Mrs Collett
Mrs Kerr

Name: ....

Teacher:..

Pymble Ladies' College

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

2014

Mathematics Extension 2

Time Allowed: 3 hours

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen. Black pen is preferred.
- Board approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- In Questions 11-16, show relevant mathematical reasoning and/or calculations.
- Start each question in a **new** booklet.

Total Marks – 100

Section I Pages 1-4

10 marks

- Attempt all Questions 1-10
- Allow about 15 mins for this section

Section II Pages 5-12

90 marks

- Attempt Questions 11-16
- Allow about 2 hour 45 minutes for this section

Mark	**/100**
Highest Mark	**/100**
Rank	

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section.

Use the multiple choice answer sheet for Questions 1-10.

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1 If $(a+bi)^2 = i$, then what are possible values for $a, b \in \mathbb{R}$?

(A) $a = \frac{1}{4},\ b = \frac{1}{4}$

(B) $a = -\frac{1}{\sqrt{2}},\ b = \frac{1}{\sqrt{2}}$

(C) $a = \frac{1}{\sqrt{2}},\ b = \frac{1}{\sqrt{2}}$

(D) $a = \frac{1}{2},\ b = \frac{1}{2}$

2 The polynomial $P(x) = x^3 + 3x^2 - 24x + 28$ has a double zero.

What is the value of the double zero?

(A) -7

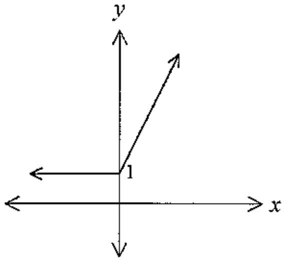
(B) -4

(C) 4

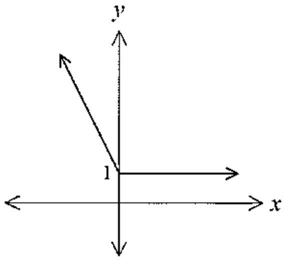
(D) 2

3 Which graph shows $y=1+x+|x|$?

(A)



(B)



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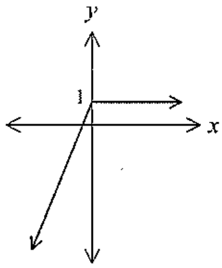
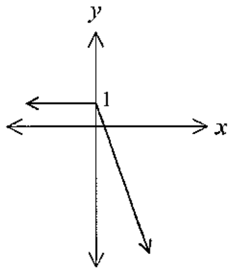


(D)



4 The graph of the ellipse $\dfrac{(x-1)^2}{9}+\dfrac{y^2}{4}=1$ and the graph of the hyperbola $x^2-y^2=4$ have

(A) no points in common.

(B) 1 point in common.

(C) 2 points in common.

(D) 3 points in common.

5 $\int \sin^{-1} 2x\, dx =$

(A) $x\sin^{-1} 2x+\frac{1}{4}\sqrt{1-4x^2}+C, \quad |x|\geq -1$

(B) $x\sin^{-1} 2x-\frac{1}{4}\sqrt{1-4x^2}+C, \quad |x|\geq -1$

(C) $x\sin^{-1} 2x+\frac{1}{2}\sqrt{1-4x^2}+C, \quad |x|\geq -1$

(D) $x\sin^{-1} 2x-\frac{1}{2}\sqrt{1-4x^2}+C, \quad |x|\geq -1$

6 Which of the following would be neither odd nor even?

(A) $y = x^2 \sin x$

(B) $y = \sin\left(x^2\right)$

(C) $y = \left(\sin x\right)^2$

(D) $y = x^2 + \sin x$

7 What is the exact value of $\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^{100}$?

(A) 1

(B) -1

(C) $\frac{1}{2^{50}}$

(D) $-\frac{1}{2^{50}}$

8 If $\frac{3x-19}{(x+3)(2x-1)} = \frac{a}{x+3} + \frac{b}{2x-1}$, then find the values of a and b.

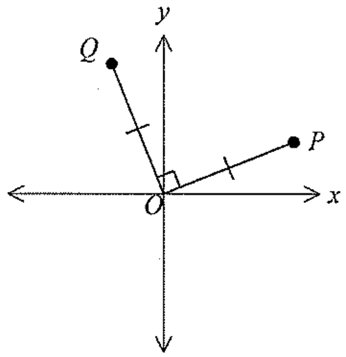
(A) $a = -4, \quad b = 5$

(B) $a = -4, \quad b = -5$

(C) $a = 4, \quad b = 5$

(D) $a = 4, \quad b = -5$

9 On the diagram P and Q represent complex numbers z and w respectively. Triangle OPQ is right angled and isosceles.



Which of the following is **false**?

(A) $|z|^2 + |w|^2 = |z+w|^2$

(B) $z^2 - w^2 = 0$

(C) $z^2 + w^2 = 0$

(D) $w = iz$

10 The ellipse $x^2 + 2ax + 2y^2 + 4by + 16 = 0$ has its centre at $(3, -2)$. Find the values of a and b.

(A) $a = -3,\ b = -2$

(B) $a = 2,\ b = -3$

(C) $a = -3,\ b = 2$

(D) $a = 3,\ b = 2$

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer each question in the appropriate writing booklet. Extra booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11. (15 marks). Use a **Separate** Booklet. **Marks**

(a) Use the substitution $u = 4 + \sin x$ to find **2**

$$\int \frac{\sin x \cos x}{4 + \sin x}\, dx.$$

(b) Let $w = -1 + \sqrt{3}\, i$ and $z = 1 - i$.

(i) Find wz in the form $a + ib$. **1**

(ii) Find w and z in mod-arg form. **2**

(iii) Hence, find the exact value of $\sin \frac{5\pi}{12}$. **2**

(c) Let polynomial $P(x) = ax^6 - bx^5 + 1$.

