Ms Lau
Mrs Kerr

Name: ……………………………………..………..

Teacher:…………………………………………….

HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION
2016

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen. Black pen is preferred.
- Board approved calculators may be used.
- A reference sheet is provided.
- In Questions 11-16, show relevant mathematical reasoning and/or calculations.
- Start each question in a new booklet.

Total Marks – 100

Section I Pages 1-6

10 marks

- Attempt all Questions 1-10
- Allow about 15 mins for this section

Section II Pages 7-15

90 marks

- Attempt Questions 11-16
- Allow about 2 hour 45 minutes for this section

Mark	**/100**
Highest Mark	**/100**
Rank	

Section I

10 marks
Attempt Questions 1-10

Use the multiple choice answer sheet for Questions 1-10.

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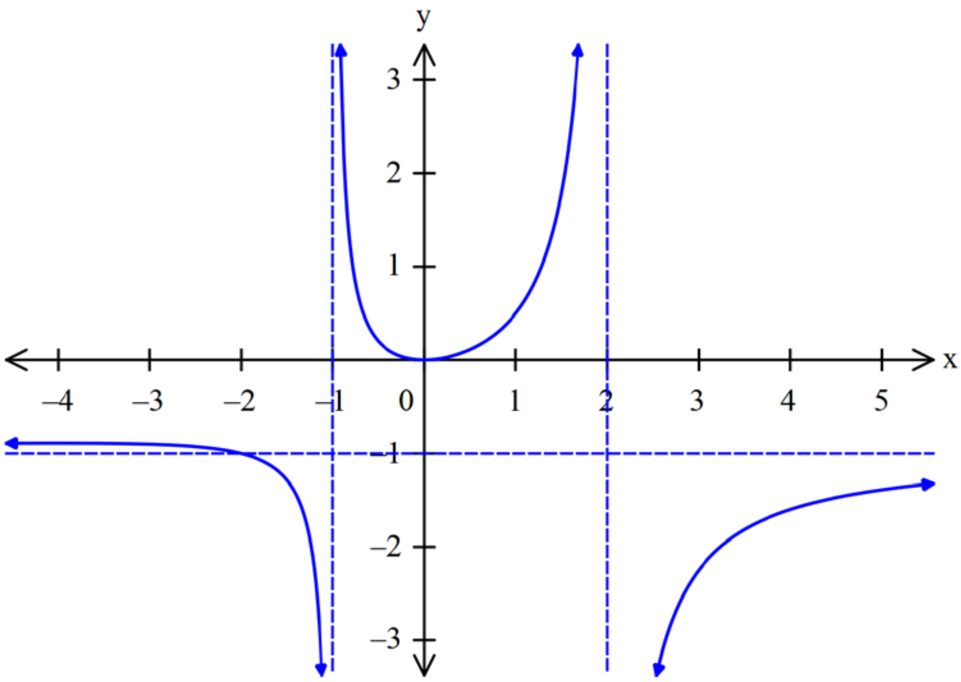
1 The graph of the function $y = \dfrac{g(x)}{(x-2)(x+1)}$ is shown below.



The function $g(x)$ could be which of the following?

