Mrs Collett
Ms Lau

Name: ....

Teacher: ....

PYMBLE LADIES' COLLEGE
ALL' ULTIMO LAVORO
SYDNEY

2011

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 120

- Attempt Questions 1–8
- All questions are of equal value

Mark		**/120**
Rank		**/18**
Highest Mark		**/120**

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Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) If $z = 3 + 2i$ and $\omega = 1 + i$, find in the form $a + ib$ where a and b are real

(i) $2z - i\omega$. **1**

(ii) $z\overline{w}$. **1**

(iii) $\frac{3}{\omega}$. **1**

(b) On the Argand diagram, sketch the locus of z described by the inequality

$|z - 2 - 3i| \geq |z + i|$. **2**

(c) Let $\alpha = -\sqrt{3} + i$.

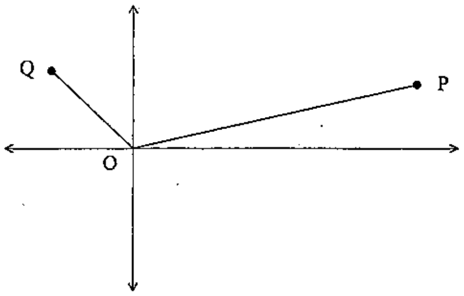
(i) Express α in modulus-argument form. **2**

(ii) Show that α is a root of the equation $z^6 + 64 = 0$. **1**

(iii) Hence, find a real quadratic factor of the polynomial $P(z) = z^6 + 64$. **2**

Question 1 continues on page 3

(d) The diagram shows a complex plane with origin O.



Points P and Q represent non-zero complex numbers z and w respectively.

(i) Write down the length of PQ in terms of z and w. **1**

(ii) Copy the diagram into your booklet. Construct point R that represents $z+w$. **2**

What type of quadrilateral is OPQR?

(iii) Prove that if $|z+w|=|z-w|$, the complex number $\frac{w}{z}$ is imaginary. **2**

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Find the value of a and b such that **2**

$$\frac{1}{(x-1)(2x+3)}=\frac{a}{x-1}+\frac{b}{2x+3}.$$

(ii) Hence find $\int \frac{dx}{(x-1)(2x+3)}$. **2**

(b) Use the substitution $t=\tan x$ to find $\int \operatorname{cosec} 2x \, dx$. **3**

(c) Evaluate $\int_{\frac{-1}{2}}^{0} \frac{dx}{2+4x+4x^2}$. **2**

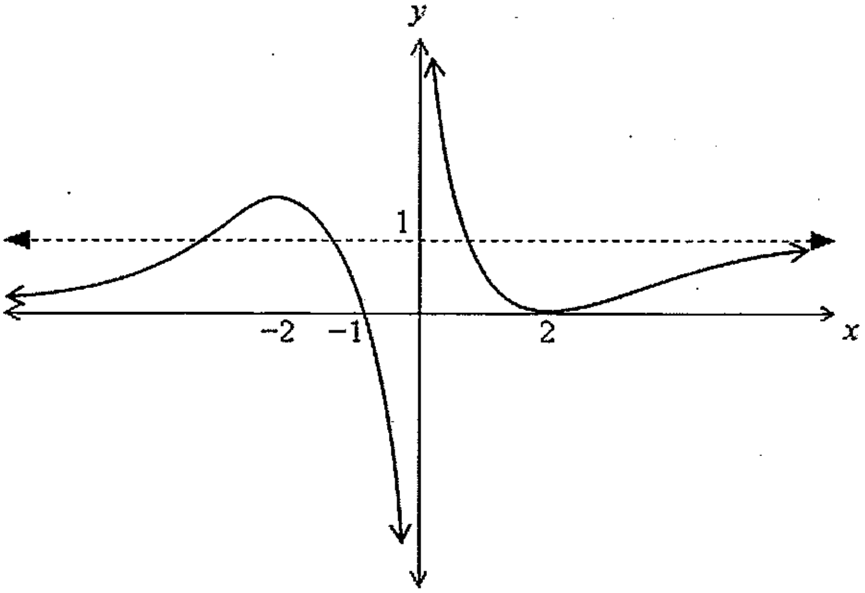
(d) Evaluate $\int x(3^x)\, dx$. **3**

(e) Use the substitution $x=u^6$ to find $\int \frac{dx}{x^{\frac{1}{2}}-x^{\frac{1}{3}}}$. **3**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The diagram below is a sketch of the function $y = f(x)$.
The lines $x = 0$, $y = 0$ and $y = 1$ are asymptotes.



Using the answer sheets provided, sketch each of the graphs below.

In each case, clearly label any maxima or minima, intercepts and the equations of any asymptotes.

(i) $y = f(|x|)$. **1**

(ii) $y = \frac{1}{f(x)}$. **2**

(iii) $y^2 = f(x)$. **2**

(iv) $y = [f(x)]^2$. **2**

(v) $y = \sin^{-1}[f(x)]$. **2**

Question 3 continues on page 6

(b) Consider the curve $y = \frac{x^3 + 4}{x^2}$.

(i) Find the coordinates of the stationary point and show that this curve is always concave up. **2**

(ii) Find the equations of any asymptotes. **1**

(iii) Sketch the curve. **2**

(iv) Find the values of k for which the equation $x^3 - kx^2 + 4 = 0$ has 3 distinct real roots. **1**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Given that $(x+1)$ is a factor of the polynomial $P(x)=x^3+2x^2+2x+1$, factorise $P(x)$ over the field of complex numbers. **2**

(b) The polynomial equation $x^3-3x^2+5x-1=0$ has roots α, β and γ.

(i) Find the value of $\alpha^2+\beta^2+\gamma^2$. **2**

(ii) Hence explain why only one root of the equation is real. **1**

(c) ω and ω^2 are the two complex cube roots of unity. **3**

If ω and ω^2 are also the roots of the equation $x^3+px^2+qx+r=0$, show that $p=q$.

(d) Given $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and using the substitution $x=\cos\theta$,

(i) Solve $8x^3-6x+1=0$. **2**

(ii) Hence prove that $\sec\frac{2\pi}{9}+\sec\frac{4\pi}{9}+\sec\frac{8\pi}{9}=6$. **2**

(e) Let the roots of $x^3-x-1=0$ be α, β and γ. **3**

Find the polynomial whose roots are $\frac{1+\alpha}{1-\alpha}$, $\frac{1+\beta}{1-\beta}$ and $\frac{1+\gamma}{1-\gamma}$.

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Consider the ellipse $\frac{x^2}{9}+\frac{y^2}{25}=1$

(i) Find the coordinates of the foci and the equations of the directrices of the ellipse. **2**

(ii) Sketch the ellipse, showing all key features, including intercepts. **2**

(b) $P\left(2p,\frac{2}{p}\right)$ and $Q\left(2q,\frac{2}{q}\right)$ are points on the rectangular hyperbola $xy=4$.

P and Q move on the hyperbola so that PQ always passes through $(6,4)$.

(i) Show that $pq=\frac{p+q-3}{2}$. **3**

(ii) If M is the midpoint of PQ, find the equation of the locus of M. **2**

(c) The tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b>0)$ at $P(a\cos\theta, b\sin\theta)$ passes through a focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with eccentricity e.

(i) Show that the tangent to the ellipse at P has equation $bx\cos\theta+ay\sin\theta=ab$. **2**

(ii) Show that P lies on the directrix of the hyperbola. **2**

(iii) Find the possible values of the gradient of the tangent at P. **2**

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) On the Argand diagram $P(z)$ is a point in the first quadrant of the circle $|z|=3$.