(i) State the conditions for α to be a zero of multiplicity two of $P(x)$. **1**

(ii) Given that $P(x)$ is divisible by $(x+1)^2$ find a and b. **3**

(d) The line $x = 1$ is a directrix and the point $(2, 0)$ is a focus of the conic whose eccentricity is $\sqrt{2}$.

(i) Derive the equation of the conic. **3**

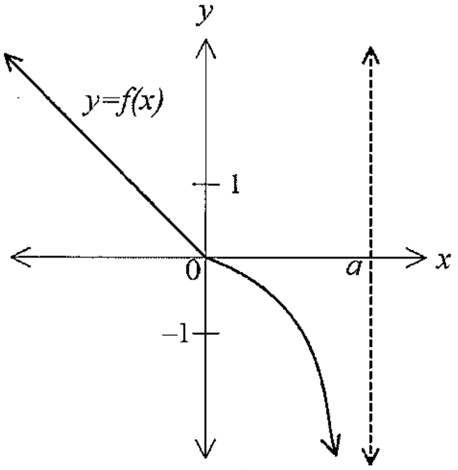
(ii) Prove that it is a rectangular hyperbola. **1**

End of Question 11

Question 12. (15 marks) Use a **Separate** Booklet. **Marks**

(a) Find $\int \frac{\ln x}{x^2}\,dx$. **2**

(b) The graph of the function $y = f(x),\ x < a$ is shown below.



Sketch the following curves on separate half-page diagrams.

(i) $y = |f(x)|$. **1**

(ii) $y = f(|x|)$. **1**

(iii) $y = \frac{1}{f(x)}$. **2**

(c) Let C be the curve $3e^{x-y} = x^2 + y^2 + 1$. **3**

Find the equation of the tangent to C at the point $(1,1)$.

Question 12 continues on page 7

(d) (i) Expand $(a-b)^3$. **1**

(ii) Solve $z^3 = -1$. **2**

(iii) Express the polynomial $z^3 - 3iz^2 - 3z + 1 + i$ in the form $(z+p)^3 + q$ where p is an imaginary number and q is a real number. **1**

(iv) Hence solve $z^3 - 3iz^2 - 3z + 1 + i = 0$ giving the solution in the form $z = x + iy$ where $x, y \in \mathbb{R}$. **2**

End of Question 12

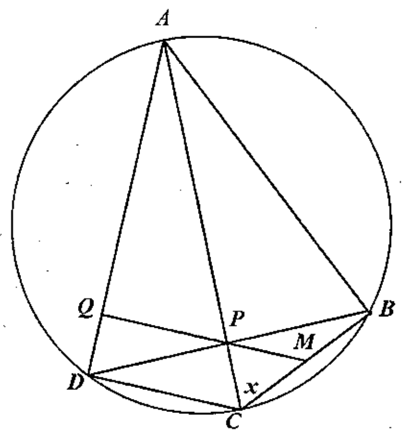
Question 13. (15 marks) Use a **Separate** Booklet. **Marks**

(a) When the polynomial $p(x) = x^4 + ax + 2$ is divided by $x^2 + 1$ the remainder is $2x + 3$. Find the value of a. **2**

(b) Using the substitution $t = \tan\frac{x}{2}$, find $\int \frac{\tan x}{1+\cos x}\,dx$. **3**

(c) Consider the region bounded by the curve $y = x^2 - 6x + 8$ and the x-axis. Use the method of cylindrical shells to find the volume of the solid formed if the region is rotated about the y-axis to form a solid of revolution. **3**

(d) $ABCD$ is a cyclic quadrilateral. Diagonals AC and BD intersect at right angles at P. M is the midpoint of BC. MP produced meets AD at Q. Let $\angle MCP = x$.



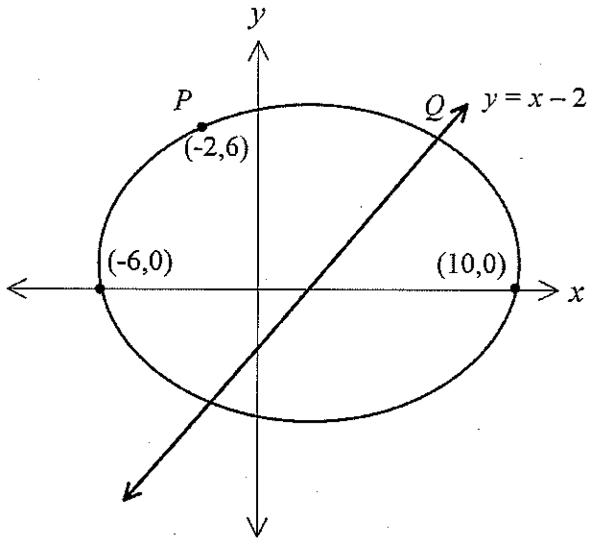
(i) Show $\angle MCP = \angle CPM$. **2**

(ii) Show $MQ \perp AD$. **2**

Question 13 continues on page 9

(e) The ellipse shown below passes through point $P(-2,6)$. **3**

The centre of the ellipse lies on the x-axis , and the ellipse passes through the points $(-6,0)$ and $(10,0)$.



The line shown is $y = x - 2$. This line intersects the ellipse at Q.

What is the x coordinate of point Q?

End of Question 13

Question 14. (15 marks) Use a **Separate** Booklet. **Marks**

(a) (i) Derive the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at point $P(x_1, y_1)$. **3**

(ii) The tangents to the ellipse $x^2+4y^2=4$ at the points $P(2\cos\theta, \sin\theta)$ and $Q(2\cos\phi, \sin\phi)$ are at right angles to each other. **2**

Show that $4\tan\theta\tan\phi=-1$.

(b) If w is one of the complex roots of $z^3=1$, simplify **3**

$(1-w)(1-w^2)(1-w^4)(1-w^8)$.