(A) x

(B) $-x$

(C) x^2

(D) $-x^2$

2 What type of conic section is represented by the equation $y^2 - 4y - x + 3 = 0$?

(A) Hyperbola

(B) Circle

(C) Parabola

(D) Ellipse

3 What are the values of real numbers p and q such that $1-i$ is a root of the equation $z^3 + pz + q = 0$?

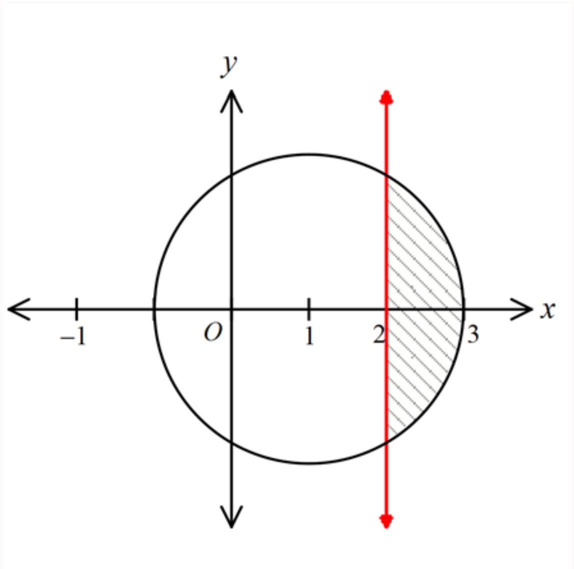
(A) $p = 2$ and $q = 4$.

(B) $p = 2$ and $q = -4$.

(C) $p = -2$ and $q = 4$.

(D) $p = -2$ and $q = -4$.

4 Consider the Argand diagram below.



Which inequality could define the shaded area?

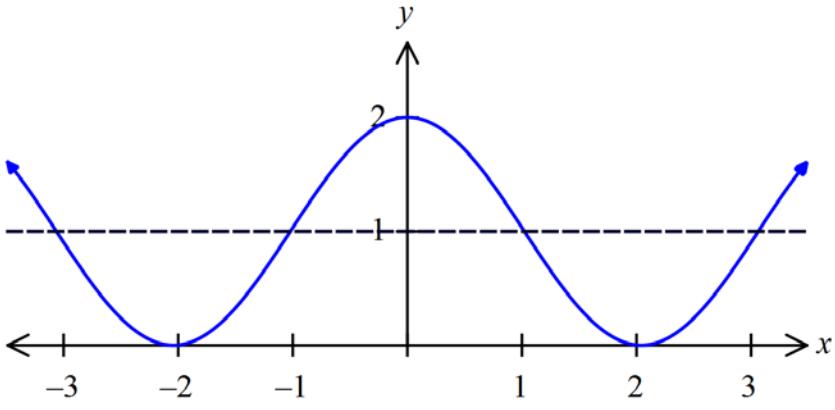
(A) $|z-1| \le 2$ and $\operatorname{Re}(z) \ge 2$.

(B) $|z-1| \le 2$ and $\operatorname{Im}(z) \ge 2$.

(C) $|z+1| \le 2$ and $\operatorname{Re}(z) \ge 2$.

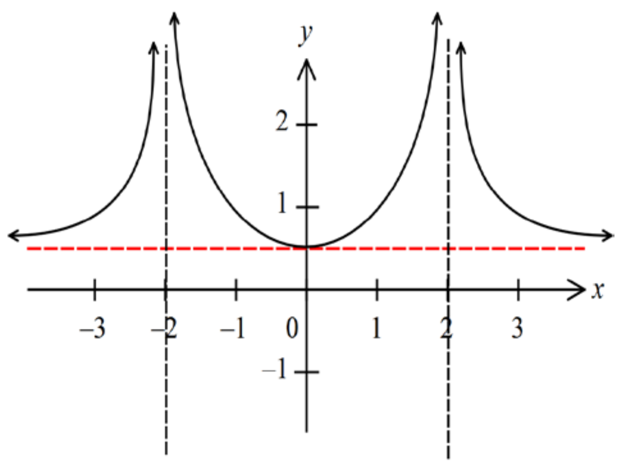
(D) $|z+1| \le 2$ and $\operatorname{Im}(z) \ge 2$.