If $arg(z)=\theta$, find in terms of θ, expressions for:

(i) $\arg z^4$ **1**

(ii) $\arg(z-3)$. **2**

(b) A stone is projected from a point on the ground and it just clears a fence d metres away. The height of the fence is h metres. The angle of projection to the horizontal is θ and the speed of projection is v m/s. The displacement equations, measured from the point of projection are

$$x = vt\cos\theta \text{ and } y = \frac{-1}{2}gt^2 + vt\sin\theta.$$

(i) Show that $v^2 = \dfrac{gd^2\sec^2\theta}{2(d\tan\theta - h)}$. **2**

(ii) Show that the maximum height reached by the stone is **3**

$$\frac{d^2\tan^2\theta}{4(d\tan\theta - h)}.$$

(iii) Show that the stone will just clear the fence at its highest point if **3**

$$\tan\theta = \frac{2h}{d}.$$

(c) (i) If $I_n = \int_0^{\frac{1}{2}} \frac{x^n}{1-x^2}\,dx$ for $n = 0,1,2,3,.......$ show that **2**

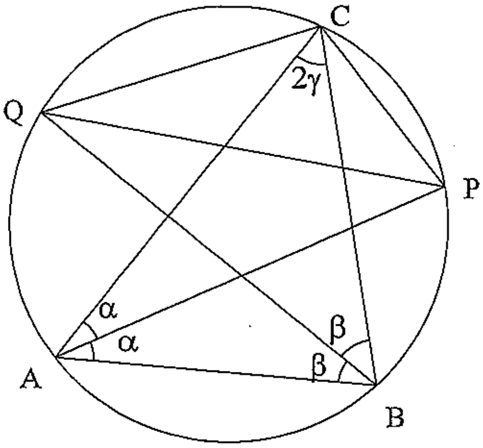
$$I_{n-2} - I_n = \frac{1}{(n-1)2^{n-1}} \quad \text{for } n = 2,3,4,............$$

(ii) Given that $\int_0^{\frac{1}{2}} \frac{1}{1-x^2}\,dx = \frac{1}{2}\log_e 3$, find the exact value of **2**

$$\int_0^{\frac{1}{2}} \frac{x^2}{1-x^2}\,dx.$$

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



In the diagram above, AB is a **fixed** chord of a circle and C is a **variable** point on the major arc AB.

The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the circle again at P and Q respectively.

Let $\angle CAB = 2\alpha$, $\angle ABC = 2\beta$ and $\angle BCA = 2\gamma$.

(i) Show that $\angle PCQ = \alpha + \beta + 2\gamma$. **1**

(ii) Hence explain why the distance PQ is constant. **2**

(iii) Use the sine rule to show that $\dfrac{AB}{PQ} = 2\sin\gamma$. **2**

(b) (i) Use DeMoivre's Theorem to show that when n is a positive integer, **2**

$$(1+i\tan\theta)^n + (1-i\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta} \quad (\cos\theta \neq 0\ ).$$

(ii) Hence show that for the equation $(1+z)^4 + (1-z)^4 = 0$ (where $\text{Re}(z) = 0$) the roots are **2**

$$z = \pm\, i\tan\frac{\pi}{8} \text{ and } z = \pm\, i\tan\frac{3\pi}{8}.$$

Question 7 continues on page 11

(c) (i) Given that $\sin x \geq \frac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$, explain why

$\int_0^{\frac{\pi}{2}} e^{-\sin x} dx < \int_0^{\frac{\pi}{2}} e^{\frac{-2x}{\pi}} dx$. **2**

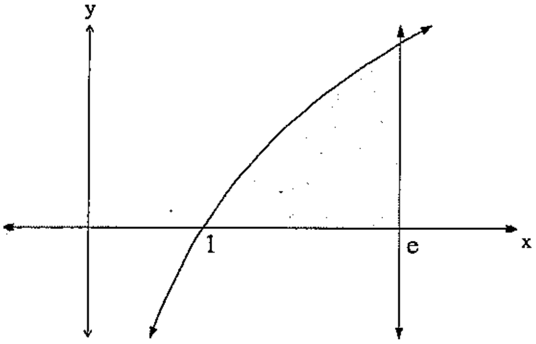
(ii) Show that $\int_{\frac{\pi}{2}}^{\pi} e^{-\sin x} dx = \int_0^{\frac{\pi}{2}} e^{-\sin x} dx$. **2**

(iii) Hence, show that $\int_0^{\pi} e^{-\sin x} dx < \frac{\pi}{e}(e-1)$. **2**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The diagram below shows the area bounded by the curve $y = \log_e x$, the x-axis and the line $x = e$.



This area is rotated about the y-axis to form a solid. By considering slices perpendicular to the y-axis, find the volume of the solid of revolution formed. **2**

(b) (i) Show that $\tan^{-1}\left(\dfrac{x}{x+1}\right) + \tan^{-1}\left(\dfrac{1}{2x+1}\right)$ is a constant for $2x+1>0$. **2**

(ii) Hence, find the exact value of the constant. **1**

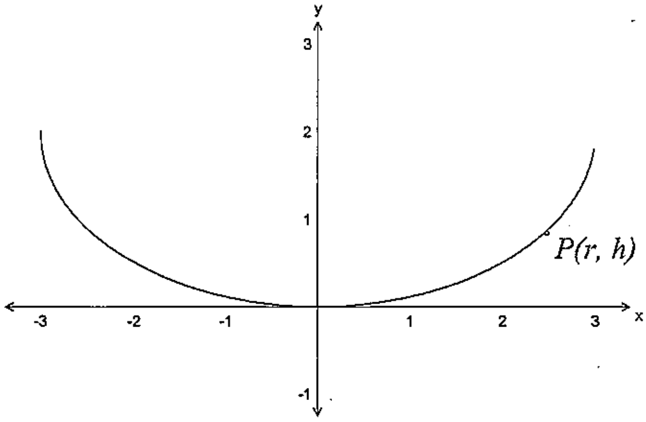
(c) (i) Prove that $\dfrac{\cos\theta - \cos(\theta + 2\alpha)}{2\sin\alpha} = \sin(\theta+\alpha)$. **2**

(ii) Hence use mathematical induction to prove that **3**

$$\sin\theta + \sin 3\theta + \sin 5\theta + \ldots\ldots + \sin(2n-1)\theta = \frac{1-\cos 2n\theta}{2\sin\theta}.$$

Question 8 continues on page 13

(d) The semi-ellipse given by $\frac{x^2}{9}+\frac{(y-2)^2}{4}=1$ where $0 \le y \le 2$ is shown below:



The point (r,h) lies on the ellipse where $r>0$ and $0<h<2$.
The tangent at P makes an angle α with the positive direction of the x-axis.

(i) Show that $\tan\alpha=\frac{4r}{9(2-h)}$. **2**

(ii) Hence, show that $\tan\alpha=\frac{2\sqrt{4-(2-h)^2}}{3(2-h)}$. **2**

(iii) Show that the acute angle between the normal at the point P and the vertical line $x=r$ is equal to the angle between the tangent at P and the positive direction of the x-axis. **1**

END OF PAPER

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

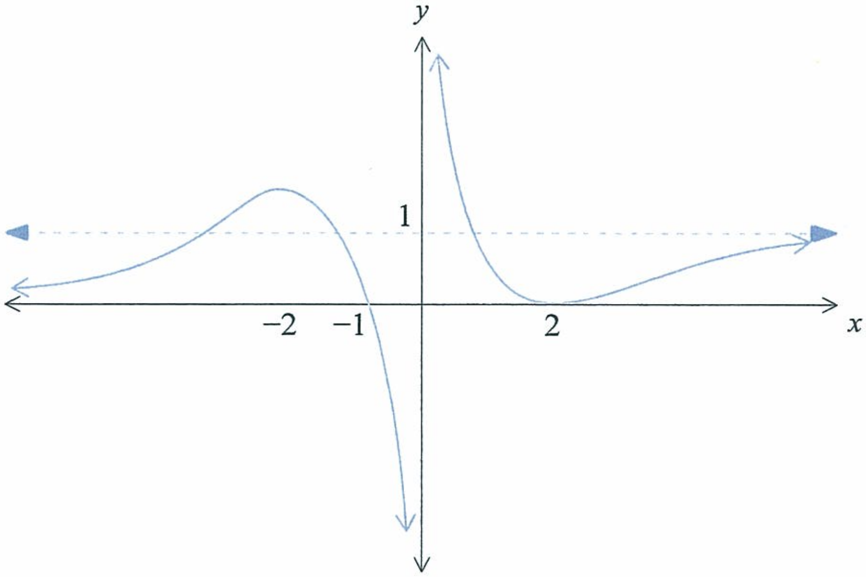
$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