(c) (i) If $I_n=\int_0^{\frac{\pi}{4}}\tan^n x\,dx$ prove that $I_n+I_{n-2}=\frac{1}{n-1}$, $n>2$. **2**

(ii) Hence, evaluate $\int_0^{\frac{\pi}{4}}\tan^5 x\,dx$. **2**

(d) Sketch the locus of z if $\frac{z-2i}{z-1}$ is purely imaginary. **3**

End of Question 14

Question 15. (15 marks) Use a **Separate** Booklet. **Marks**

(a) The base of a solid is given by the region in the xy-plane enclosed by the curve $y = x^2$ and $y = 8 - x^2$.
Each cross-section perpendicular to the x-axis is a square.

(i) Show that the area of the square cross-section at $x = h$ is $\left(8 - 2h^2\right)^2$. **1**

(ii) Hence, find the volume of the solid. **3**

(b) Show that $\displaystyle\int_0^1 \frac{dx}{x^2 - x + 1} = \frac{2\sqrt{3}\pi}{9}$. **2**

(c) Let $f(x) = \dfrac{4}{x-1} - \dfrac{4}{x+1} - 1$, where $x \neq \pm 1$.

(i) Find the x and y intercepts of the graph of $y = f(x)$. **2**

(ii) Show that $y = f(x)$ is an even function. **1**

(iii) Find the equation of the horizontal asymptote. **1**

(iv) Sketch the graph of $y = f(x)$. **2**

(v) Let S be the area bound by the graph of $y = f(x)$, the straight lines $x = 3$, $x = a$ $(a > 3)$ and $y = -1$. **3**

Find S in terms of a and deduce that $S < 4\ln 2$.

End of Question 15

Question 16. (15 marks) Use a **Separate** Booklet. **Marks**

(a) The locus of w is described by the equation $|w+3| = |w-2+5i|$.

(i) Sketch on an Argand Diagram the locus of w. **2**

(ii) Find the Cartesian equation of the locus of w. **2**

(b) (i) Given that $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$ explain why $\frac{1}{(n+1)^2} < \frac{1}{n(n+1)}, n \in \mathbb{Z}^+$. **1**

(ii) Using induction, prove $S_n = \sum_{r=1}^{n} \frac{1}{r^2} \leq 2 - \frac{1}{n}, n \geq 1$. **3**

(c) Consider the quadratic equation $x^2 - x + k = 0$ where k is a real number.
The equation has 2 distinct positive roots α and β.

(i) Show $0 < k < \frac{1}{4}$. **2**

(ii) Show that $\frac{1}{\alpha^2} + \frac{1}{\beta^2} > 8$. **2**

(d) Given $I = \int_{-1}^{1} \frac{x^2 e^x}{e^x + 1} dx$ and $J = \int_{-1}^{1} \frac{x^2}{e^x + 1} dx$.

(i) Use the substitution $u = -x$ in I to show $I = J$. **1**

(ii) Hence evaluate I and J. **2**

End of Paper

Solutions

1. $a^2 - b^2 + 2abi$

$a^2 - b^2 = 0$

$2ab = 1$

$ab = 1/2$

$\therefore$ C

2. $P(x) = x^3 + 3x^2 - 24x + 28$

$P'(x) = 3x^2 + 6x - 24$

When $P'(x) = 0$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

$\therefore x = 2, -4$

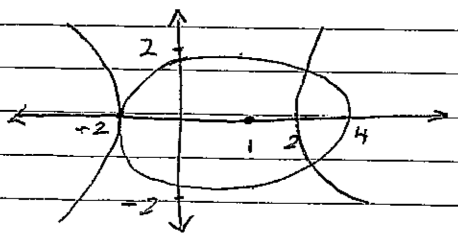
$P(2) = 8 + 12 - 48 + 28$

$= 0$

$\therefore$ D

3. A.

4. $\dfrac{(x-1)^2}{9} + \dfrac{y^2}{4} = 1$ D



5. $\int \sin^{-1} 2x \, dx =$ $u = \sin^{-1} 2x$ $dv = 1$

$du = \dfrac{2}{\sqrt{1-4x^2}} dx$ $v = x$

$= x \sin^{-1} 2x - \int \dfrac{2x}{\sqrt{1-4x^2}} dx$

$= x \sin^{-1} 2x + \dfrac{1}{2}\sqrt{1-4x^2} + C$

$\therefore$ C

6. D

7. $\left(\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}} i\right)^{100} = \left(\text{cis}\left(-\pi/4\right)\right)^{100}$

$= \text{cis}(-25\pi)$

$= \text{cis}\, \pi$

$= -1$

8. D.

9. B.

10. C

Question 11

a) $\displaystyle\int \frac{\sin x \cos x}{4+\sin x}\,dx$ $\quad u = 4+\sin x$, $\quad du = \cos x\,dx$

$= \displaystyle\int \frac{(u-4)\,du}{u}$

$= \displaystyle\int \left(1-\frac{4}{u}\right) du$

$= u - 4\ln|u| + C$

$= 4+\sin x - 4\ln|4+\sin x| + C$

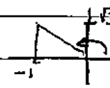
b) i) $w = -1+\sqrt{3}i \qquad z = 1-i$

$wz = (-1+\sqrt{3}i)(1-i)$

$= -1 + i + \sqrt{3}i + \sqrt{3}$

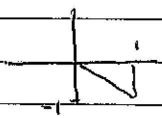
$= \sqrt{3}-1 + (1+\sqrt{3})i$

ii) $w = 2\operatorname{cis} 2\pi/3$



$\arg = \tan^{-1}(-\sqrt{3})$
$= 2\pi/3$

$z = \sqrt{2}\operatorname{cis}(-\pi/4)$



iii) $\sin \frac{5\pi}{12}$

$wz = 2\sqrt{2}\operatorname{cis}\left(\frac{2\pi}{3}-\frac{\pi}{4}\right)$

$= 2\sqrt{2}\operatorname{cis}\frac{5\pi}{12}$

$wz = \sqrt{3}-1 + (1+\sqrt{3})i$ from i)

$\therefore \sqrt{3}-1+(1+\sqrt{3})i = 2\sqrt{2}\cos\frac{5\pi}{12} + 2\sqrt{2}\,i\sin\frac{5\pi}{12}$

