**PYMBLE LADIES' COLLEGE**

**YEAR 12**

**4 UNIT MATHEMATICS**

**TRIAL HIGHER SCHOOL CERTIFICATE**

**Time Allowed: 3 HOURS**
**(Plus 5 minutes' reading time)**

**AUGUST,1999**

**DIRECTIONS TO CANDIDATES**

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in each question.
- Standard integrals are attached.
- Board approved calculators may be used.
- Start each question on a new page.
- There are eight (8) questions

**<u>QUESTION 1.</u>**

(a) Find $\int \frac{2x}{1+x^4}\,dx$ 1

(b) Find $\int \sin^4 x\,dx$ 3

(c) Find $\int \frac{e^{-2x}}{e^{-x}+1}\,dx$ 3

(d) Given that $\int_1^6 \frac{dx}{\sqrt{x^2+2x}} = \ln(a+\sqrt{b})$, find the values of $a$ and $b$ where $a$ and $b$ are rational. 4

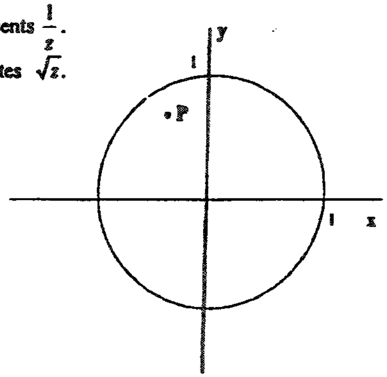
(e) Evaluate $\int_1^{e^{\pi/4}} \cos\log x\,dx$ 4

**)UESTION 2.** Start a new page

1) Point P as shown represents the complex number z. **2**
Copy the diagram onto your paper, and indicate on the diagram

(i) the point Q which represents $\frac{1}{z}$.

(ii) the point R which indicates $\sqrt{z}$.




) Sketch the locus of the point $z$ such that $z\bar{z} \geq 4$ and $2 \leq z + \bar{z} \leq 4$. **2**

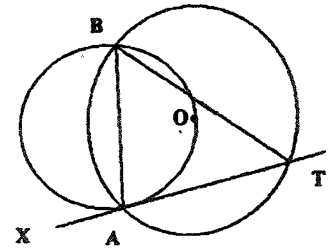
Find the equation of the locus of $z$ such that $\frac{z-3i}{z-2}$ is a real number. **3**
Sketch this locus.

If $z$ is any complex number such that $|z| = 1$, show using an Argand diagram or otherwise that $-\frac{\pi}{6} \leq \arg(z+2) \leq \frac{\pi}{6}$. **4**
Write down the greatest and least values of $|z+2|$.

By using de Moivre's theorem or otherwise find the three cube roots of $27i$, giving your answer in the form $a + ib$. **4**

**QUESTION 3.** Start a new page

(a) **4**




Two circles intersect at A and B as shown, such that the smaller circle passes through the centre O of the larger circle.

The tangent XA to the smaller circle at A cuts the larger circle at T.

Show that $\Delta$ ABT is isosceles.

(b) (i) Solve $x^2 + 6x + 13 = 0$, giving your answers in the form $a+ib$. **6**

(ii) The polynomial $P(x) = x^4 + 2x^3 - 6x^2 - 22x + 65$ has one zero $2+i$.
Factorise $P(x)$ ($\alpha$) as a pair of quadratic factors
($\beta$) as linear factors over the complex field.

(c) The circle $x^2 + y^2 + 2gx + 2fy + h = 0$ cuts the rectangular hyperbola $x = ct$, $y = \frac{c}{t}$ in four distinct points P,Q,R and S. **5**
Show that the parameters at these four points are the roots of the equation

$$c^2t^4 + 2gct^3 + ht^2 + 2fct + c^2 = 0.$$

If the midpoint of the chord joining two of these points is the centre of the circle, show that the midpoint of the chord joining the other two points is the origin.

**QUESTION 4.** Start a new page

(a) An ellipse has equation $\frac{x^2}{9}+\frac{y^2}{5}=1$. 4

(i) Find the eccentricity of the ellipse.

(ii) Show that the chord of contact from any point on the directrix passes through the focus.

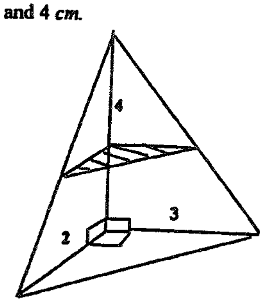
(Use the fact that the equation of a chord of contact to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from the point $(x_0, y_0)$ is $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$)

(b) The curve $f(x)=x^2+b$ $(x \geq 0)$ and its inverse $y=f^{-1}(x)$ are tangential to each other. 3

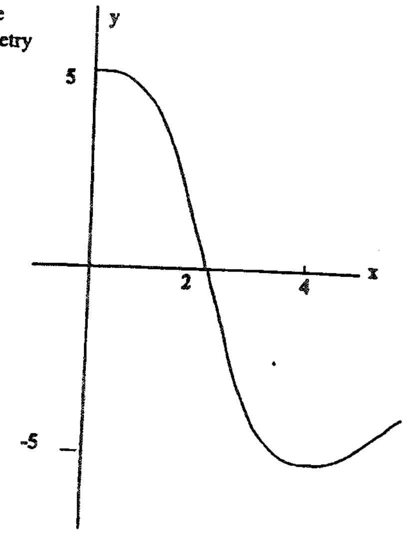
Find the coordinates of the point of contact and the value of $b$ for this to occur.