5 The graph of the function $y = f(x)$ is shown below.

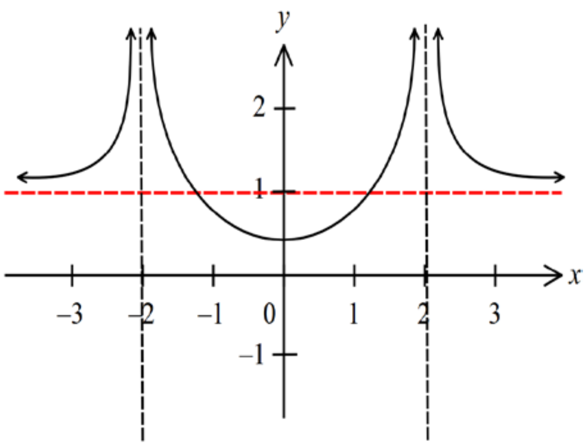


Which graph shows the most accurate representation of the function $y = \dfrac{1}{f(x)}$?

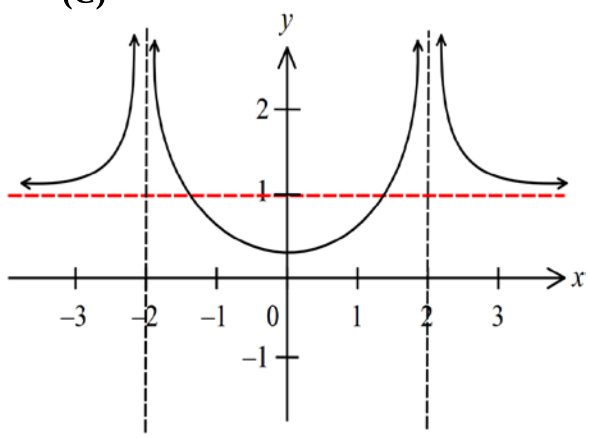
(A)



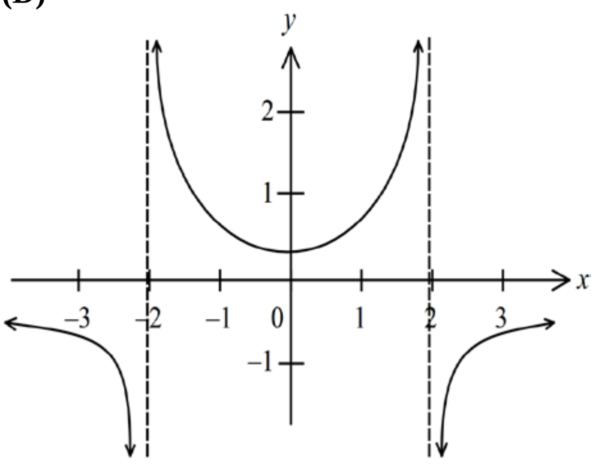
(B)



(C)



(D)