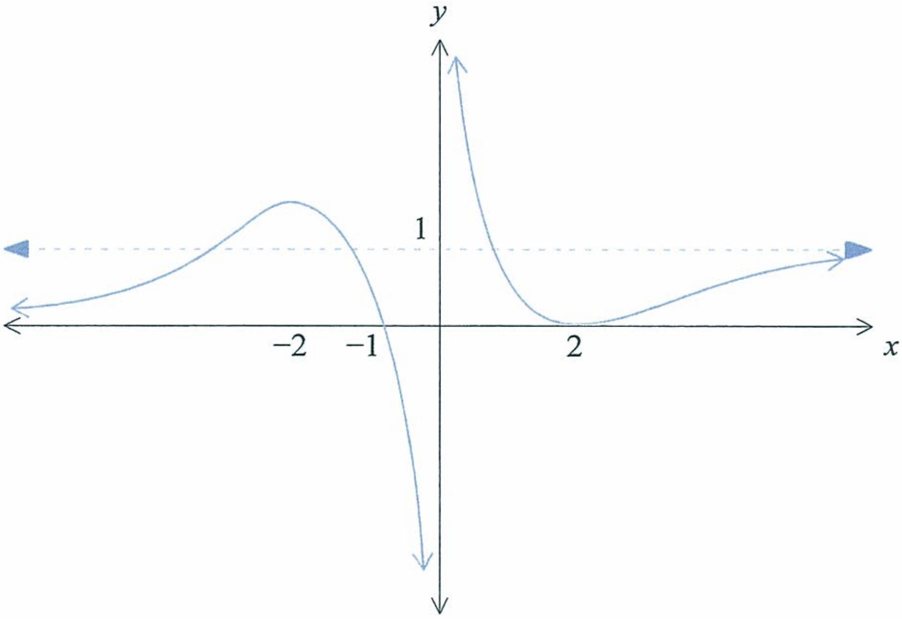
Answer Sheets For Question 3(a) Name:.......................................

(page 1) Teacher: ....

(i) $y = f(|x|)$



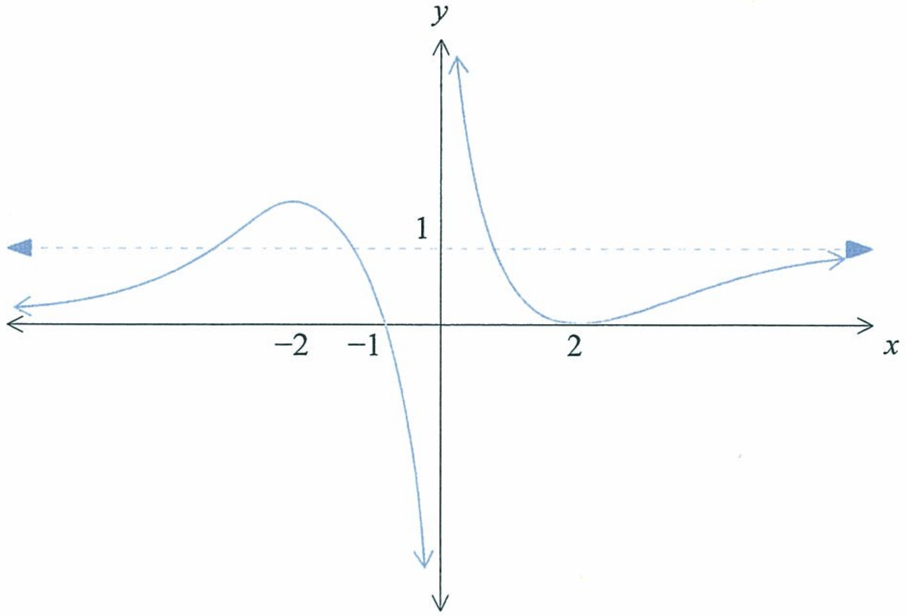
(ii) $y = \dfrac{1}{f(x)}$



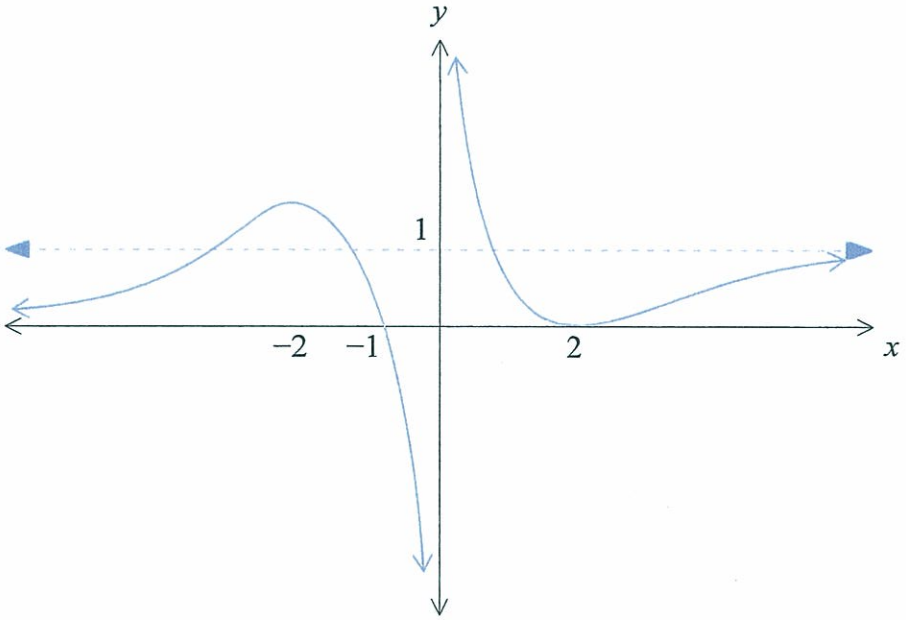
Answer Sheets For Question 3(a) Name:......................................

(page 2) Teacher: ....

(iii) $y^2 = f(x)$



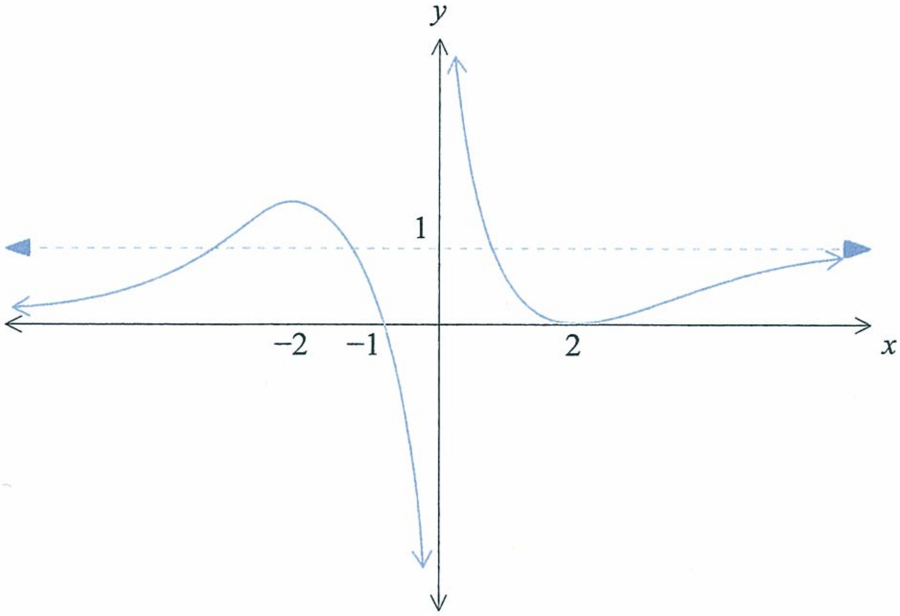
(iv) $y = [f(x)]^2$



Answer Sheets For Question 3(a) Name:.......................................

(page 3) Teacher: ....