(c) A triangular pyramid has three mutually perpendicular faces and three mutually perpendicular edges of length 2, 3 and 4 $cm$. 4

By slicing parallel to the base and summing such slices, calculate the volume of the pyramid.




(d) The diagram shows part of the graph of the function $y=f(x)$ which has point symmetry about the point (2,0). 4




Draw a separate sketch over the domain $0 \leq x \leq 4$ for each of the following.

(i) $y=f(x-1)$

(ii) $|y|=f(x)$

(iii) $y=\frac{1}{f(x)}$

(iv) $y=\int_0^x f(x)\,dx$

## QUESTION 5. Start a new page

a) $1$, $\omega$ and $\omega^2$ are the cube roots of 1. **6**

Show that $(1+\omega)$ is a root of the equation $P(z) = z^3 - 3z^2 + 3z - 2 = 0$.

(i) Find the rational root of $P(z) = 0$ and hence find the third root in terms of $\omega$.

(ii) Draw the circle which passes through these three roots and write down its equation in the form $|z - a| = b$.

b) A loop has equation $4y^2 = x(x-3)^2$. **9**

(i) Use implicit differentiation to find the turning points of the curve. (You do not need to test for the nature of the turning points)

(ii) Find where the tangents to the curve at $x = 1$ cross the curve again.

(iii) Sketch the curve, showing its essential features and indicating clearly its behaviour in the vicinity of the origin.

(iv) Find the area of the loop in the form $a\sqrt{3}$.

## QUESTION 6. Start a new page

(a) $\beta$ is the acute angle between the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(a > b)$ with eccentricity $e$. **2**

Show that $e = \sec\frac{\beta}{2}$.

(b) Find the equation of the tangent and the normal to the curve $xy = c^2$ at the point $T(ct, \frac{c}{t})$ $(t \neq 1)$. **8**

This tangent cuts the $x$-axis at X and the $y$-axis at Y. The normal cuts $y = x$ at L and $y = -x$ at M.

Show that LXMY is a rhombus.

(c) The function $f(x)$ is defined as $f(x) = \sin(2x+1) + \sin(2x-3)$ **5**

Find the maximum and minimum values of $f(x)$ over the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, and the values of $x$ for which these occur.

**QUESTION 7.** Start a new page

(a) The equation $x^3+3x-5=0$ has roots $\alpha, \beta,$ and $\gamma$, **2**

Write down the equation with roots $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$.

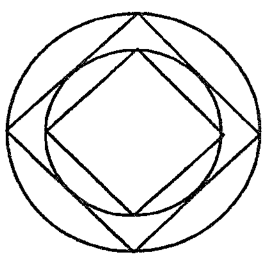
(b) A particle of unit mass moves in a straight line so that at time $t$ its displacement from a fixed origin is $x$ m and its speed is $v$ m/sec. **8**

The acceleration of the particle has magnitude $4-v^2$ m/sec$^2$.

(i) Given that initially $v=0$ and $x=0$ find an expression for $v$ in terms of $t$.

(ii) Find also an expression for $x$ in terms of $v$ and discuss whether the particle again comes to rest, if so where, and if not describe what does happen.

(c) In a circle of radius $d$ cm a square is inscribed. In the square a second circle is inscribed and in this a second square as shown. **5**



The process is repeated until there are $n$ squares.

Find a sequence which gives the perimeters of the squares and write down the perimeter of the $n$th square in simplified form.

Calculate the greatest value of $n$ such that the perimeter is more than 10% of the original radius.

**QUESTION 8.** Start a new page

(a) Write down, without proof, the value of **3**

(i) $\lim_{n\to\infty} \frac{\log n}{n}$

(ii) $\lim_{n\to -\infty} ne^n$

(iii) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

(b) Show that for $n \geq 0$ **5**

$$\tan^{-1}(n+1)-\tan^{-1}(n)=\cot^{-1}(n^2+n+1).$$

Hence find the exact value of the sum

$$\cot^{-1}1+\cot^{-1}3+\cot^{-1}7+\dots\dots+\cot^{-1}31$$

(c) Certain functions may be expressed as an infinite series called Taylor's Series by using the equation **7**

$$f(x)=f(0)+f'(0)x+f''(0)\frac{x^2}{2!}+\dots\dots+f^k(0)\frac{x^k}{k!}+\dots$$

Use this equation to find the Taylor Series for the functions

(i) $f(x)=\sin x$

(ii) $f(x)=\cos x$

(iii) $f(x)=e^{ix}$

Using these series, can you find a relationship between these three given functions?

Write down the value of $e^{i\pi}$.