6 What is $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x \cos x \, dx$ equivalent to?

(A) $2\int_{0}^{\frac{\pi}{2}} x \cos x \, dx$

(B) 0

(C) $\pi - 2$

(D) $\frac{\pi}{2} - 1$

7 Which expression is equal to $\frac{1}{(x+1)(x^2+4)}$?

(A) $\frac{1}{3(x+1)} - \frac{x+1}{3(x^2+4)}$

(B) $\frac{1}{5(x+1)} - \frac{x-1}{5(x^2+4)}$

(C) $\frac{1}{5(x+1)} + \frac{x-1}{5(x^2+4)}$

(D) $\frac{1}{x+1} - \frac{x+4}{x^2+4}$

8 Which expression is equal to $\int \cos^3 x \, dx$?

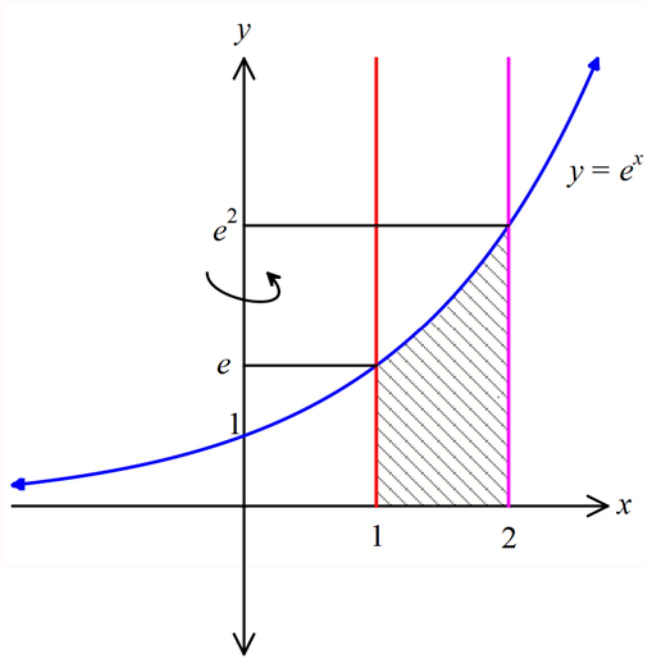
(A) $\frac{\cos^4 x}{4} + C$

(B) $3\cos^2 x + \sin x + C$

(C) $\sin x - \frac{\sin^3 x}{3} + C$

(D) $x - \frac{\sin^3 x}{3} + C$

9 The region bounded by the curve $y = e^x$, the x-axis, and the lines $x = 1$ and $x = 2$, is rotated around the y-axis to form a solid with volume V.



Which of the following is correct?

(A) $V = \pi \int_{e}^{e^2} \left[4 - (\ln y)^2\right] dy.$

(B) $V = 3\pi e + \pi \int_{e}^{e^2} \left[4 - (\ln y)^2\right] dy.$

(C) $V = 3\pi e + \pi \int_{e}^{e^2} (4 - 2\ln\, y)\, dx.$

(D) $V = \pi \int_{1}^{2} e^{2x}\, dx.$

10

$$J = \int_0^1 \sqrt{1 - x^4}\ dx$$

$$K = \int_0^1 \sqrt{1 + x^4}\ dx$$

$$L = \int_0^1 \sqrt{1 - x^8}\ dx$$

Which of the following is true for the definite integrals shown above?

(A) $J < L < 1 < K$

(B) $J < L < K < 1$

(C) $L < J < 1 < K$

(D) $L < J < K < 1$

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer each question in the appropriate writing booklet. Extra booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks). Use a Separate Booklet. **Marks**

(a) Write the complex number $\left(\dfrac{4i^7 - i}{1+2i}\right)^2$ in the form $a+bi$ where a and b are real numbers. **2**

(b) Using integration by parts, find $\int_0^1 x\, e^{2x} dx$. **2**

(c) Find the value of $\dfrac{dy}{dx}$ at the point $(1,\ 4)$ on the curve $4x^3 + xy^2 = 5xy$. **3**

(d) If $z = 4+2i$ and $w = -1+3i$, find $\arg(zw)$. **2**

Question 11 continues on page 8.

(e) On an Argand diagram shade the region that is satisfied by both the conditions. **2**

$$0 \le Arg\left(z+i\right) \le \frac{\pi}{4} \text{ and } \left|z-1\right| \le \sqrt{2}\ .$$

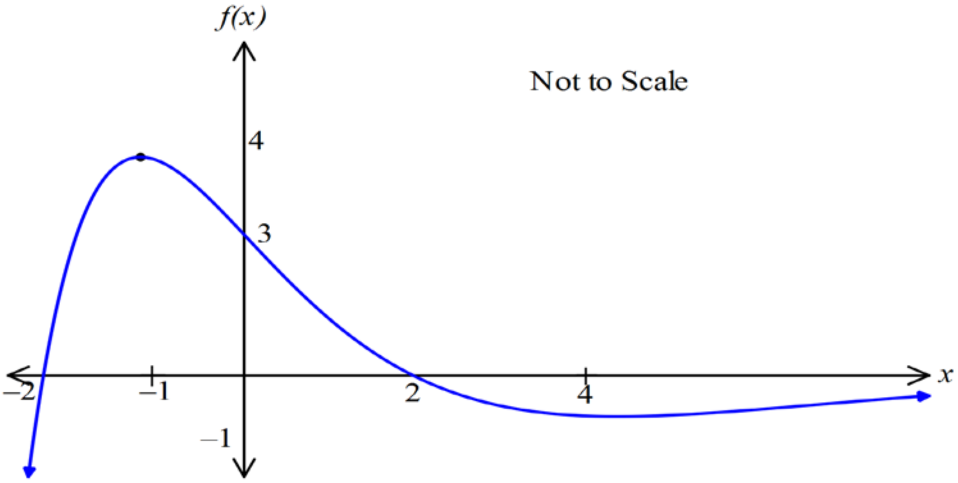
(f) If α, β and γ are roots of the equation $x^3+6x+1=0$, find the polynomial equation whose roots are $\alpha\beta, \beta\gamma$ and $\alpha\gamma$. **2**