Equating parts

$2\sqrt{2}\sin\frac{5\pi}{12} = 1+\sqrt{3}$

$\therefore \sin\frac{5\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$

c) $P(x) = ax^6 - bx^5 + 1$

i) $P(\alpha) = 0 = P'(\alpha)$

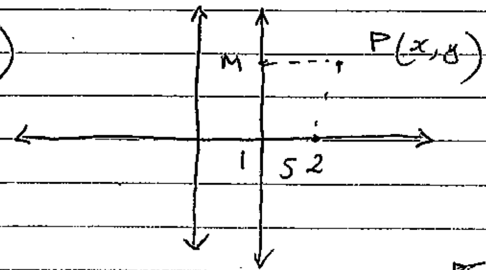
ii) $P(-1) = a+b+1 = 0 \quad$ — ①

$P'(x) = 6ax^5 - 5bx^4$

$P'(-1) = -6a - 5b = 0 \quad$ — ②

$5a + 5b = -5$
$-6a - 5b = 0$
$-a = -5$
$a = 5$
$b = -6$

d)



Let $P(x,y)$ be a point on the conic
Let S be the focus
M be the foot of the perpendicular at the directrix

$\frac{PS}{PM} = \sqrt{2}$

$d_{PS} = \sqrt{(x-2)^2 + y^2}$

$d_{PM} = \sqrt{(x-1)^2}$

$\frac{d_{PS}}{d_{PM}} = \sqrt{2}$

$\frac{\sqrt{(x-2)^2 + y^2}}{\sqrt{(x-1)^2}} = \sqrt{2}$

$\frac{(x-2)^2 + y^2}{(x-1)^2} = 2$

$x^2 - 4x + 4 + y^2 = 2x^2 - 4x + 2$

$x^2 - y^2 = 2 \qquad \frac{x^2}{2} - \frac{y^2}{2} = 1$

ii) ($e = \sqrt{2}$ ∴ hyperbola is rectangular.)

$\frac{x^2}{2} - \frac{y^2}{2} = 1$

Gradients of asymptotes $\frac{\sqrt{2}}{\sqrt{2}} \times -\frac{\sqrt{2}}{\sqrt{2}}$

$= -1$

∴ Asymptotes are perpendicular

∴ Rectangular

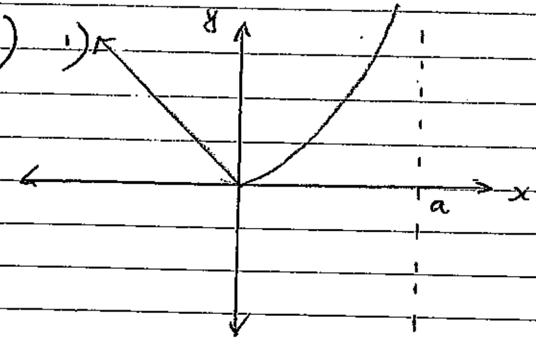
Question 12

a) $\int \frac{\ln x}{x^2}\,dx \qquad u = \ln x \quad dv = x^{-2} \qquad du = \frac{1}{x} \quad v = -x^{-1}$

$= -\frac{\ln x}{x} + \int x^{-2}\,dx$

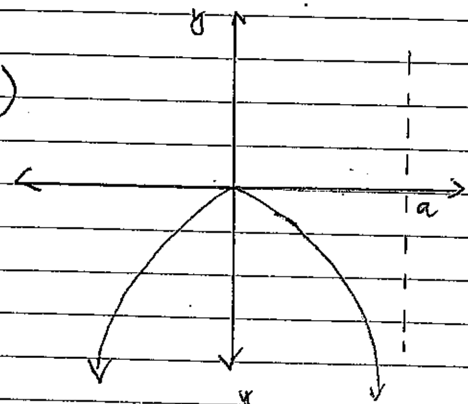
$= -\frac{\ln x}{x} - \frac{1}{x} + C$

b) i)



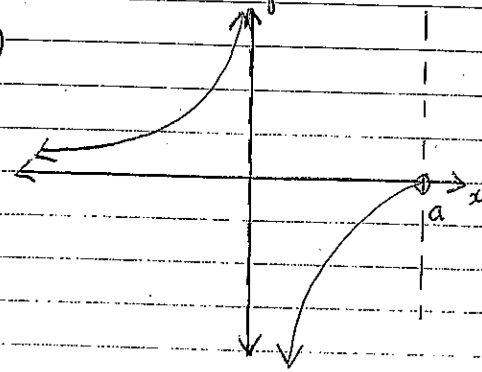
$y = |f(x)|$

ii)



$y = f(|x|)$

iii)