**END OF PAPER**

## STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Q1.

a) $\int \frac{2x}{1+x^4}\,dx = \tan^{-1}x^2 + C$ [Use $u = x^2$ if nec'y] (1)

b) $\int \sin^4 x\,dx = \int \sin^2 x(1-\cos^2 x)\,dx$

$= \int \sin^2 x\,dx - \int \sin^2 x \cos^2 x\,dx$

$= \int \tfrac{1}{2}(1-\cos 2x)\,dx - \tfrac{1}{8}(1-\cos 4x)\,dx$

$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$ (3)

c) $\int \frac{e^{-2x}}{e^{-x}+1}\,dx$ Let $u = e^{-x}$, $\ln u = -x$, $\frac{1}{u}\,du = -dx$

$= \int \frac{u^2}{u+1}\cdot\frac{-1}{u}\,du$ (3)

$= \int \frac{-u}{u+1}\,du = \int\left(-1+\frac{1}{u+1}\right)du$

$= -u + \ln(u+1) + C$

$= -e^{-x} + \ln(e^{-x}+1) + C$

(d) $\int_1^6 \frac{dx}{\sqrt{x^2+2x}} = \int_1^6 \frac{dx}{\sqrt{(x+1)^2-1}}$

$= \left[\ln\left\{x+1+\sqrt{(x+1)^2-1}\right\}\right]_1^6$ (4)

$= \ln(7+\sqrt{48}) - \ln(2+\sqrt{3})$

$= \ln\left(\frac{7+4\sqrt{3}}{2+\sqrt{3}}\cdot\frac{2-\sqrt{3}}{2-\sqrt{3}}\right)$

$= \ln(2+\sqrt{3}) = \ln(a+\sqrt{b})$

$\therefore a = 2,\ b = 3$

(e) $\int_1 \cos\log x\,dx$ Let $\log x = u$, $x = e^u$, $dx = e^u\,du$, when $x = e^{\frac{\pi}{4}}$, $u =$ when $x = 1$, $u =$

$= \int_0^{\frac{\pi}{4}} \cos u \cdot e^u\,du$

$= \int_0^{\frac{\pi}{4}} \cos u\cdot\frac{d}{du}(e^u)\,du$

$= e^u\cos u\Big]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \sin u\cdot\frac{d}{du}(e^u)\,du$

$= \Big[e^u\cos u + e^u \sin u\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} e^u\cos u\,du$

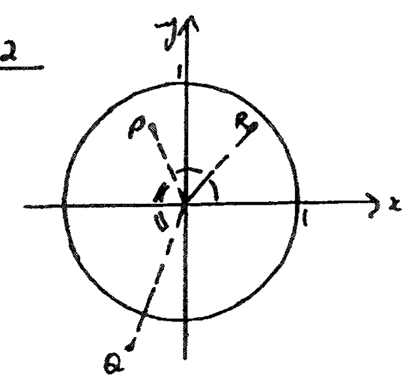
$\therefore 2\int_0^{\frac{\pi}{4}} e^u \cos u\,du = e^{\frac{\pi}{4}}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right) - e^0(1+$

$= e^{\frac{\pi}{4}}(\sqrt{2}) - 1$

$\therefore \int_1^{e^{\frac{\pi}{4}}} \cos\log x\,dx = \frac{1}{2}\left(\sqrt{2}\,e^{\frac{\pi}{4}} - 1\right)$

Q2

(a) (2)



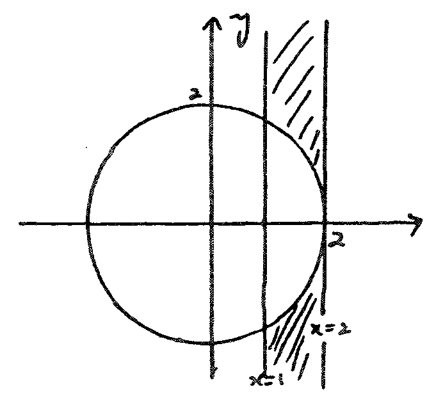

(b) $z\bar{z} \geq 4$

$x^2+y^2 \geq 4$

and $2 \leq z+\bar{z} \leq 4$

$2 \leq 2x \leq 4$

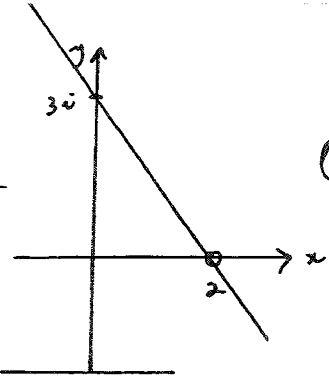
$1 \leq x \leq 2$



(i) $\frac{z-3i}{z-2}$ is real

$\therefore \arg(z-3i)-\arg(z-2)=0$ or $\pi$

$\therefore$ locus of $z$ is $3x+2y=6$ (except $x=2, y=0$)




(3)

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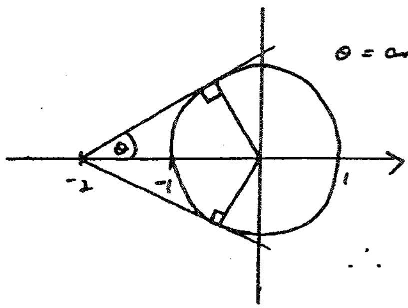
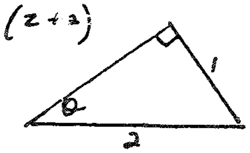