(g) Find $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \cos^3 x \, dx$. **2**

End of Question 11

Question 12 (15 marks). Use a Separate Booklet. **Marks**

(a) Given the sketch of the function $f(x)$, draw a one-third page sketch of each of the following on separate diagrams.



(i) $y = -f(x)$. **1**

(ii) $y = \sqrt{f(x)}$. **1**

(iii) $y = f(x^2)$. **2**

(b) If $2x^4 + 9x^3 + 6x^2 - 20x - 24 = 0$ has a root of multiplicity of 3, factorise $2x^4 + 9x^3 + 6x^2 - 20x - 24$ fully. **2**

Question 12 continues on page 10.

Question 12 (continued). **Marks**

(c) The roots of the equation $z^3 + 2z^2 + 3z - 4 = 0$ are $\alpha,\ \beta$ and γ.

(i) *(1)* Write down the value of $\alpha + \beta + \gamma$ and the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. **1**

(2) Hence show that $\alpha^2 + \beta^2 + \gamma^2 = -2$. **1**

(ii) Find the value of $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$. **2**

(iii) Find a cubic equation whose roots are $\alpha + \beta,\ \beta + \gamma$ and $\gamma + \alpha$. **2**

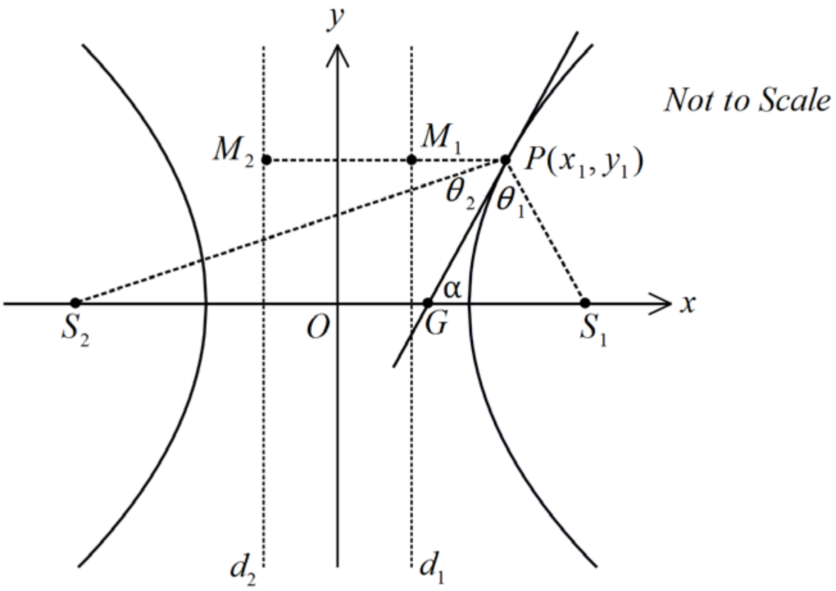
(d) Using the substitution $n = 1 - x$, evaluate $\displaystyle\int_{-1}^{0} \frac{x}{(1-x)^4}\, dx$. **3**

End of Question 12

Question 13 (15 marks). Use a Separate Booklet. **Marks**

(a) The area enclosed by the curve $y=(x-3)^2$ and the line $y=9$ is rotated about the y-axis. Use the method of cylindrical shells to find the exact volume of the solid formed. **4**

(b)



The point $P(x_1, y_1)$ lies on the hyperbola $\frac{x^2}{25}-\frac{y^2}{9}=1$.

