Pymble ladies 2005 Ext. 2 trial

Question 1 (15 marks) Use a separate writing booklet

MARKS

(a) Find $\int \tan 2x \sec 2x \, dx$. **1**

(b) Find $\int \frac{1}{x} \sec^2 (\ln x) \, dx$. **1**

(c) Find $\int \frac{4x - x^2}{(x+1)(x^2+4)} \, dx$. **3**

(d) Find $\int \cos 5x \sin 2x \, dx$. **2**

(e) Evaluate $\int_0^{\frac{\pi}{3}} \frac{1}{1 - \sin x} \, dx$ using the substitution $t = \tan x$. **3**

(f) Find $\int_0^{\frac{\pi}{2}} x \sin x \cos x \, dx$. **3**

(g) Using the result $\int_0^a f(x) \, dx = \int_0^a f(a - x) \, dx$, show that

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} \, dx = \frac{\pi}{4}.$$ **2**

Question 2 (14 marks) Use a separate writing booklet

MARKS

(a) Use the graph of $y = \ln x$ to sketch the graphs of:

(i) $y = \ln(-x)$ **1**

(ii) $y = -\ln x$ **1**

(iii) $y = |\ln x|$ **1**

(b) Use the graphs of $y = x$ and $y = e^{-x}$ to sketch the graph of $y = xe^{-x}$. **2**

(c) Use the graph of $y = x^2 - 1$ to sketch the graph of $y = (x^2 - 1)^2$. **2**

(d) For the function $f(x) = 3x - \frac{x^3}{4}$, use the graph of $y = f(x)$ to sketch the graph of $y^2 = f(x)$. **2**

(e) Use the graphs of $y = 2^u$ and $u = \cos x$ $(0 \le x \le 2\pi)$ to sketch the graph of $y = 2^{\cos x}$ $(0 \le x \le 2\pi)$. **2**

(f) Sketch the graph of $y = \sin 2x$ for $0 \le x \le 2\pi$. Use this graph to solve the inequality $|\sin 2x| \ge \frac{1}{2}$, for $0 \le x \le 2\pi$. **3**

Question 3 (15 marks) Use a separate writing booklet

MARKS

(a) Solve for z where $z \in \mathbb{C}$ — **2**

$$z^2 + 2iz + 2 = 0.$$

(b) Form a quadratic equation whose roots are $4i$ and $3+i$. — **2**

(c) If $w = 1+2i$ and $z = 2-3i$, express in the form $a + ib$.

(i) $w + z$ — **1**

(ii) $w\bar{z}$ — **2**

(iii) $\dfrac{w}{z}$ — **2**

(d) Express $\sqrt{3} - i$ in the form $r(\cos\theta + i\sin\theta)$ and plot on the Argand diagram showing θ, r and the Cartesian coordinates. — **3**

(e) By expanding $(\cos\theta + i\sin\theta)^4$, find expressions for $\cos 4\theta$ and $\sin 4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$. Hence deduce an expression for $\tan 4\theta$ in terms of powers of $\tan\theta$. — **3**

Question 4 (15 marks) Use a separate writing booklet

MARKS

(a) For the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ find:

(i) the eccentricity; **1**

(ii) the coordinates of the foci; **1**

(iii) the equations of the directrices. **1**

(iv) Sketch the ellipse showing essential features. **1**

(b) Find the equation of the tangent to the hyperbola $\frac{x^2}{12}-\frac{y^2}{27}=1$ at the point (4, 3). **2**

(c) A point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. **4**

The line through P perpendicular to the x-axis meets an asymptote at Q and the normal at P meets the x-axis at N.
Show that QN is perpendicular to the asymptote.

(d) The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. **5**

The normal at P meets the hyperbola again at Q.
M is the midpoint of PQ. Find the equation of the locus of M.

Question 5 (15 marks) Use a separate writing booklet

MARKS

(a) Find $P(x)$, given that $P(x)$ is monic, of degree 3, with 5 as a single zero and -2 as a zero of multiplicity 2. **2**

(b) Find the remainder when $P(x)=x^3+2x^2+1$ is divided by $x+i$. **1**

(c) If $P(x)=x^4-2x^3-x^2+6x-6$ has a zero $1-i$, find the zeros of $P(x)$ over $\mathbb{C}$, and factorise $P(x)$ fully over $\mathbb{R}$. **3**

(d) Solve the equation $18x^3+27x^2+x-4=0$, given the roots are in arithmetic progression. **3**

(e) The equation $x^3+x^2-2x-3=0$ has roots α, β and γ. Find the equations with roots:

(i) $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$; **1**

(ii) $\alpha+2$, $\beta+2$ and $\gamma+2$. **1**

(f) The equation $x^3+x^2+2=0$ has roots α, β and γ. Evaluate:

(i) $\alpha^3+\beta^3+\gamma^3$ **2**

(ii) $\alpha^4+\beta^4+\gamma^4$ **2**

Question 6 (15 marks) Use a separate writing booklet

MARKS

(a) If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$ for $n \geq 0$, show that $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$ for $n \geq 2$. Hence evaluate I_6. **5**

(b) By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region, determined by $0 \leq x \leq 2$ and $0 \leq y \leq x^3$, about the line $y = 8$. **5**

(c) (i) Let R be the region in the plane for which $0 \leq x \leq \frac{\pi}{2}$ and $0 \leq y \leq \sin x$. Sketch R. **1**

(ii) A solid is formed by rotating the region R about the y-axis. Use the method of cylindrical shells to find the volume of the solid. **4**

Question 7 (15 marks) Use a separate writing booklet

MARKS

(a) The rise and fall of the tide at Bedrock Harbour may be taken as simple harmonic, the interval between successive high tides being 12½ hours. The harbour entrance has a depth of 15m at high tide and 7m at low tide. **7**
If low tide occurs at 11am on a certain day, find the earliest time thereafter that a ship requiring a minimum depth of 13m of water can pass through the entrance.