$\theta=\arg(z+2)$




$\sin\theta=\frac{1}{2}\ \therefore\ \theta=\frac{\pi}{6}$

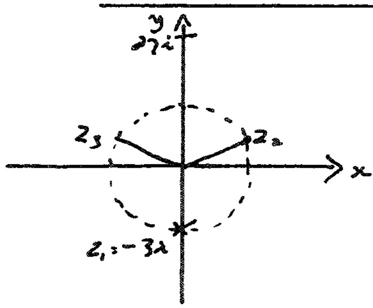
$\therefore\ -\frac{\pi}{6}\le\arg(z+2)\le\frac{\pi}{6}$

(4)

max $|z+2|=3$ when $z=1$

min $|z+2|=1$ when $z=-1$

)




$z^3=27i$

$z_1=-3i$

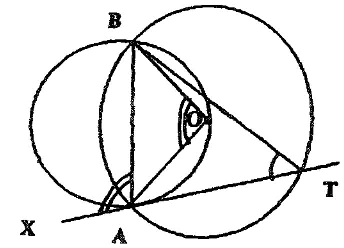
$z_2=3\,\text{cis}\,\frac{\pi}{6}=\frac{3\sqrt{3}}{2}+i\frac{3}{2}$

$z_3=3\,\text{cis}\,\frac{5\pi}{6}=-\frac{3\sqrt{3}}{2}+i\frac{3}{2}$

(4)

## Q3.

(a)




(4)

$\angle XAB=\angle AOB$ ( $\angle$ between tangent + chord $=\angle$ in alt. segment

$\angle AOB=2\angle ATB$ ( $\angle$ at centre $=2\angle$ at circum on arc

in $\Delta ABT$, ext $\angle XAB=\angle ATB+\angle ABT$ (ext $\angle=\Sigma$ int opp $\angle$s

i.e. $\angle XAB=\frac{1}{2}\angle XAB+\angle ABT$

$\therefore \angle ABT=\frac{1}{2}\angle XAB=\angle ATB$

since base $\angle$s of $\Delta ABT$ are equal, $\Delta ABT$ is isosceles

(b)(i) $x^2+6x+13=0$

$x=\frac{-6\pm\sqrt{36-52}}{2}=-3\pm 2i$

(6)

(ii) $P(x)=x^4+2x^3-6x^2-22x+65$

$P(2+i)=0\ \therefore P(2-i)=0$ since real coeffs.

$\therefore (x-2-i)(x-2+i)$ is factor

$x^2-4x+5$ ......

$\therefore P(x)=(x^2-4x+5)(x^2+6x+13)$ by inspection

$=(x-2-i)(x-2+i)(x+3-2i)(x+3+2i)$

(c) $x^2+y^2+2gx+2fy+k=0$ circle centre $(-g,-f)$

Subst $x=ct,\ y=\frac{c}{t}$

$\therefore (ct)^2+\left(\frac{c}{t}\right)^2+2g(ct)+2f\left(\frac{c}{t}\right)+k=0$

$c^2t^4+c^2+2gct^3+2fct+kt^2=0$

$c^2t^4+2gct^3+kt^2+2fct+c^2=0 \quad (A)$

$\therefore$ 4 roots of this eqn are parameters at $P,Q,R,S$.

say $t_1,t_2,t_3,t_4$

From eqn (A) $\sum t_1 = \frac{-2gc}{c^2} = \frac{-2g}{c}$ (5)

but given $\frac{c(t_1+t_2)}{2}=-g$

$t_1+t_2=\frac{-2g}{c}$

$\therefore t_3+t_4=0$

so $c\left(\frac{t_3+t_4}{2}\right)=0$ and $\frac{c}{2}\left(\frac{1}{t_3}+\frac{1}{t_4}\right)=0$

so midpoint of other chord is $(0,0)$

Q4

(a) (i) $\frac{x^2}{9}+\frac{y^2}{5}=1 \quad \therefore a^2=9,\ b^2=5$

$e^2=1-\frac{b^2}{a^2}$

$=1-\frac{5}{9}$

$=\frac{4}{9}$

$\therefore e=\frac{2}{3} \quad (e>0)$ (4)

(ii) Focus $(2,0)$ directrix $x=\frac{9}{2}$

C.C: $\frac{x\cdot\frac{9}{2}}{9}+\frac{yy_0}{5}=1$

$\frac{x}{2}+\frac{yy_0}{5}=1$

Subst $(2,0)$: LHS $=\frac{2}{2}+0=1=$ RHS

$\therefore$ C.C. passes thru focus.