The two foci of the hyperbola are S_1 and S_2 and the two directrices are d_1 and d_2, as shown.

(i) Show that the length $S_1P=\frac{\sqrt{34}}{5}x_1-5$. **2**

(ii) Show that the equation of the tangent at P is $\frac{x_1 x}{25}-\frac{y_1 y}{9}=1$. **2**

(iii) The tangent at P intersects the transverse axis at point G. Find the coordinates of point G. **1**

(iv) Given $\angle S_1PG=\theta_1$, $\angle GPS_2=\theta_2$ and $\angle S_1GP=\alpha$,

(1) By using the sine rule, show that $\sin\alpha=\frac{x_1\sin\theta_1}{5}$. **2**

(2) Hence, show that $\sin\theta_1=\sin\theta_2$. **2**

(3) Hence, deduce that GP bisects $\angle S_1PS_2$. **2**

End of Question 13

Question 14 (15 marks). Use a Separate Booklet. **Marks**

(a) Find all real x such that $3\sqrt{x(1-x)} < |x-2|$. **3**

(b) (i) Show that $x^4 + y^4 \geq 2x^2y^2$. **1**

(ii) If $P(x, y)$ is any point on the curve $x^4 + y^4 = 1$, prove that $OP \leq 2^{\frac{1}{4}}$ where O is the origin. **3**

(c) (i) *(1)* Use De Moivre's Theorem to show that if $z = \cos\theta + i\sin\theta$, then **1**

$$z^n - \frac{1}{z^n} = 2i\sin n\theta.$$

(2) Write down a similar expression for $z^n + \dfrac{1}{z^n}$. **1**

(ii) *(1)* Expand $\left(z - \dfrac{1}{z}\right)^2 \left(z + \dfrac{1}{z}\right)^2$ in terms of z. **1**

(2) Hence, show that $8\sin^2\theta\cos^2\theta = A + B\cos 4\theta$, where A and B are integers. **2**

(iii) Hence, by means of the substitution $x = 2\sin\theta$, find the exact value of **3**

$$\int_1^2 x^2\sqrt{4-x^2}\,dx.$$

End of Question 14

Question 15 (15 marks). Use a Separate Booklet. **Marks**

(a) Draw a half page sketch of $y = \log_e |\tan x|$ for the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. **2**

(b) (i) Use the factor theorem to show that $2x - 1$ is a factor of $8x^3 - 4x + 1$. **1**

(ii) Show that $4\cos 2\theta \cos\theta + 1$ can be written as $8x^3 - 4x + 1$ where $x = \cos\theta$. **1**

(iii) Given that $\theta = 72°$ is a solution of $4\cos 2\theta \cos\theta + 1 = 0$, use the results from parts (i) and (ii) to show that the exact value of $\cos 72°$ is $\frac{\left(\sqrt{5} - 1\right)}{p}$ where p is a constant. **3**

(c) (i) Express $(k+1)^2 + 5(k+1) + 8$ in the form $k^2 + ak + b$, where a and b are constants. **1**

(ii) Prove by induction that, for all integers $n \ge 1$, **3**

$$\sum_{r=1}^{n} r(r+1)\left(\frac{1}{2}\right)^{r-1} = 16 - \left(n^2 + 5n + 8\right)\left(\frac{1}{2}\right)^{n-1}.$$

(d) Given that $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^n x \, dx$, for $n = 1, 2, \ldots$.