(v) $y = \sin^{-1}\left[f(x)\right]$



QUESTION 1

(a)(i) $2z - iw$

$= 6+4i - i(1+i)$

$= 6+4i-i+1$

$= 7+3i$ 1 R|W

(ii) $z\bar{w} = (3+2i)(1-i)$

$= 3-3i+2i+2$

$= 5-i$ 1 R|W

(iii) $\frac{3}{w} = \frac{3\bar{w}}{|w|^2}$

$= \frac{3(1-i)}{(\sqrt{2})^2}$ 1 R|W

$= \frac{3-3i}{2}$

(b) $|z-2-3i| \geq |z+i|$

$M_{PQ} = (1, 1)$

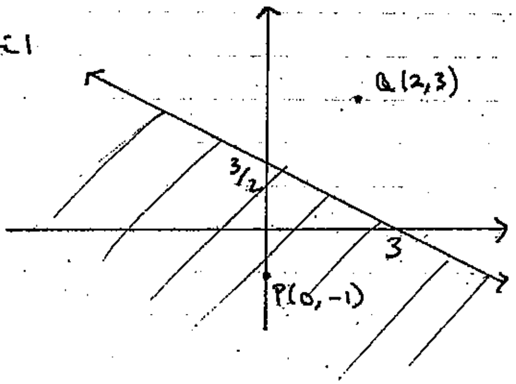
$m_{PQ} = 2$

$\therefore$ perp bisector

$y-1 = -\frac{1}{2}(x-1)$

$2y-2 = -x+1$

$x+2y-3=0$



2m - all correct

1m - Correct line incorrect shading

(c) $\alpha = -\sqrt{3}+i$

(i) $r = \sqrt{3+1} = 2$

$\theta = \frac{5\pi}{6}$

$\therefore \alpha = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

2m - both r and θ correct

1m - one of r or θ correct

0m - both incorrect

(ii) $z^6+64=0$

Subst α in LHS

$= 2^6\left(\text{cis}\frac{5\pi}{6}\right)^6 + 64$ 1 R|W

$= 64(\text{cis}\,5\pi) + 64$ (De Moivre)

$= 64 \times -1 + 64$

$= 0$ $\therefore \alpha$ is a root of $z^6+64=0$

(iii) Since α a root of $z^6+64=0$ and z^6+64 has real coefficients then $\bar{\alpha}$ also a root.

$\therefore (x-\alpha)(x-\bar{\alpha})$ is a factor of z^6+64

$= (x^2 - (\alpha+\bar{\alpha})x + \alpha\bar{\alpha})$

$= (x^2 - 2\text{Re}(\alpha)x + |\alpha|^2)$

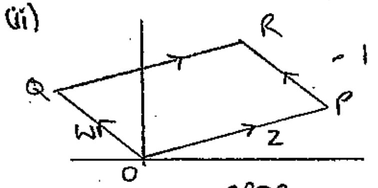
$= x^2 + 2\sqrt{3}x + 4$ is real quadratic factor.

2m - all correct including $\alpha, \bar{\alpha}$

1m - used another factor (ie not "Hence") or incorrect expansion of $(x-\alpha)(x-\bar{\alpha})$

(d) (i) $PQ = |w-z|$ 1 R|W

(ii)



1

1

OPQR is parallelogram.

(iii) If $|z+w| = |z-w|$ diagonals equal

$\therefore$ OPQR is rectangle

$\therefore$ OP and OQ perpendicular

1

1

$\therefore w = kiz$ (k = const) and $\frac{w}{z} = \frac{kiz}{z} = ki$ $\therefore$ imaginary.

Q2

a) i) $a(2x+3)+b(x-1)=1$

When $x=1$, $5a=1$

$a=\frac{1}{5}$ 1

When $x=0$ & $a=\frac{1}{5}$;

$\frac{3}{5}+(-b)=1$

$b=\frac{-2}{5}$ 1 ②

ii) $\int\frac{dx}{(x-1)(2x+3)}$

$=\frac{1}{5}\int\left(\frac{1}{x-1}-\frac{2}{2x+3}\right)dx$ 1

$=\frac{1}{5}\left[\ln|x-1|-\ln|2x+3|\right]+c$

$=\frac{1}{5}\ln\left|\frac{x-1}{2x+3}\right|+c$ 1 ②

b) $\int\operatorname{cosec}2x\,dx$; $t=\tan x$

$=\int\frac{1}{\sin 2x}dx$ $dt=\sec^2x\,dx$

$dt=(\tan^2x+1)dx$

$=\int\frac{1+t^2}{2t}\cdot\frac{dt}{t^2+1}$ 1 $\frac{dt}{t^2+1}=dx$ 1

$=\frac{1}{2}\ln|t|+c$

$=\frac{1}{2}\ln|\tan x|+c$ 1 ③

c) $\int_{-\frac{1}{2}}^{0}\frac{dx}{2+4x+4x^2}$ 1

$=\int_{-\frac{1}{2}}^{0}\frac{dx}{(2x+1)^2+1}$

$=\frac{1}{2}\tan^{-1}(2x+1)\Big]_{-\frac{1}{2}}^{0}$ 1

$=\frac{1}{2}(\tan^{-1}1-\tan^{-1}0)$

$=\frac{1}{2}(\frac{\pi}{4}-0)$

$=\frac{\pi}{8}$ 1 ②

d) $\int(x\cdot3^x)dx$

$=\frac{x\cdot3^x}{\ln3}-\int\frac{3^x}{\ln3}dx$ 2

$=\frac{1}{\ln3}\left(x\cdot3^x-\frac{3^x}{\ln3}\right)+c$ 1

$=\frac{3^x}{\ln3}\left(x-\frac{1}{\ln3}\right)+c$ ③

e) $\int\frac{dx}{x^{\frac{1}{2}}-x^{\frac{1}{3}}}$; $x=u^6$

$=\int\frac{6u^5}{u^3-u^2}du$ 1 $dx=6u^5du$

$=6\int\frac{u^3}{u-1}du$ 1

$=6\int\left(u^2+u+1+\frac{1}{u-1}\right)du$

$=2u^3+3u^2+6u+6\ln|u-1|+c$

$=2x^{\frac{1}{2}}+3x^{\frac{1}{3}}+6x^{\frac{1}{6}}+6\ln|x^{\frac{1}{6}}-1|+c$ 1 ③

$y=3^x$

$x\ln3=\ln y$

$x=\frac{\ln y}{\ln3}$

$\frac{dx}{dy}=\frac{1}{y\ln3}$

$\frac{dy}{dx}=3^x\cdot\ln3$

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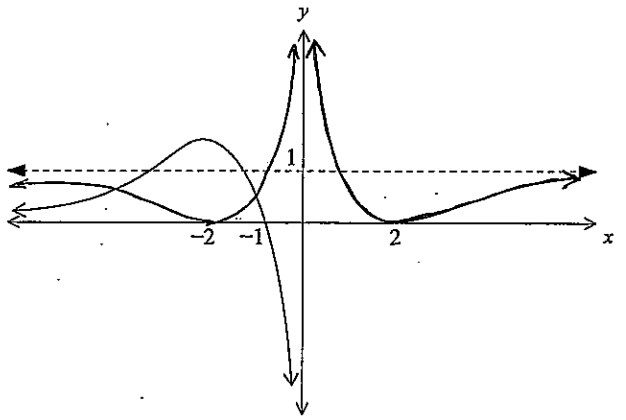
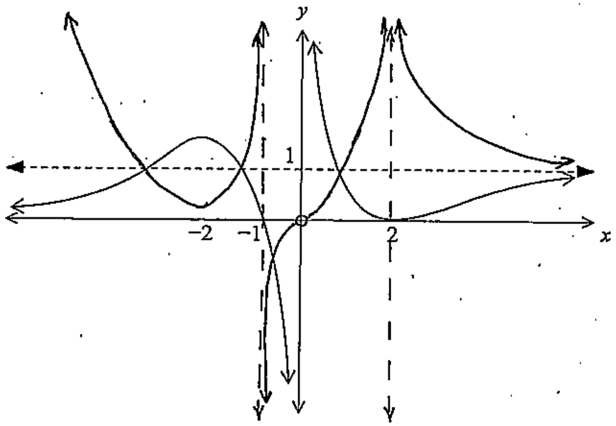
ANSWER SHEET FOR QUESTION 2(a)

(ii) Given $y = f(x)$, sketch $y = f(|x|)$.



①

(ii) Given $y = f(x)$, sketch $y = \frac{1}{f(x)}$.