$y = \frac{1}{f(x)}$

c) $3e^{x-y} = x^2 + y^2 + 1$

$$3e^{x-y} \times \left(1 - \frac{dy}{dx}\right) = 2x + 2y\frac{dy}{dx}$$

$$3e^{x-y} - \frac{dy}{dx} \times 3e^{x-y} - 2y\frac{dy}{dx} = 2x$$

$$-\frac{dy}{dx}\left(3e^{x-y} + 2y\right) = 2x - 3e^{x-y}$$

$$\frac{dy}{dx} = \frac{3e^{x-y} - 2x}{3e^{x-y} + 2y}$$

When $x = 1$, $y = 1$ $\quad \frac{dy}{dx} = \frac{3e^0 - 2}{3e^0 + 2}$

$$= \frac{1}{5}$$

Eq^n of tangent $\quad y - 1 = \frac{1}{5}(x - 1)$

$$5y - 5 = x - 1$$

$$x - 5y + 4 = 0$$

d) i) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

ii) $z^3 = -1$

$$z^3 + 1 = 0$$

$$(z+1)(z^2 - z + 1) = 0$$

$\therefore z = -1 \quad z = \frac{1 \pm \sqrt{1-4}}{2}$

$$= \frac{1 \pm \sqrt{3}\,i}{2}$$

iii) $z^3 - 3iz^2 - 3z + 1 + i = (z - i)^3 + 1$

iv) $z^3 - 3iz^2 - 3z + 1 + i = 0$

$$(z - i)^3 + 1 = 0$$

$$(z - i + 1)\left((z-i)^2 - (z-i) + 1\right) = 0$$

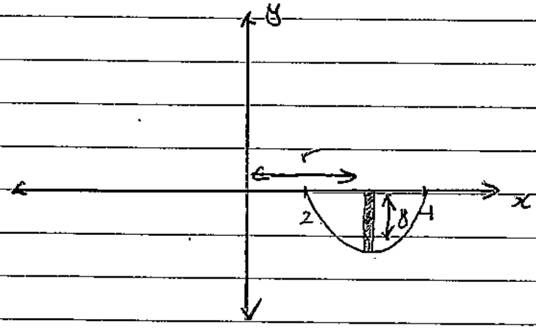
$\therefore z - i = -1 \quad$ or $\quad z - i = \frac{1 \pm \sqrt{3}\,i}{2}$

$z = i - 1$

$$z = i + \frac{1 \pm \sqrt{3}\,i}{2}$$

Question 13

a)



$y = x^2 - 6x + 8$

$= (x-4)(x-2)$

Thickness $= dx \qquad r = x$

$V_{slice} = 2\pi r h\, dx \qquad h = y = |x^2 - 6x + 8|$

$$dV = \sum_2^4 2\pi x \left|(x^2 - 6x + 8)\right| dx$$

$$V = \lim_{dx \to 0} \sum_2^4 2\pi x \left|(x^2 - 6x + 8)\right| dx$$

$$= \left|2\pi \int_2^4 (x^3 - 6x^2 + 8x)\, dx\right|$$

$$= \left|2\pi \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{8x^2}{2}\right]_2^4\right|$$

$$= \left|2\pi \left\{\frac{4^4}{4} - 2 \times 4^3 + 4 \times 4^2 - \left(\frac{2^4}{4} - 2 \times 2^3 + 4 \times 2^2\right)\right\}\right|$$

$$= \left|2\pi \{64 - 128 + 64 - 4 + 16 - 16\}\right|$$

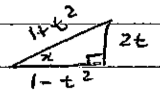
$$= 8\pi \text{ units}^3$$

b) $t = \tan \frac{x}{2}$ $\quad dx = \frac{2}{1+t^2}\,dt$ $\quad \int \frac{\tan x}{1+\cos x}\,dx$

$\frac{x}{2} = \tan^{-1} t$

$x = 2\tan^{-1} t$

$\tan x = \frac{2t}{1-t^2}$



$$= \int \frac{\frac{2t}{1-t^2}}{1+\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int \frac{\frac{2t}{1-t^2}}{\frac{1+t^2+1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2t}{1-t^2} \times \frac{1+t^2}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{1+t^2}}\,dt$$

$$= \int \frac{2t}{1-t^2}\,dt$$

$$= -\ln|1-t^2| + C$$

$$= -\ln\left|1-\tan^2 \frac{x}{2}\right| + C$$

c) $P(x) = x^4 + ax + 2$

$P(x) = (x^2+1)A(x) + 2x + 3$

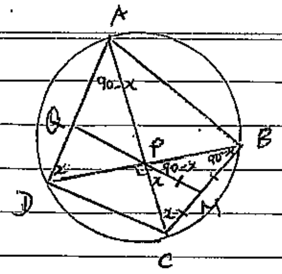
$P(x) = (x^2+1)(x^2-1) + 2x + 3$

$= x^4 - 1 + 2x + 3$

$= x^4 + 2x + 2$

$\therefore a = 2$

d)



i) $\angle BPC = 90^\circ$

$\therefore$ BC is a diameter of a circle passing through BPC, M is the midpoint of BC

$\therefore$ M is the centre of this circle.

$\therefore$ PM = MC radii of this circle

$\therefore$ $\triangle PMC$ is isosceles

$\therefore$ $\angle MCP = \angle CPM = x$, equal angles are opposite equal sides in an isosceles triangle

ii) $\angle CPM = x$ (from i)

$\angle CPD = 90^\circ$ (given)

$\angle QPD = 90 - x$ (since QM is a straight line)

$\angle ADB = x$ (angles at the circumference standing on the same arc AB are equal)

$\therefore \angle PQD = 90^\circ$ (angle sum $\triangle PQD$)

e) $2a = 16$

$\therefore \quad a = 8$

Eq^n of ellipse

$$\frac{(x-2)^2}{64} + \frac{y^2}{b^2} = 1$$

Passes through $(-2, 6)$

$$\therefore \frac{(-2-2)^2}{64} + \frac{6^2}{b^2} = 1$$

$$\frac{16}{64} + \frac{36}{b^2} = 1$$

$$\frac{36}{b^2} = \frac{48}{64}$$

$$b^2 = \frac{64 \times 36}{48}$$

$$b^2 = 48$$

$$\therefore \frac{(x-2)^2}{64} + \frac{y^2}{48} = 1 \qquad y = x - 2$$

$$\frac{(x-2)^2}{64} + \frac{(x-2)^2}{48} = 1$$

$$48(x^2 - 4x + 4) + 64(x^2 - 4x + 4) = 3072$$

$$112x^2 - 448x + 448 = 3072$$

$$x^2 - 4x + 4 = \frac{3072}{112}$$

$$(x-2)^2 = \frac{3072}{112}$$

$$x - 2 = \pm \frac{\sqrt{3072}}{\sqrt{112}}$$

$$x = 2 \pm \frac{32\sqrt{3}}{4\sqrt{7}}$$

$$= 2 \pm \frac{8\sqrt{3}}{} \times \sqrt{7}$$

$x = 2 + \frac{8\sqrt{21}}{7}$ is the x coordinate of Q

Question 14

a) i)