(b) $f(x)=x^2+b \quad (x\geq 0)$

$f(x)$ and $f^{-1}(x)$ ~~at~~ touch on $y=x$ where both have gradient of 1.

so $f'(x)=2x$

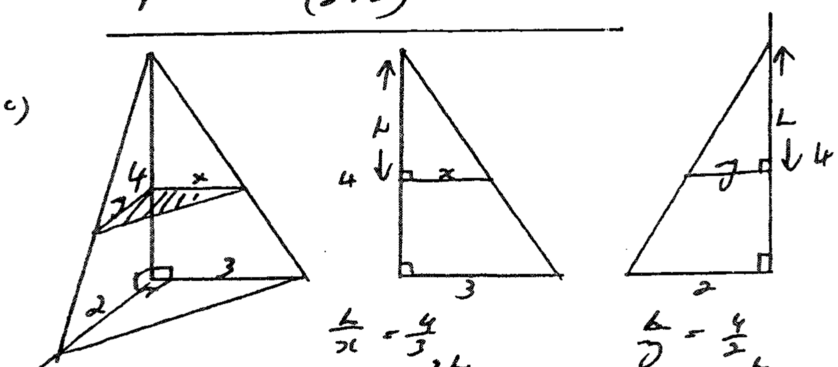
$=1$ when $x=\frac{1}{2},\ y=\frac{1}{2}$

$\therefore f(x)=x^2+b \Rightarrow \frac{1}{2}=\left(\frac{1}{2}\right)^2+b$

$\therefore b=\frac{1}{4}$

Point of contact $\left(\frac{1}{2},\frac{1}{2}\right)$ when $b=\frac{1}{4}$.

(c)




$\frac{h}{x}=\frac{4}{3}$ $\qquad$ $\frac{h}{y}=\frac{4}{2}$

$\therefore x=\frac{3h}{4}$ $\qquad$ $y=\frac{h}{2}$

$\therefore$ Area of slice $=\frac{1}{2}\left(\frac{3h}{4}\right)\left(\frac{h}{2}\right)=\frac{3h^2}{16}$

$\therefore \delta V=\frac{3h^2}{16}\delta h$

$V=\lim_{\delta h\to 0}\sum_{h=0}^{4}\frac{3h^2}{16}\delta h$

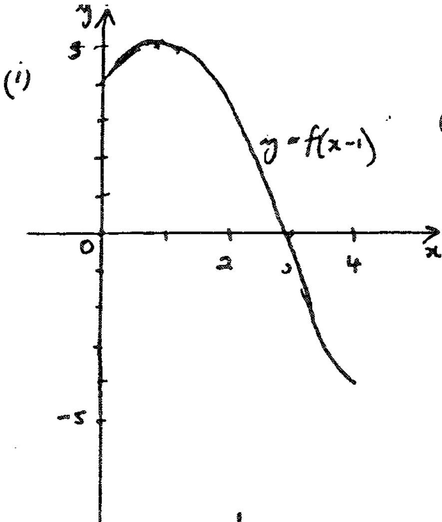
$=\int_0^4 \frac{3h^2}{16}\,dh$

$=\frac{3}{16}\left[\frac{h^3}{3}\right]_0^4$

$=\frac{64}{16}-0=4$

1)

(a)(i)

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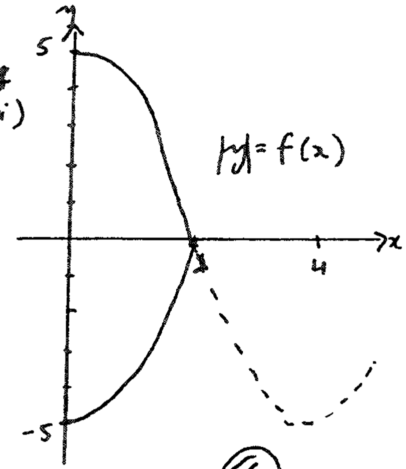
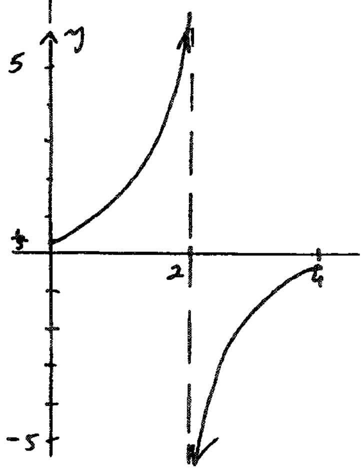
(ii)

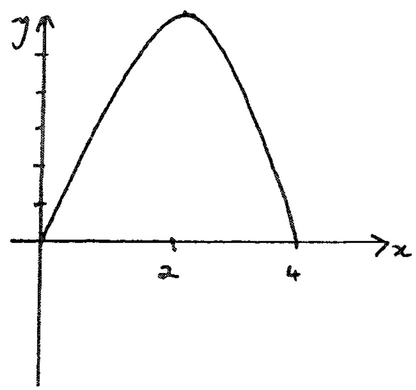



(4)

(iii)




(iv)




Q2

(a) $1, \omega, \omega^2$ cube roots of $1 \Rightarrow 1+\omega+\omega^2 = 0$

and $\omega^3 = 1$

$\therefore 1+\omega = -\omega^2$

$P(z) = z^3 - 3z^2 + 3z - 2$

$P(1+\omega) = (-\omega^2)^3 - 3(-\omega^2)^2 + 3(-\omega^2) - 2$

$= -\omega^6 - 3\omega^4 - 3\omega^2 - 2$

$= -\omega^2(\omega^4 + 3\omega^2 + 3) - 2$

$= -\omega^2(\omega + 3(-\omega)) - 2$

$= +2\omega^3 - 2$

$= 0$

$\therefore P(1+\omega)$ is a root

(6)

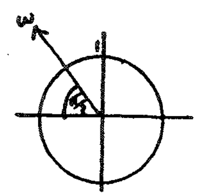
(i) $P(2) = 0 \quad \therefore 2$ is rational root.