(b) Use Mathematical Induction to prove DeMoivre's Theorem ie. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all positive integer values of n. **4**

(c) Let $f(x) = \begin{pmatrix} \dfrac{\sin x}{x} & \text{for} & 0 < x < \dfrac{\pi}{2} \\ 1 & \text{for} & x = 0 \end{pmatrix}$

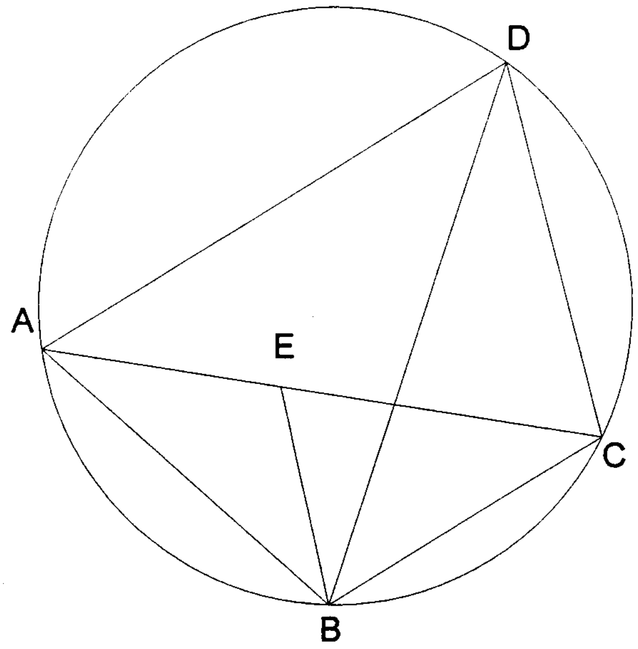
(i) Find the derivative of $f(x)$ for $0 < x < \frac{\pi}{2}$ and prove that $f'(x)$ is negative in this interval. **2**

(ii) Sketch the graph of $y = f(x)$ for $0 < x < \frac{\pi}{2}$ and deduce that $\sin x > \frac{2x}{\pi}$ in this interval. **2**

Question 8 (15 marks) Use a separate writing booklet

MARKS

(a)

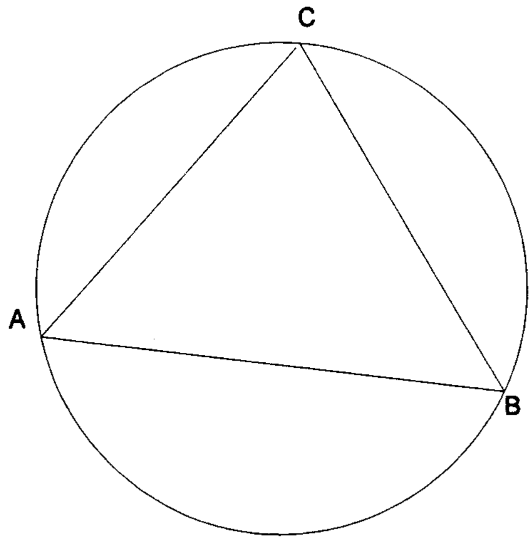


In the diagram $ABCD$ is a cyclic quadrilateral. E is a point on AC such that $\angle ABE = \angle DBC$.

(i) Show that $\Delta ABE \, ||| \, \Delta DBC$ and $\Delta ABD \, ||| \, \Delta EBC$. **2**

(ii) Hence show that $AB.DC + AD.BC = AC.DB$ **2**

(iii)



In the diagram ABC is an equilateral triangle inscribed in a circle.
P is a point on the minor arc AB of the circle. **2**
Use the result in part (ii) to show that $PC = PA + PB$.

Question 8 (continued)

MARKS

(b) (i) Prove that $a^2 + b^2 \geq 2ab$ where a, b are any two real numbers. **2**

(ii) If a, b and c are three real, positive numbers all less than 1, such that $a + b + c > abc$, prove that $a^2 + b^2 + c^2 > abc$. **2**

(c) When a particle is projected vertically upwards from the moon's surface, its distance x from the centre of the moon is given by

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -f\frac{R^2}{x^2}$$

where v is the upward speed, R is the radius of the moon and f is the acceleration due to gravity at the moon's surface and any possible atmospheric resistance is neglected. If v_0 is the speed of projection, show that:

(i) $v^2 = \dfrac{2f\,R^2}{x} + {v_0}^2 - 2f\,R$; **2**

(ii) the maximum height H, above the moon's surface, to which the particle will ascend is given by **2**

$$H = \frac{R\,{v_0}^2}{2f\,R - {v_0}^2}.$$

(iii) Taking $R \approx 1800\ km$, $f \approx 1.6\ ms^{-2}$, estimate the escape velocity of the particle from the moon in kms^{-1}. **1**