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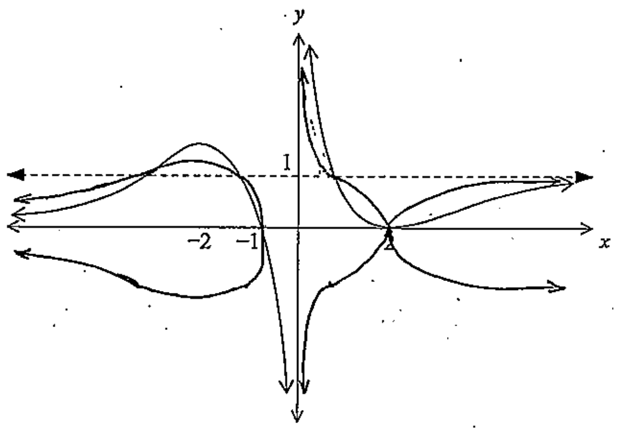
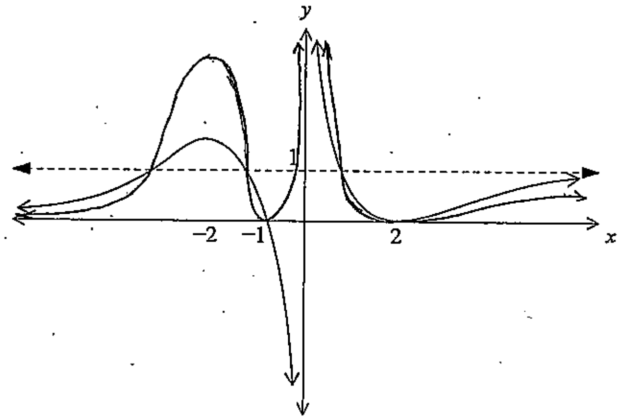
(iii) Given $y = f(x)$, sketch $y^2 = f(x)$.



②

(iv) Given $y = f(x)$, sketch $y = [f(x)]^2$.



②

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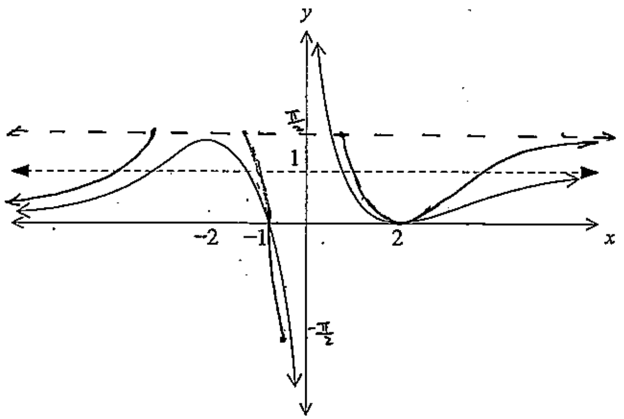
(v) Given $y = f(x)$, sketch $y = \sin^{-1}[f(x)]$.



②

b) $y = \frac{x^3 + 4}{x^2}$, $x \neq 0$

$= x + 4x^{-2}$

i) $y' = 1 - 8x^{-3}$

$y' = 1 - \frac{8}{x^3} = 0$

$x^3 = 8$

$x = 2$

$y = \frac{8+4}{4} = 3$

Stationary pt $= (2, 3)$

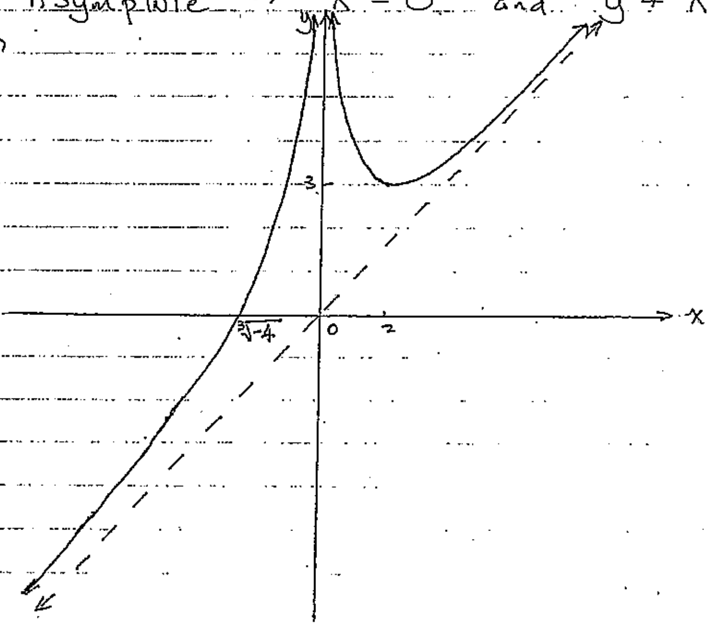
$y'' = 24x^{-4}$

Since $x^{-4} > 0$ for all values of x, $x \neq 0$,

so $y'' > 0$ hence curve is always concave up. ②

ii) Asymptote $\Rightarrow$ $x = 0$ and $y \neq x$. ①

iii)



1 for right asymptots and turning pt
1 for shape and x intercept

②

iv) $k > 3$ ①

Question 4.

) $P(x) = x^3+2x^2+2x+1$

Since $x+1$ is a factor

$P(x) = (x+1)(x^2+x+1)$ ✓1 (by inspection)

Now zeros of x^2+x+1 $\quad x = \dfrac{-1 \pm \sqrt{1-4}}{2}$

$x = \dfrac{-1 \pm \sqrt{3}i}{2} = \dfrac{-1}{2} \pm \dfrac{\sqrt{3}}{2}i$

$\therefore P(x) = (x+1)\left(x+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$ ✓1

$x^3-3x^2+5x-1=0$ has roots α, β, γ

(i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\beta\alpha)$

$= \left(\frac{-b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$ ✓1

$= 3^2 - 2(5)$

$= -1$ ✓1

(ii) Since sum of squares of roots <0, at least one must be complex. ✓1

Since coefficients real, complex roots occur in conjugate pairs.

Since polynomial degree 3, only 3 roots + 2 must be complex and therefore only one real root.

If ω root, $\omega^3+p\omega^2+q\omega+r=0$ (1)

If ω^2 root, $\omega^6+p\omega^4+q\omega^2+r=0$ (2)

From ①, $1+p\omega^2+q\omega+r=0$ ✓1

Fm ②, $1+p\omega+q\omega^2+r=0$

Subtracting $\omega^2(p-q) - \omega(p-q) = 0$

(c)(cont) $(\omega^2-\omega)(p-q)=0$ ✓1

$\therefore \omega^2-\omega=0$

$\omega(\omega-1)=0$ or $p-q=0$

$\omega=0$ (not poss)

or $\omega=1$ (not poss since ω complex) ✓1 $\therefore p=q$.

$\therefore p=q$.

(d) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(i) $8x^3-6x+1=0$

Subst $x=\cos\theta$,

$8\cos^3\theta - 6\cos\theta + 1 = 0$

$2(4\cos^3\theta - 3\cos\theta) = -1$

$4\cos^3\theta - 3\cos\theta = -1/2$

S | A
T | C

$\cos 3\theta = -1/2$

$\therefore 3\theta = \pm\frac{2\pi}{3} + 2n\pi$ ✓1 $\quad 3\theta = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$\theta = \pm\frac{2\pi}{9} + \frac{2n\pi}{3}$

3 unique solutions for $\cos\theta$: $\cos\frac{2\pi}{9}, \cos\frac{8\pi}{9}, \cos\frac{14\pi}{9}$ ✓1

$\therefore x = \cos\frac{2\pi}{9}, \cos\frac{8\pi}{9}, \cos\frac{14\pi}{9}$

$= \cos\frac{2\pi}{9}, \cos\frac{8\pi}{9}, \cos 4\pi/9$

(ii) Now $\cos\frac{14\pi}{9} = \cos\left(2\pi - \frac{4\pi}{9}\right) = \cos\frac{4\pi}{9}$.

Let roots be α, β, γ then $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \dfrac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}$

5

$\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{c/a}{-d/a}$ 2m - all correct

1m - correct exp for $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

incorrect subst

$= \frac{c}{-d}$

$= 6$

$\therefore \frac{1}{\cos 2\pi/9} + \frac{1}{\cos 4\pi/9} + \frac{1}{\cos 8\pi/9} = 6$

$\sec 2\pi/9 + \sec 4\pi/9 + \sec 8\pi/9 = 6.$

) $x^3 - x - 1 = 0$ Roots α, β and γ.