~~(iii)~~ $\Sigma$ roots $= 3$

ie $1+\omega+2+\alpha = 3$

$\therefore \alpha = -\omega$ so third root is $-\omega$

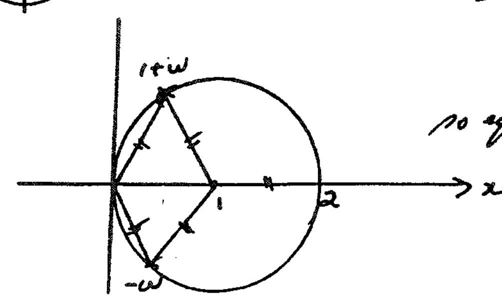
(ii)




say $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

then $1+\omega = \frac{1}{2} + \frac{\sqrt{3}}{2} i$ $\quad |1+\omega| =$

$-\omega = \frac{1}{2} - \frac{\sqrt{3}}{2} i$




so eqn is $|z-1| =$

(i) $4y^2 = x(x-3)^2$ (9)

$$8y\frac{dy}{dx} = x(2)(x-3) + (x-3)^2$$
$$= (x-3)(2x + x - 3)$$
$$\frac{dy}{dx} = \frac{3(x-3)(x-1)}{8y}$$
$$= 0 \text{ when } x = 1 \quad \left(\text{not when } \begin{matrix} x=3 \\ y=0 \end{matrix}\right)$$ (2)

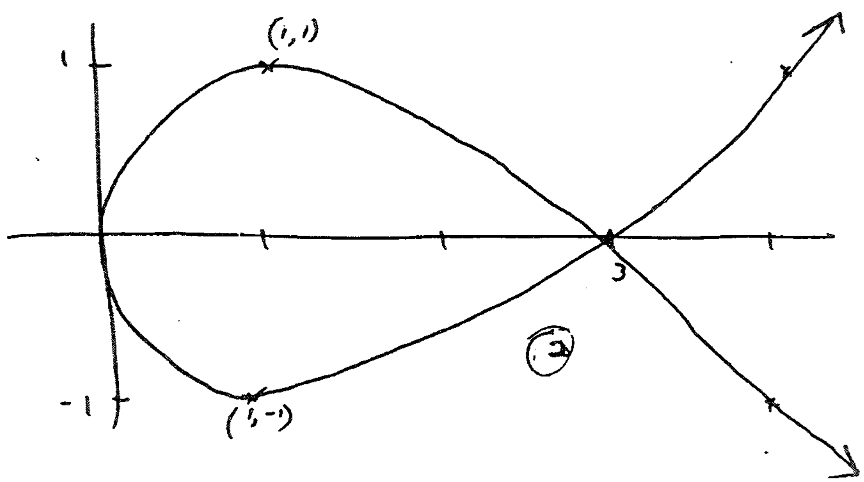
∴ TP's $(1,1)$, $(1,-1)$

Tangents have eqn $y=1$ & $y=-1$

so $x(x-3)^2 = 4$ (1)

$$x^3 - 6x^2 + 9x - 4 = 0$$
$$(x-1)(x^2 - 5x + 4) = 0$$ (2)
$$(x-1)(x-1)(x-4) = 0$$ (1)

cuts again at $(4,1)$ & $(4,-1)$




$4y^2 = x(x-3)^2$

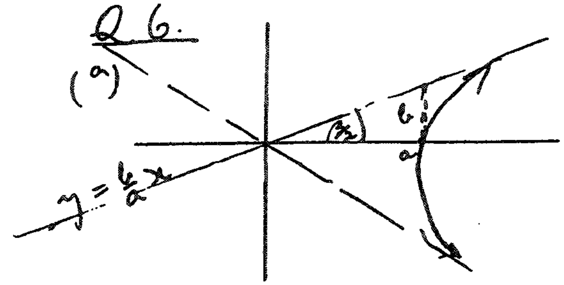
$$2y = \sqrt{x(x-3)^2}$$

for $0 < x < 3$, $\sqrt{(x-3)^2} = 3 - x$

$$\therefore 2y = \sqrt{x}(3-x)$$
$$y = \frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}}$$
$$\therefore \text{Area} = 2\int_0^3 \left(\frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}}\right)dx$$
$$= 2\left[x^{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{5}\right]_0^3$$
$$= 2\left(3\sqrt{3} - \frac{1}{5}\cdot 9\sqrt{3}\right)$$ (2)
$$= \frac{12\sqrt{3}}{5}$$

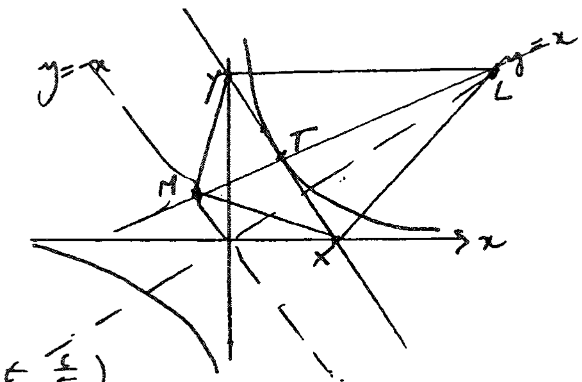
Q 6.

