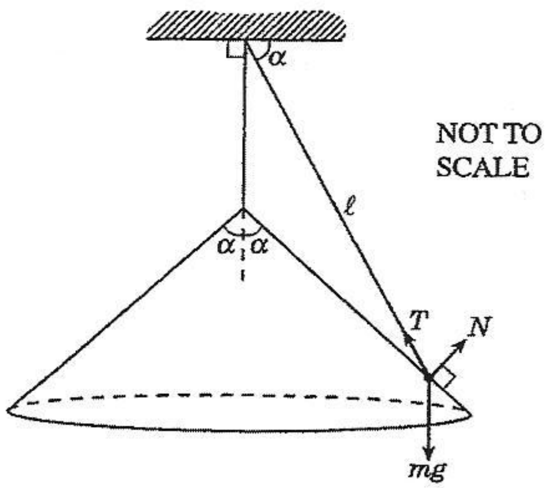
(a) A submarine of mass m is travelling underwater at maximum power. At maximum power, its engines deliver a force F on the submarine.
The water exerts a resistive force proportional to the square of the submarine's speed v.

(i) Explain why $\dfrac{dv}{dt}=\dfrac{1}{m}\left(F-kv^2\right)$, where k is a positive constant. **1**

(ii) The submarine increases its speed from v_1 to v_2. Show that the distance travelled during this period is

$$\frac{m}{2k}\log_e\left(\frac{F-kv_1^{\ 2}}{F-kv_2^{\ 2}}\right)$$ **3**

(b) A particle of mass m is suspended by a string of length l from a point directly above the vertex of a smooth cone, which has a vertical axis. The particle remains in contact with the cone and rotates as a conical pendulum with angular velocity w.



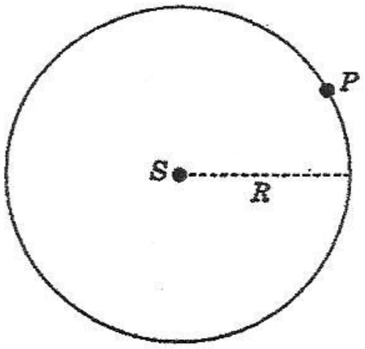
The angle of the cone at its vertex is 2α, where $\alpha>\dfrac{\pi}{4}$ and the string makes an angle of α with the horizontal as shown in the diagram.
The forces acting on the particle are the tension in the string T, the normal reaction to the cone N and the gravitational force mg.

(i) Find expressions for T and N in terms of m, g, α, l and w. **4**

(ii) Find the condition for the particle not to lose contact with the cone. **2**

Marks

(c)



A planet P of mass m kilograms moves in a circular orbit of radius R metres around a star S. Coordinate axes are taken in the plane of the motion, centred at S. The position of the planet at time t seconds is given by the equations

$$x = R\cos\frac{2\pi t}{T} \text{ and } y = R\sin\frac{2\pi t}{T},$$

where T is a constant.

(i) Show that the planet is subject to a force of constant magnitude, F newtons. **3**

(ii) It is known that the magnitude of the gravitational force pulling the planet towards the star is given by

$$F = \frac{GMm}{R^2},$$

where G is a constant and M is the mass of the star S in kilograms.

Find an expression for T in terms of R, M and G. **2**

QUESTION 8. (START A NEW BOOKLET) ***Marks***

(a) The hyperbolic trig functions cosh$x$ and sinh$x$ are defined by the following:

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$$

(i) Show that $\cosh^2 x = 1 + \sinh^2 x$ **2**

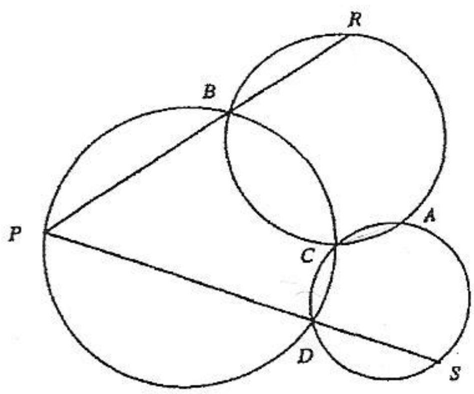
(ii) Show that if $y = \cosh x$ then $\frac{dy}{dx} = \sinh x$ **1**

(b) Given that a, b and c are real numbers:

(i) Show that $2ab \le a^2 + b^2$ **1**

(ii) Hence, deduce that $3(ab + bc + ac) \le (a + b + c)^2$ **2**

(c)



In the diagram above, three circles intersect at a common point C. PBR and PDS are straight lines.

(i) copy the diagram into your booklet.

(ii) Show that R, A, S are collinear points. **4**

(iii) If CA is perpendicular to RAS explain where the centre of the circle through P, B, C, D is located relative to the line PC. **3**

QUESTION 8. (Continued) *Marks*

(d) A modern supercomputer can calculate 1000 billion (i.e. 10^{12}) basic arithmetical operations per second. Use Stirling's formula to estimate how many years such a computer would take to calculate 100! basic arithmetical operations. Stirling's formula states that $n!$ is approximately equal to 2

$$\sqrt{2\pi} \times n^{n+\frac{1}{2}} \times e^{-n}$$

Leave your answer in scientific notation.

END OF PAPER

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$