Require eq with roots $\frac{1+\alpha}{1-\alpha}, \frac{1+\beta}{1-\beta}, \frac{1+\gamma}{1-\gamma}$

Let $y = \frac{1+\alpha}{1-\alpha}$

$y(1-\alpha) = (1+\alpha)$

$y - 1 = \alpha + \alpha y$

$y - 1 = \alpha(1+y)$

$\alpha = \frac{y-1}{1+y}$ - 1

$\therefore$ Eq becomes

$\left(\frac{y-1}{1+y}\right)^3 - \left(\frac{y-1}{1+y}\right) - 1 = 0$

$(y-1)^3 - (y-1)(1+y)^2 - (1+y)^3 = 0$ - 1

$(y^3 - 3y^2 + 3y - 1) - ((y-1)(1+2y+y^2)) - (y^3+3y^2+3y+1) = 0$

$(\cancel{y^3} - 3y^2 + \cancel{3y} - 1) - (\cancel{y^3} + \cancel{2y^2} + \cancel{y} - 1 - \cancel{2y} - \cancel{y^2}) - (\cancel{y^3} + \cancel{3y^2} + \cancel{3y} + 1) = 0$

$-y^3 - 7y^2 + y - 1 = 0.$ - 1

$y^3 + 7y^2 - y + 1 = 0$

6

Question 5

(a) $\frac{x^2}{9} + \frac{y^2}{25} = 1$

(i) $a^2 = 9 \quad b^2 = 25$

$9 = 25(1-e^2)$

$e^2 = 16/25$

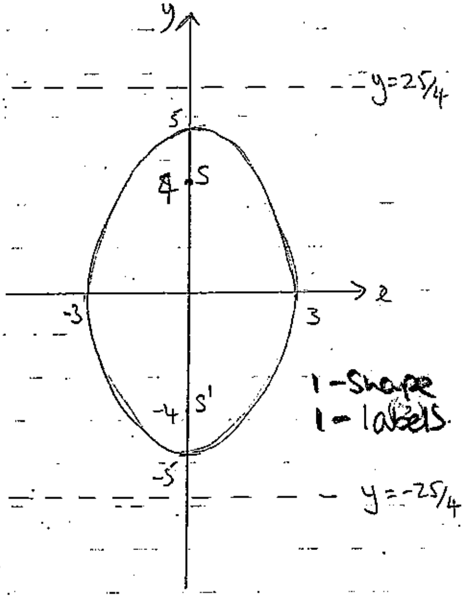
$\therefore e = 4/5$ — (1)

$\therefore$ foci $(0, \pm be)$

$(0, \pm 4)$

directrices $y = \pm \frac{b}{e}$

$= \pm \frac{25}{4}$ ← (1)



1 - Shape

1 - labels

(b) $P(2p, 2/p)$ $Q(2q, 2/q)$ $xy = 4$

$m_{PQ} = \frac{2/q - 2/p}{2(q-p)}$

$= \frac{2p - 2q}{2pq(q-p)}$

$= \frac{-1}{pq}$ - 1

Eq of PQ: $y - 2/p = \frac{-1}{pq}(x - 2p)$ - 1

Since passes through $(6, 4)$

$4 - 2/p = \frac{-1}{pq}(6 - 2p)$

$4pq - 2q = -6 + 2p$

3m - all correct
2m - correct gradt line
1m - correct grad.

) cont. $2pq - q = -3 + 2p$. 1

$2pq = p + q - 3$

$pq = \frac{p+q-3}{2}$ as required ①

(ii) $M = \left(\frac{2p+2q}{2}, \frac{\frac{2}{p}+\frac{2}{q}}{2}\right)$

$= \left(p+q, \frac{1}{p}+\frac{1}{q}\right)$

$= \left(p+q, \frac{p+q}{pq}\right)$

$\therefore$ $x = p+q$ ② $y = \frac{p+q}{pq}$ ③

$\therefore$ Subst ② into ③,

$y = \frac{x}{pq}$

Subst ①, ~~into~~

$y = \frac{x}{\frac{p+q-3}{2}}$

$y = \frac{x}{\frac{x-3}{2}}$

$y = \frac{2x}{x-3}$.

2m - all correct
1m - if midpoint + correct subst of x/y with incorrect subst of pq.

Question 5 (cont)

(c) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P(a\cos\theta, b\sin\theta)$

$x = a\cos\theta$ $y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta$ $\frac{dy}{d\theta} = b\cos\theta$

$\frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$ 1

Eq of tangent,

$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$ay\sin\theta + bx\cos\theta = ab(\sin^2\theta + \cos^2\theta)$

$ay\sin\theta + bx\cos\theta = ab$. // 1

(ii) Since tangent passes through focus $(ae, 0)$ or $(-ae, 0)$

then $\pm abe\cos\theta = ab$

$e\cos\theta = \pm 1$

$\cos\theta = \pm\frac{1}{e}$ ① 1

ignored ± error

Now at P $x = a\cos\theta$

Sub in ① $= \pm a \cdot \frac{1}{e}$

$= \pm\frac{a}{e}$ 1

Hence P lies on directrix of hyperbola.

(iii) At P, $b^2 = a^2(e^2-1)$ ② and $e\cos\theta = \pm 1$ ③

and $\frac{dy}{dx} = -\frac{b}{a}\frac{\cos\theta}{\sin\theta}$

c) (iii) cont

$$\left(\frac{dy}{dx}\right)^2 = \frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}$$

$$= (e^2-1)\frac{\cos^2\theta}{\sin^2\theta} \quad \text{(from ②)}$$

$$= \frac{e^2\cos^2\theta - \cos^2\theta}{1-\cos^2\theta}$$

$$= \frac{1 - \cos^2\theta}{1-\cos^2\theta} \quad \text{(from ③)} \quad \text{ie } (e\cos\theta = \pm 1)$$

$$= 1$$

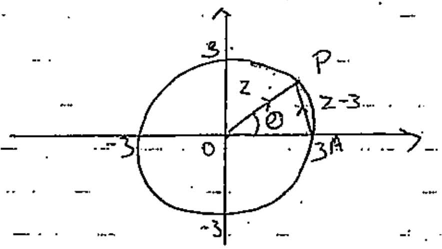
$$\therefore \frac{dy}{dx} = \pm 1.$$

Hence tangent to ellipse at P has gradient ±1.

2m - Correctly using
$e\cos\theta = \pm 1$ and
$b^2 = a^2(e^2-1)$
to obtain gradients

1m - sub either of (2) +
(3) into grad
expression +
finding grad
in terms of e, a
or b +
not simplifying
to ±1.

[illegible] 6



(i) $\arg z^4 = 4\arg z$

$= 4\theta$ ①

(ii) $\arg(z-3)$

$\angle OPA = \frac{\pi - \theta}{2}$ (sum of isosceles Δ) 1

$\therefore \arg(z-3) = \pi - \left(\frac{\pi-\theta}{2}\right)$ (angles on a straight line $= \pi$) 1

$= \frac{\pi}{2} + \frac{\theta}{2}$

$= \frac{\pi+\theta}{2}$ ②

Q6

b) $\left[\begin{array}{l} x = vt\cos\theta \\ y = \frac{-1}{2}gt^2 + vt\sin\theta \end{array}\right.$

i) When $x = d$; $y = h$;

$d = vt\cos\theta \Rightarrow t = \frac{d}{v\cos\theta}$ — ①

$h = \frac{-1}{2}gt^2 + vt\sin\theta$ — ②

Sub ① into ② :

$h = \frac{-gd^2}{2v^2\cos^2\theta} + \frac{dv\sin\theta}{v\cos\theta}$ 1

$2v^2h = -gd^2\sec^2\theta + 2d\tan\theta\, v^2$ 1

$2v^2(h - d\tan\theta) = -gd^2\sec^2\theta$

$2v^2(d\tan\theta - h) = gd^2\sec^2\theta$

$v^2 = \frac{gd^2\sec^2\theta}{2(d\tan\theta - h)}$ ②

ii) $\dot{y} = -gt + v\sin\theta$

Max height when $\dot{y} = 0$:

$gt = v\sin\theta$

$t = \frac{v\sin\theta}{g}$ 1

Max height, y

$= \frac{-g}{2}\,\frac{v^2\sin^2\theta}{g^2} + \frac{v^2\sin^2\theta}{g}$

$= \frac{v^2\sin^2\theta}{2g}$ 1

$= \frac{\sin^2\theta}{2g} \cdot \frac{gd^2\sec^2\theta}{2(d\tan\theta - h)}$; from (i)

$= \frac{d^2\tan^2\theta}{4(d\tan\theta - h)}$; $\sin^2\theta\sec^2\theta = \sin^2\theta \cdot \frac{1}{\cos^2\theta} = \tan^2\theta$] 1 ③

iii) Stone clears the fence at its height pt. $\Rightarrow y = h$

$4h(d\tan\theta - h) = d^2\tan^2\theta$ 1

$d^2\tan^2\theta - 4hd\tan\theta + 4h^2 = 0$

$(d\tan\theta - 2h)^2 = 0$ 1

$d\tan\theta = 2h$ 1

$\tan\theta = \frac{2h}{d}$ ③

Q6

c) i) $I_n = \int_0^{\frac{1}{2}} \frac{x^n}{1-x^2}\,dx$

$I_{n-2} = \int_0^{\frac{1}{2}} \frac{x^{n-2}}{1-x^2}\,dx$

$I_{n-2} - I_n$

$= \int_0^{\frac{1}{2}} \frac{x^{n-2} - x^n}{1-x^2}\,dx$

$= \int_0^{\frac{1}{2}} \frac{x^{n-2}(1-x^2)}{1-x^2}\,dx$ 1

$= \int_0^{\frac{1}{2}} x^{n-2}\,dx$

$= \left.\frac{x^{n-1}}{n-1}\right]_0^{\frac{1}{2}}$ 1

$= \frac{1}{n-1}\left[\frac{1}{2}^{n-1} - 0\right]$

$= \frac{1}{(n-1)\,2^{n-1}}$ ②

ii) $\int_0^{\frac{1}{2}} \frac{x^2}{1-x^2}\,dx = I_2$

$= I_{2-2} - \frac{1}{(2-1).2^{2-1}}$

$= I_0 - \frac{1}{2}$ 1

$= \int_0^{\frac{1}{2}} \frac{x^0}{1-x^2}\,dx - \frac{1}{2}$

$= \int_0^{\frac{1}{2}} \frac{1}{1-x^2}\,dx - \frac{1}{2}$

$= \frac{1}{2}\log_e 3 - \frac{1}{2}$ 1

$= \frac{1}{2}(\ln 3 - 1)$ ②

Question 7

(i) $\angle PCQ = \angle PCB + \angle BCA + \angle ACQ$ (adjacent angles).

Now $\angle PCB = \angle PAB$ (angles at the circumference subtended by same arc (PB) are equal)

$= \alpha$ (given)

Similarly, $\angle ACQ = \angle ABQ$ (angles at the circumference subtended by same arc (AQ) are equal)

$= \beta$ (given)

And $\angle BCA = 2\gamma$ (given)

1 R/W (correct reasoning)

$\therefore \angle PCQ = \alpha + \beta + 2\gamma$.

(ii) AB is fixed $\therefore$ angle subtended by AB is fixed.

$\therefore 2\gamma$ = constant.

$\therefore \gamma$ = constant.

In ΔABC, $2\alpha + 2\beta + 2\gamma = 180°$ (angle sum of $\Delta = 180°$)

$\therefore \alpha + \beta + \gamma = 90$

1m - showing $\alpha+\beta$ const

$\therefore \alpha + \beta = 90 - \gamma$

= constant since, γ constant

1m - correct conclusion.

Since $\alpha+\beta$ and 2γ are constants

$\alpha + \beta + 2\gamma$ = constant

$\therefore$ PQ is constant since subtends a fixed angle at the circumference $\angle PCQ = \alpha + \beta + 2\gamma$.

) Consider ΔAPB, $\angle APB = 2\gamma$ (angles at circumference subtended by same arc are equal)

and $\dfrac{\sin 2\gamma}{AB} = \dfrac{\sin\alpha}{BP}$ ① (sine rule)

1(

Question 7 (cont)

(a) (iii) cont

Consider ΔQPB, $\dfrac{\sin(\alpha+\beta)}{PQ} = \dfrac{\sin\alpha}{BP}$ ②

From ① and ②,

$$\frac{\sin(\alpha+\beta)}{PQ} = \frac{\sin 2\gamma}{AB}$$

1m - for expressions for AB and PQ using sine rule

$$\frac{AB}{PQ} = \frac{\sin 2\gamma}{\sin(\alpha+\beta)}$$

1m - combining + subst $\alpha+\beta = 90 - \gamma$.

$$= \frac{2\sin\gamma\cos\gamma}{\sin(90-\gamma)} \quad \text{(from ii)}$$

$$= \frac{2\sin\gamma\cos\gamma}{\cos\gamma}$$

$$= 2\sin\gamma.$$

(b) (i) LHS $= (1+i\tan\theta)^n \pm (1-i\tan\theta)^n$

$$= \left(1 + i\frac{\sin\theta}{\cos\theta}\right)^n + \left(1 - i\frac{\sin\theta}{\cos\theta}\right)^n$$

$$= \left(\frac{1}{\cos\theta}(\cos\theta + i\sin\theta)\right)^n + \left(\frac{1}{\cos\theta}(\cos\theta - i\sin\theta)\right)^n$$ 1

$$= \frac{1}{\cos^n\theta}\left[(\cos\theta \pm i\sin\theta)^n + (\cos\theta \mp i\sin\theta)^n\right]$$

$$= \frac{1}{\cos^n\theta}\left[\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta\right]$$ (De Moivre)

$$= \frac{1}{\cos^n\theta} 2\cos n\theta$$ 1

= RHS

2m - must apply De Moivre correctly

stion 7 (cont)

ii) Let $z = i\tan\theta$

From (i) $(1+z)^n + (1-z)^n = \frac{2\cos n\theta}{\cos^n\theta} = 0$

$\therefore \frac{2\cos n\theta}{\cos^n\theta} = 0$

4 $(1+z)^4 + (1-z)^4 = 0$

$\frac{2\cos 4\theta}{\cos^4\theta} = 0$

1 Correct geeal soln

$\therefore \cos 4\theta = 0$

$4\theta = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2 \ldots$

$\theta = \pm\pi/8, \pm 3\pi/8, \pm\frac{5\pi}{8} \ldots$ 1

$\therefore$ 4 unique solutions $z = \pm i\tan\pi/8, \pm i\tan\frac{3\pi}{8}$.

1 U 3

7(c) i) Since e^{-x} is a decreasing function and 1

as $\sin x \geq \frac{2x}{\pi}$ for $0 \leq x \leq \frac{\pi}{2}$,

1 $e^{-\sin x} \leq e^{\frac{-2x}{\pi}}$ for $0 \leq x \leq \frac{\pi}{2}$,

$\therefore \int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx < \int_0^{\frac{\pi}{2}} e^{\frac{-2x}{\pi}}\,dx$

ii) $\int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}\,dx$; let $u = \pi - x$

$= \int_{\frac{\pi}{2}}^{0} (-e^{-\sin(\pi-u)})\,du$ $\quad du = -dx$

$x = \pi,\ u = \pi - \pi = 0$

$x = \frac{\pi}{2},\ u = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

$= \int_0^{\frac{\pi}{2}} e^{-\sin u}\,du$

$= \int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx$

iii) $\int_0^{\pi} e^{-\sin x}\,dx$

$= \int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}\,dx + \int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx$