(a)




(2)

$$\tan\frac{\beta}{2} = \frac{b}{a}$$
$$\therefore e^2 = 1 + \frac{b^2}{a^2}$$
$$= 1 + \tan^2\frac{\beta}{2}$$
$$= \sec^2\frac{\beta}{2}$$
$$\text{so } e = \sec\frac{\beta}{2}$$




$T\left(ct, \frac{c}{t}\right)$

$x = ct$ $\qquad$ $y = \frac{c}{t}$

$\frac{dx}{dt} = c$ $\qquad$ $\frac{dy}{dt} = -\frac{c}{t^2}$

$\frac{dy}{dx} = \frac{-c}{t^2} \cdot \frac{1}{c} = \frac{-1}{t^2}$

(8)

$\therefore$ Eqn of tangent $y - \frac{c}{t} = \frac{-1}{t^2}(x - ct)$

$t^2y - ct = -x + ct$

$x + t^2 y = 2ct$

Eqn of normal

$y - \frac{c}{t} = t^2(x - ct)$

$ty - c = t^3(x - ct)$

$t^3x - ty = ct^4 - c$

$X$: when $y=0$, $x = 2ct$ $\qquad X = (2ct, 0)$

$Y$: when $x=0$, $y = \frac{2c}{t}$ $\qquad Y = \left(0, \frac{2c}{t}\right)$

$\therefore$ $t^3x - tx = ct^4 - c$

$x(t^3 - t) = c(t^4 - 1)$

$x = \frac{c(t^2-1)(t^2+1)}{t(t^2-1)} = \frac{c(t^2+1)}{t}$

$\therefore L = \left(\frac{c}{t}(t^2+1), \frac{c}{t}(t^2+1)\right)$

$M$: $t^3x + tx = ct^4 - c$

$x = \frac{c}{t}(t^2-1)$ $\quad \therefore M = \left(\frac{c}{t}(t^2-1), \frac{-c}{t}(t^2-1)\right)$

LM normal
XY tangent $\}$ perpendicular

Midpoint $XY = \left(ct, \frac{c}{t}\right)$

Midpoint $LM = \left[\frac{c}{2t}(t^2+1+t^2-1), \frac{c}{2t}(t^2+1-t^2+1)\right]$

$= \left[\frac{c}{2t}(2t^2), \frac{c}{2t}(2)\right]$

$= \left(ct, \frac{c}{t}\right)$

So since diagonals bisect each other at rt. L.

$\therefore$ LXMY is rhombus

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(c) $f(x) = \sin(2x+1) + \sin(2x-3)$

$= 2\sin\frac{4x-2}{2}\cos\frac{4}{2}$

$= 2\cos 2 \cdot \sin(2x-1)$

(5)

since $\cos 2 < 0$, max $f(x)$ is $-2\cos 2$ when

$2x - 1 = -\frac{\pi}{2}$

$2x = 1 - \frac{\pi}{2}$

$x = \frac{1}{2} - \frac{\pi}{4}$

min $f(x)$ is $2\cos 2$ when $2x - 1 = \frac{\pi}{2}$

$2x = 1 + \frac{\pi}{2}$

$x = \frac{1}{2} + \frac{\pi}{4}$

Q7.

(a) $x^3 + 3x - 5 = 0$ roots $\alpha, \beta, \gamma$

$\therefore \alpha\beta\gamma = 5$

so $\alpha\beta = \frac{5}{\gamma}$ etc.

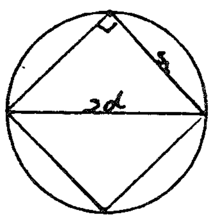
we need $y = \frac{5}{x} \Rightarrow x = \frac{5}{y}$ (2)

$\therefore \left(\frac{5}{y}\right)^3 + 3\left(\frac{5}{y}\right) - 5 = 0$

$125 + 15y^2 - 5y^3 = 0$

$y^3 - 3y^2 - 25 = 0$

(c)



$2s_1^2 = (2d)^2$

$s_1^2 = 2d^2$

so $P_1 = 4s_1 = 4\sqrt{2}d = 2^{\frac{5}{2}}d$

$2s_2^2 = s_1^2$

$s_2^2 = \frac{1}{2}s_1^2 = d^2$

$\therefore P_2 = 4s_2 = 4d = 2^2 d$

$2s_3^2 = s_2^2 = d^2$

$s_3^2 = \frac{1}{2}d^2$

$P_3 = 4s_3 = 4 \cdot \frac{1}{\sqrt{2}}d = 2^{\frac{3}{2}}d$ (5)