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{a^2} \times \frac{b^2}{2y}$$

$$= \frac{-b^2x}{a^2y}$$

At (x_1, y_1) $\quad \frac{dy}{dx} = \frac{-b^2x_1}{a^2y_1}$

Eq^n of tangent

$$y - y_1 = \frac{-b^2x_1}{a^2y_1}(x - x_1)$$

$$a^2y_1y - a^2y_1^2 = -b^2x_1x + b^2x_1^2$$

$$a^2y_1y + b^2x_1x = a^2y_1^2 + b^2x_1^2$$

ii) $x^2 + 4y^2 = 4$

$$\frac{x^2}{4} + y^2 = 1 \qquad P(2\cos\theta, \sin\theta) \quad Q(2\cos\phi, \sin\phi)$$

from i) $\quad y = \frac{-b^2x_1x}{a^2y_1} + \frac{a^2y_1^2}{a^2y_1} + \frac{b^2x_1^2}{a^2y_1}$

$$\therefore m_P = \frac{-b^2 \times 2\cos\theta}{a^2\sin\theta}$$

$$m_Q = \frac{-b^2 \times 2\cos\phi}{a^2\sin\phi}$$

$m_P \times m_Q = -1$ since tangent at P is perpendicular to tangent at Q

$$\therefore \frac{-b^2 \times 2\cos\theta}{a^2\sin\theta} \times \frac{-b^2 \times 2\cos\phi}{a^2\sin\phi} = -1$$

$$\frac{4b^4\cos\theta\cos\phi}{a^4\sin\theta\sin\phi} = -1$$

$a^2 = 4$, $b^2 = 1$

$$\therefore \frac{4\times 1^2\cos\theta\cos\phi}{4^2\sin\theta\sin\phi} = -1$$

$$\cos\theta\cos\phi = -4\sin\theta\sin\phi$$

$$1 = \frac{-4\sin\theta\sin\phi}{\cos\theta\cos\phi}$$

$$\therefore 4\tan\theta\tan\phi = -1$$

b)

$$z^3 = 1$$

$$z^3 - 1 = 0$$

$$(z-1)(z^2+z+1) = 0$$

$$\therefore z = 1 \quad z = \frac{-1\pm\sqrt{-3}}{2}$$

$$z = \frac{-1\pm\sqrt{3}\,i}{2}$$

$z_1 = 1$

$z_2 = \text{cis}\, 2\pi/3$

$z_3 = \text{cis}(-2\pi/3)$

Let $w = \text{cis}\,2\pi/3$.

$$w^2 = \text{cis}\,4\pi/3 = \text{cis}(-2\pi/3)$$

$$w^4 = \text{cis}\,8\pi/3 = \text{cis}\,2\pi/3$$

$$w^8 = \text{cis}\,16\pi/3 = \text{cis}(-2\pi/3)$$

$$\therefore (1-\text{cis}\,2\pi/3)(1-\text{cis}(-2\pi/3))(1-\text{cis}\,2\pi/3)(1-\text{cis}(-2\pi/3))$$

$$= (1-\text{cis}\,2\pi/3)^2(1-\text{cis}(-2\pi/3))^2$$

$$= \left(1-\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\right)^2\left(1-\left(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right)\right)^2$$

$$= \left(\tfrac{3}{2}-\tfrac{\sqrt{3}}{2}i\right)^2\left(\tfrac{3}{2}+\tfrac{\sqrt{3}}{2}i\right)^2$$

$$= \left(\left(\tfrac{3}{2}-\tfrac{\sqrt{3}}{2}i\right)\left(\tfrac{3}{2}+\tfrac{\sqrt{3}}{2}i\right)\right)^2 = \left(\tfrac{9}{4}+\tfrac{3}{4}\right)^2 = 9$$

c) i)

$$I_n = \int_0^{\pi/4} \tan^n x\,dx$$

$$I_{n-2} = \int_0^{\pi/4} \tan^{n-2} x\,dx$$

$$I_n + I_{n-2} = \int_0^{\pi/4} \tan^n x + \tan^{n-2} x\,dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x(\tan^2 x + 1)\,dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x\,dx$$

$u = \tan^{n-2} x$, $dv = \sec^2 x$

$du = (n-2)\tan^{n-3} x \sec^2 x\,dx$, $v = \tan x$

$$= \tan^{n-2} x \tan x\Big|_0^{\pi/4} - \int_0^{\pi/4} \tan x \times (n-2)\tan^{n-3} x \sec^2 x\,dx$$

$$= \tan^{n-1} x\Big|_0^{\pi/4} - (n-2)\int_0^{\pi/4} \tan^{n-2} x \sec^2 x\,dx$$

$$= 1 - (n-2)\int_0^{\pi/4} \tan^n x + \tan^{n-2} x\,dx$$

$$I_n + I_{n-2} = 1 - (n-2)I_n - (n-2)I_{n-2}$$

$$I_n + I_{n-2} = 1 - nI_n + 2I_n - nI_{n-2} + 2I_{n-2}$$

$$I_n + nI_n - 2I_n + I_{n-2} + nI_{n-2} - 2I_{n-2} = 1$$

$$(n-1)I_n + (n-1)I_{n-2} = 1$$

$$I_n + I_{n-2} = \frac{1}{n-1} \quad \text{As req'd}$$

ii) $\int_0^{\pi/4} \tan^5 x \, dx = I_5$

$I_5 + I_3 = \frac{1}{4}$

$I_3 + I_1 = \frac{1}{2}$

$I_1 = \int_0^{\pi/4} \tan x \, dx$

$= -\ln|\cos x| \Big|_0^{\pi/4}$

$= -\ln|\cos \frac{\pi}{4}| + \ln|\cos 0|$

$= -\ln \frac{1}{\sqrt{2}}$

$= \ln\sqrt{2}$

$I_3 + \ln\sqrt{2} = \frac{1}{2}$

$I_3 = \frac{1}{2} - \frac{1}{2}\ln 2$

$I_5 + \frac{1}{2} - \frac{1}{2}\ln 2 = \frac{1}{4}$

$I_5 = \frac{1}{2}\ln 2 - \frac{1}{4}$

d) $\frac{z-2i}{z-1}$ let $z = x+iy$

$\frac{x+iy-2i}{x+iy-1}$

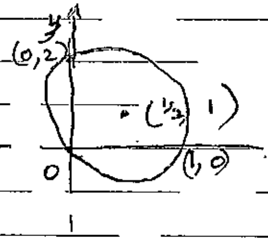
$= \frac{x+(y-2)i}{x-1+iy} \times \frac{x-1-iy}{x-1-iy}$