$= 2\int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx$ 1

$< 2\int_0^{\frac{\pi}{2}} e^{-\frac{2x}{\pi}}\,dx$

$= 2\left[-\frac{\pi}{2} e^{\frac{-2x}{\pi}}\right]_0^{\frac{\pi}{2}}$

$= -\pi\left(e^{\frac{-\pi}{\pi}} - e^{-0}\right)$

$= -\pi\left(e^{-1} - 1\right)$ 1

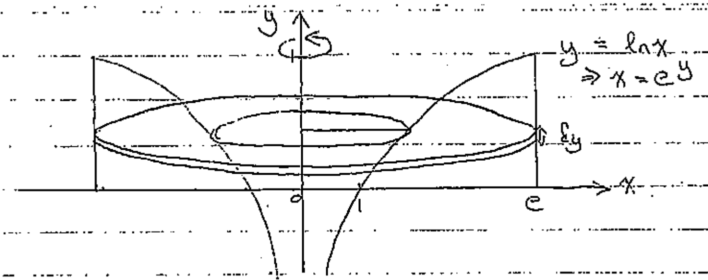
$= \pi\left(1 - \frac{1}{e}\right)$

$= \pi\left(\frac{e-1}{e}\right)$

$= \frac{\pi}{e}(e-1)$

Q8

a)



Area $= \pi(e^2 - x^2)$

$\delta V = \pi(e^2 - e^{2y})\,\delta y$

$V = \lim_{\delta y \to 0} \sum_{y=0}^{1} \pi(e^2 - e^{2y})\,\delta y$

$= \int_0^1 \pi(e^2 - e^{2y})\,dy$

$= \pi\left[e^2 y - \frac{1}{2}e^{2y}\right]_0$

$= \pi(e^2 - \frac{1}{2}e^2 - 0 + \frac{1}{2}e^0)$

$= \pi(\frac{1}{2}e^2 + \frac{1}{2})$

$= \frac{\pi}{2}(e^2 + 1)$ cubic units

②

Q 8

b) i) $\frac{d}{dx}\tan^{-1}\left(\frac{x}{x+1}\right)$

$= \frac{1}{1+\left(\frac{x}{x+1}\right)^2} \cdot \frac{1(x+1) - 1 \cdot x}{(x+1)^2}$

$= \frac{(x+1)^2}{(x+1)^2 + x^2} \cdot \frac{1}{(x+1)^2}$

$= \frac{1}{2x^2 + 2x + 1}$

$\frac{d}{dx}\tan^{-1}\left(\frac{1}{2x+1}\right)$

$= \frac{1}{1+\left(\frac{1}{2x+1}\right)^2} \cdot \frac{-2}{(2x+1)^2}$

$= \frac{(2x+1)^2}{(2x+1)^2 + 1} \cdot \frac{-2}{(2x+1)^2}$

$= \frac{-2}{4x^2 + 4x + 2}$

$= \frac{-1}{2x^2 + 2x + 1}$

$\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{x+1}\right) + \tan^{-1}\left(\frac{1}{2x+1}\right)\right]$

$= \frac{1}{2x^2+2x+1} - \frac{1}{2x^2+2x+1}$

$= 0$

$\therefore \tan^{-1}\left(\frac{x}{x+1}\right) + \tan^{-1}\left(\frac{1}{2x+1}\right)$ is a constant. ②

ii) When $x = 0$,

$\tan^{-1}(0) + \tan^{-1}(1)$

$= 0 + \frac{\pi}{4}$

$= \frac{\pi}{4}$

$\therefore$ Constant $= \frac{\pi}{4}$. ①

c) i) $LHS = \frac{\cos\theta - \cos(\theta+2\alpha)}{2\sin\alpha}$

$= \frac{\cos\theta - \cos\theta\cos 2\alpha + \sin\theta\sin 2\alpha}{2\sin\alpha}$

$= \frac{\cos\theta - \cos\theta(1-2\sin^2\alpha) + 2\sin\theta\sin\alpha\cos\alpha}{2\sin\alpha}$

$= \frac{\cos\theta - \cos\theta + 2\sin^2\alpha\cos\theta + 2\sin\theta\sin\alpha\cos\alpha}{2\sin\alpha}$

$= \sin\alpha\cos\theta + \sin\theta\cos\alpha$

$= \sin(\theta+\alpha)$

$= RHS$ ③

ii) Let the statement be

$\sin\theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2n-1)\theta) = \frac{1-\cos 2n\theta}{2\sin\theta}$.

Step 1: Test the statement is true for $n=1$.

$LHS = \sin[2(1)-1]\theta$

$= \sin\theta$

$RHS = \frac{1-\cos 2\theta}{2\sin\theta}$

$= \frac{1-(1-2\sin^2\theta)}{2\sin\theta}$

$= \sin\theta$

$= LHS$

Step 2: Assume statement is true for $n=k$;

i.e. $\sin\theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2k-1)\theta$

$= \frac{1-\cos 2k\theta}{2\sin\theta}$

Step 3: Prove statement is true for $n=k+1$;

i.e. prove $\sin\theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2k+1)$

$= \frac{1-\cos(2k+2)\theta}{2\sin\theta}$.

$LHS = \sin\theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2k-1)\theta + \sin(2k+1$

$= \frac{1-\cos 2k\theta}{2\sin\theta} + \cancel{\sin(2k+1)\theta}$

$= \frac{1-\cos 2k\theta}{2\sin\theta} + \cancel{\sin(2k\theta+\theta)}$

$= \frac{1-\cos 2k\theta}{2\sin\theta} + \frac{\cos 2k\theta - \cos(2k\theta+2\theta)}{2\sin\theta}$, from (i)

$= \frac{1-\cos 2k\theta + \cos 2k\theta - \cos(2k+2)\theta}{2\sin\theta}$

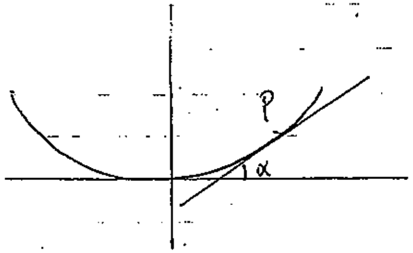
$= \frac{1-\cos(2k+2)\theta}{2\sin\theta}$

$= RHS$

Step 4: Since the statement is true for $n=1$, assumed true for $n=k$ and proved true for $n=k+1$, by mathematical induction it is true for $n=1+1=2$, $n=2+1=3$, and so on. Hence the statement is true for all positive integers of n.

1 for Steps 2 and 4 ③

stion 8 (cont)



$$\frac{x^2}{9}+\frac{(y-2)^2}{4}=1$$

Differentiating $\frac{2x}{9}+\frac{2(y-2)}{4}\frac{dy}{dx}=0$

$$\frac{dy}{dx}=\frac{-2x}{9}\times\frac{2}{y-2}$$

$$=\frac{4x}{9(2-y)}$$ 1

At P, $m_P=\frac{4r}{9(2-h)}$

$m_P=\tan\alpha$ 1

$\therefore \tan\alpha=\frac{4r}{9(2-h)}$ ②

i) (r, h) lies on ellipse, $\frac{r^2}{9}+\frac{(h-2)^2}{4}=1$

$$\frac{r^2}{9}=1-\frac{(h-2)^2}{4}$$

$$=\frac{4-(h-2)^2}{4}$$

$$r^2=9\left(\frac{4-(2-h)^2}{4}\right)$$ 1 $(h-2)^2=(2-h)^2$

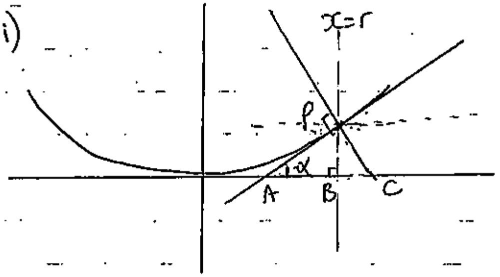
$$r=\frac{3\sqrt{4-(2-h)^2}}{2}$$ $r>0$

Question 8c (cont)

$$\therefore \tan\alpha=\frac{4}{9(2-h)}\times\frac{3\sqrt{4-(2-h)^2}}{2}$$ 1

$$=\frac{2\sqrt{4-(2-h)^2}}{3(2-h)}$$ as required. ②

(iii)



∠APB = 90 − α (angle sum of ΔAPB = 180°)
∠APC = 90° (angle between tangent and normal)
∠CPB = ∠APC − ∠APB
= 90° − (90 − α)
= α.

∴ acute angle between normal and x = r, ∠CPB = α as required. ①