$\therefore$ for nth square, $P_n = \left(2^{3-\frac{n}{2}}\right)d$

$2^{3-\frac{n}{2}} > 0.1$

$\left(3 - \frac{n}{2}\right)\ln 2 > \ln 0.1$

$\frac{n}{2} < 3 - \frac{\ln 0.1}{\ln 2}$

$n < 6 - \frac{2\ln 0.1}{\ln 2} < 12.64$

7(b) $\ddot{x} = 4 - v^2$

$\frac{dv}{dt} = 4 - v^2$

$\frac{dt}{dv} = \frac{1}{4-v^2}$ (1)

$\frac{A}{2+v} + \frac{B}{2-v} = \frac{1}{4-v^2}$

$2(A+B) = 1$

$-A + B = 0$

$\therefore A = \frac{1}{4}, B = \frac{1}{4}$

$= \frac{1}{4(2+v)} + \frac{1}{4(2-v)}$ (1)

$t = \frac{1}{4}\left[\ln(2+v) - \ln(2-v)\right] + C$

when $t=0, v=0$ $\therefore 0 = \frac{1}{4}(\ln 2 - \ln 2) + C$

$\therefore C = 0$

$t = \frac{1}{4}\left(\ln\frac{2+v}{2-v}\right)$ (1)

$e^{4t} = \frac{2+v}{2-v}$

$2e^{4t} - ve^{4t} = 2 + v$

$v = 2\left(\frac{e^{4t}-1}{e^{4t}+1}\right)$ (1)

also $v\frac{dv}{dx} = 4 - v^2$ (8)

$\int_0^v \frac{v\,dv}{4-v^2} = \int_0^x dx$ (1)

$\left[-\frac{1}{2}\ln(4-v^2)\right]_0^v = x$ (1)

$2x = \ln 4 - \ln(4-v^2)$ (1)

$= \ln\frac{4}{4-v^2}$ $\quad x = \frac{1}{2}\ln\left(\frac{4}{4-v^2}\right)$

$4e^{-2x} = 4 - v^2$

$\cancel{v^2 = 4 - 4e^{-2x}}$

$v = 0$ when $t = 0$ only. otherwise $v > 0$ (1)

as $t \to \infty$, $x \to \infty$ and $v \to 2$ .. 2

8 (a) (i) $\lim_{n\to\infty} \frac{\log n}{n} = 0$

(ii) $\lim_{n\to -\infty} n e^{n} = 0$ 3

(iii) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e$

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b) $\tan^{-1}(n+1) - \tan^{-1} n = \alpha - \beta$ say

$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$= \frac{n+1-n}{1+n(n+1)}$

$= \frac{1}{1+n+n^2}$

$\therefore \cot(\alpha-\beta) = n^2+n+1$ 2

so $\tan^{-1}(n+1) - \tan^{-1}(n) = \cot^{-1}(n^2+n+1)$

find $\cot^{-1}1 + \cot^{-1}3 + \cdots + \cot^{-1}31$

when $n=0$, $\tan^{-1}1 - \tan^{-1}0 = \cot^{-1}1$

$=1$, $\tan^{-1}2 - \tan^{-1}1 = \cot^{-1}3$

$=5$, $\tan^{-1}6 - \tan^{-1}5 = \cot^{-1}31$

$\cot^{-1}1 + \cot^{-1}3 + \cdots + \cot^{-1}31$ 2

$(\tan^{-1}1 - \tan^{-1}0) + (\tan^{-1}2 - \tan^{-1}1) + \cdots + (\tan^{-1}6 - \tan^{-1}5)$

$\tan^{-1}6 - \tan^{-1}0$

$= \tan^{-1}6$. 1

(i) $f(x) = \sin x$ $\quad f(0) = 0$

$f'(x) = \cos x$ $\quad f'(0) = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$

$f^3(x) = -\cos x$ $\quad f^3(0) = -1$ 2

$f^4(x) = \sin x$ $\quad f^4(x) = 0$

$\therefore \sin x = 1x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

(ii) $f(x) = \cos x$ $\quad f(0) = 1$

$f'(x) = -\sin x$ $\quad f'(0) = 0$ 1

$f''(x) = -\cos x$ $\quad f''(0) = -1$

$f^3(x) = \sin x$ $\quad f^3(0) = 0$

$f^4(x) = +\cos x$ $\quad f^4(0) = 1$ 7

$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

(iii) $f(x) = e^{ix}$ $\quad f(0) = 1$

$f'(x) = i e^{ix}$ $\quad f'(0) = i$ 2

$f''(x) = -e^{ix}$ $\quad f''(0) = -1$

$f^3(x) = -i e^{ix}$ $\quad f^3(0) = -i$

$f^4(x) = e^{ix}$ $\quad f^4(0) = 1$

$\therefore e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \cdots$ 1

$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$

$\underline{e^{ix} = \cos x + i\sin x}$

$e^{i\pi} = \cos\pi + i\sin\pi$ 1