$= \frac{x^2 - x - ixy + x(y-2)i - (y-2)i + y(y-2)}{(x-1)^2 + y^2}$

If locus is imaginary then

$\frac{x^2 - x + y^2 - 2y}{(x-1)^2 + y^2} = 0$

$\therefore (x - \frac{1}{2})^2 + (y-1)^2 = \frac{5}{4}$



When $x = 0$

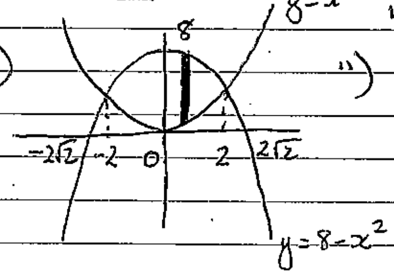
$(y-1)^2 = 1$

$y - 1 = \pm 1$

$y = 2, 0$

Question 15

a)



i) $A = (y_2 - y_1)^2 = (8 - x^2 - x^2)^2 = (8-2x^2)^2$

ii) Thickness is dx

$V = (y_2 - y_1)^2 dx$

$dV = \sum_{-2}^{2} (8 - x^2 - x^2)^2 dx$

$= \sum_{-2}^{2} (8 - 2x^2)^2 dx$

$y = x^2$

$y = 8 - x^2$

$x^2 = 8 - x^2$

$2x^2 = 8$

$x^2 = 4$

$x = \pm 2$

$V = \lim_{dx \to 0} 2 \sum_{0}^{2} (8 - 2x^2)^2 dx$

$= 8 \int_0^2 (4 - x^2)^2 \, dx$

$= 8 \int_0^2 16 - 8x^2 + x^4 \, dx$

$= 8 \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_0^2$

$$= 8\left\{16\times 2 - \frac{8\times 2^3}{3} + \frac{2^5}{5} - 0\right\}$$

$$= \frac{7168}{15}\ \text{units}^3$$

b) $$\int_0^1 \frac{dx}{x^2 - x + 1}$$

$$= \int_0^1 \frac{dx}{x^2 - x + \frac{1}{4} + \frac{3}{4}}$$

$$= \int_0^1 \frac{dx}{(x-\frac{1}{2})^2 + \frac{3}{4}}$$

$$= \left[\frac{2}{\sqrt{3}} \tan^{-1} \frac{2(x-\frac{1}{2})}{\sqrt{3}}\right]_0^1$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2(1-\frac{1}{2})}{\sqrt{3}} - \frac{2}{\sqrt{3}} \tan^{-1} \frac{2(0-\frac{1}{2})}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} \tan^{-1} \frac{1}{\sqrt{3}}$$

$$= \frac{4}{\sqrt{3}} \times \frac{\pi}{6}$$

$$= \frac{2\pi}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{2\sqrt{3}\pi}{9}$$ As req'd

c) $f(x) = \frac{4}{x-1} - \frac{4}{x+1} - 1 \qquad x \neq \pm 1$

i) $f(0) = \frac{4}{0-1} - \frac{4}{0+1} - 1$

$= -4 - 4 - 1$

$= -9$

when $f(x) = 0$

$\frac{4}{x-1} - \frac{4}{x+1} - 1 = 0$

$\frac{4(x+1) - 4(x-1)}{(x^2-1)} = 1$

$4x + 4 - 4x + 4 = x^2 - 1$

$x^2 - 9 = 0$

$x = \pm 3$

ii) $f(a) = \frac{4}{a-1} - \frac{4}{a+1} - 1$

$f(-a) = \frac{4}{-a-1} - \frac{4}{-a+1} - 1$

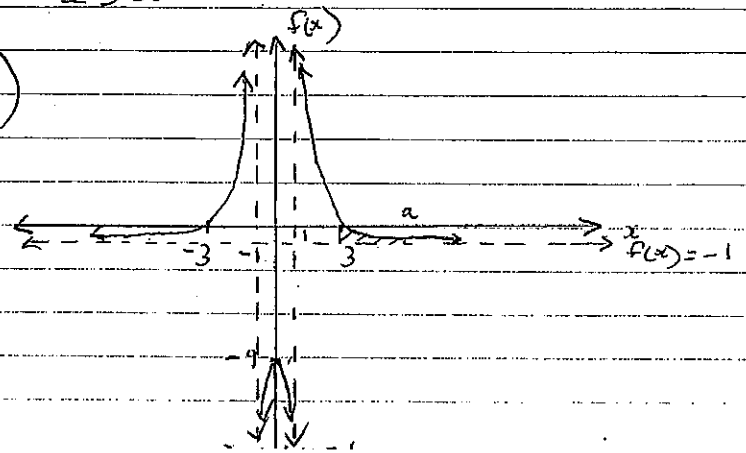
$= \frac{-4}{a+1} + \frac{4}{a-1} - 1$

$= \frac{4}{a-1} - \frac{4}{a+1} - 1$

Since $f(a) = f(-a)$ $f(x)$ is an even f^n

iii) $\lim_{x\to\infty} f(x) = -1$

iv)



v) $S = \int_3^a \frac{4}{x-1} - \frac{4}{x+1} - 1 \, dx$

$= \left[4\ln|x-1| - 4\ln|x+1| - 1 \right]_3^a$

$= \{ 4\ln|a-1| - 4\ln|a+1| - 1 - (4\ln 2 - 4\ln 4 - 1) \}$

$= 4\ln\left|\frac{a-1}{a+1}\right| - 1 - 4\ln\frac{1}{2} + 1$

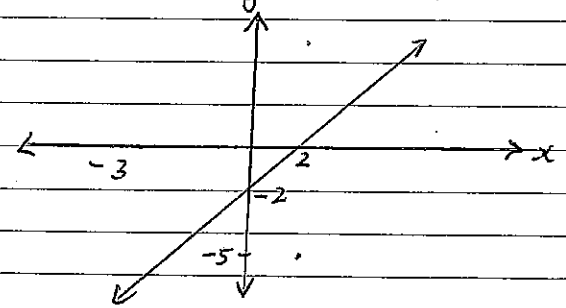
$= 4\ln\left|\left(\frac{a-1}{a+1}\right)\right| + 4\ln 2$

Since $a > 3$ $\frac{a-1}{a+1} < 1$ $\therefore \ln\left|\frac{a-1}{a+1}\right| < 0$

$\therefore S < 4\ln 2$

Q16

a) i) $|w+3| = |w-2+5i|$



Let $w = x+iy$

ii) $|x+3+iy| = |x-2+i(y+5)|$

$\sqrt{(x+3)^2 + y^2} = \sqrt{(x-2)^2 + (y+5)^2}$

$x^2 + 6x + 9 + y^2 = x^2 - 4x + 4 + y^2 + 10y + 25$

$10x - 10y - 20 = 0$

$x - y - 2 = 0$

b) i) $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$

Multiply b.s by $\frac{1}{n+1}$

$\frac{1}{n(n+1)} - \frac{1}{(n+1)^2} = \frac{1}{n(n+1)^2}$

$\frac{1}{(n+1)^2} = \frac{1}{n(n+1)} - \frac{1}{n(n+1)^2}$

Since $n \in \mathbb{Z}^+$ $\frac{1}{n(n+1)^2} > 0$

$\therefore \frac{1}{(n+1)^2} < \frac{1}{n(n+1)}$

Let the statement be

ii) $S_n = \sum_{r=1}^{n} \frac{1}{r^2} \le 2 - \frac{1}{n}, \quad n \ge 1$

When $n = 1$

LHS $S_1 = 1$ RHS $= 2 - \frac{1}{1}$

$= 1$

$\therefore$ Statement True for $n = 1$

Assume the statement is true for $n = k$

i.e $S_k = \sum_{r=1}^{k} \frac{1}{r^2} \le 2 - \frac{1}{k}; \quad k \ge 1$

RTP true for $n = k+1$

i.e $S_{k+1} = \sum_{r=1}^{k+1} \frac{1}{r^2} \le 2 - \frac{1}{k+1}; \quad k \ge 1$

Now $S_{k+1} = S_k + T_{k+1}$

$= \sum_{r=1}^{k} \frac{1}{r^2} + \frac{1}{(k+1)^2}$

$$\le 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

$$\le 2 - \frac{1}{k} + \frac{1}{k(k+1)} \quad \text{from } 1)$$

$$\le 2 - \frac{k-1+1}{k(k+1)}$$

$$\le 2 - \frac{k}{k(k+1)}$$

$$\le 2 - \frac{1}{k+1} \quad k \geq 1$$

Hence since the statement is true for $n=1$ and $n=k$ and $n=k+1$, the statement is true for $n=1+1=2$ and so on by the principle of mathematical induction

Q 16

(c) (i) $x^2 - x + k = 0$

since 2 distinct positive roots

$b^2 - 4ac > 0$

$(-1)^2 - 4 \times 1 \times k > 0$

$1 - 4k > 0 \quad (1)$

$4k < 1$

$\therefore \; k < 1/4$

Product of roots $= +k > 0$ (since both roots positive)

$\therefore \; 0 < k < 1/4$.

$\alpha^2 \quad \beta^2$

ie. $\frac{1}{\alpha^2} + \frac{1}{\beta^2} - 8 > 0$

$$\frac{\alpha^2 + \beta^2 - 8\alpha^2\beta^2}{\alpha^2\beta^2} > 0$$

$\therefore$ show numerator > 0 since $\alpha^2\beta^2 > 0$.

Now $\alpha^2 + \beta^2 - 8\alpha^2\beta^2$

$= (\alpha+\beta)^2 - 2\alpha\beta - 8\alpha^2\beta^2$

$= (1)^2 - 2k - 8k^2 \qquad (\alpha\beta = k, \; \alpha+\beta = 1)$

$= 1 - 2k - 8k^2$

$= (1 - 4k)(1 + 2k)$

From (1) $1 - 4k > 0$ and $1 + 2k > 0$ $(k > 0)$

$\therefore \; 1 - 2k - 8k^2 > 0$

and $\frac{1}{\alpha^2} + \frac{1}{\beta^2} > 8$

(d) (i) $I = \int_{-1}^{1} \frac{x^2 e^x}{e^x + 1}\,dx \qquad J = \int_{-1}^{1} \frac{x^2}{e^x+1}\,dx$

Let $u = -x \quad \frac{du}{dx} = -1$ when $x = -1 \; u = 1$, $x = 1 \; u = -1$

$$\therefore I = \int_{1}^{-1} \frac{u^2 e^{-u}}{e^{-u}+1} \, -du$$

$$= \int_{-1}^{1} \frac{u^2 e^{-u}}{e^{-u}+1}\,du$$

$$= \int_{-1}^{1} \frac{u^2 e^{-u}}{e^{-u}+1} \times \frac{e^u}{e^u}\,du$$

$$= \int_{-1}^{1} \frac{u^2}{1+e^u}\,du = J.$$

(ii) $I + J = \int_{-1}^{1} \frac{x^2 e^x + x^2}{e^x + 1} \, dx$

$= \int_{-1}^{1} \frac{x^2 (e^x + 1)}{(e^x + 1)} \, dx$

$= \left[\frac{x^3}{3} \right]_{-1}^{1}$

$= \frac{1}{3} - \frac{-1}{3} = \frac{2}{3} = 2I = 2J$

$\therefore \quad I = J = \frac{1}{3}.$