(i) Show that $I_1 = \frac{1}{2}\ln 2$. **1**

(ii) Show that $I_{n-2} + I_n = \frac{1}{n-1}\left(3^{\frac{n-1}{2}} - 1\right)$ for $n = 3, 4, 5\ldots$ **3**

End of Question 15

Question 16 (15 marks). Use a Separate Booklet. **Marks**

(a) The equation $z^4 + 4iz^3 - 4iz^2 + (8 + 8i)z - 32(1+i) = 0$ has roots $\alpha, \beta, -2\alpha, \gamma$ which represent the vertices A, B, C and D of a parallelogram in the Argand plane.

(i) Using the properties of a parallelogram, show that $\alpha + \beta + \gamma = 0$. **2**

(ii) Hence, or otherwise, show that $\alpha = 2i$. **1**

(iii) Given $\sqrt{3 + 4i} = \pm(2 + i)$, find the vertices of the parallelogram $ABCD$. **3**

(b) (i) Show that $1 + \cos 2\theta + i \sin 2\theta = 2\cos\theta(\cos\theta + i\sin\theta)$. **1**

(ii) Hence, prove that $\left(1 + \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n = -2^n\cos^n\frac{\pi}{n}$ where n is any integer. **1**

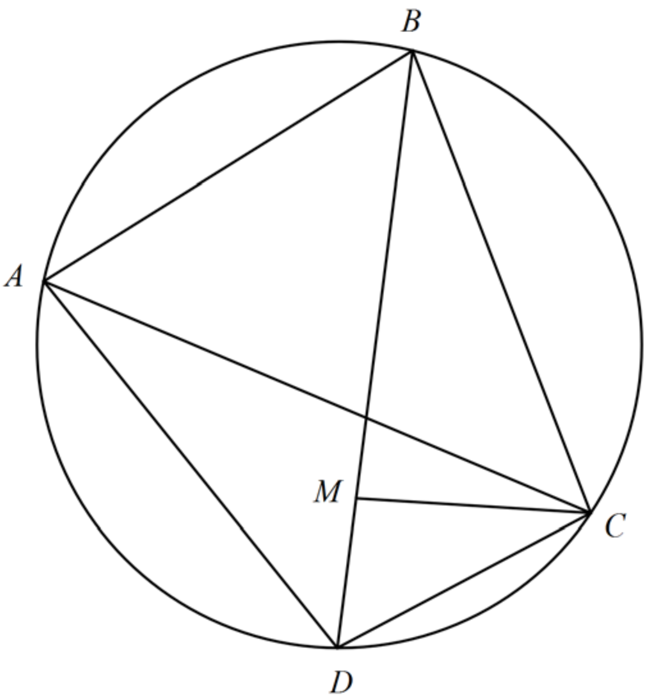
(iii) Hence, simplify $\left(1 + \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n - \left(1 + \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}\right)^n$. **2**

Question 16 continues on page 15

(c) (i) Ptolemy's Theorem states that in a cyclic quadrilateral the product of the diagonals is equal to the sum of the products of the pairs of opposite sides, i.e. $AC \times BD = AB \times CD + BC \times AD$. **3**

M is the point on BD such that $\angle ACB = \angle DCM$.

Prove Ptolemy's theorem.



(ii) Hence, if $AB = AD$, $\angle BCD$ is a right angle and the area of the quadrilateral $ABCD$ is 18 cm^2, find the length of AC. **2**

End of Paper

Extension 2 2016 Trial solutions

1. D
2. C
3. C
4. A
5. A
6. B
7. B
8. C
9. B
10. A

Question 11

a) $\left(\frac{4i^7 - i}{1+2i}\right)^2$

$= \left(\frac{-5i}{1+2i}\right)^2$ ① → For getting to a point where realisation can happen

$= \left[\frac{-5i(1-2i)}{1+4}\right]^2$

$= [-i(1-2i)]^2$

$= (-2-i)^2$

$= 4 - 1 + 4i$

$= 3 + 4i$ ① Answer

b) $\int_0^1 x e^{2x}\, dx$

$= \left[\frac{1}{2}x e^{2x}\right]_0^1 - \frac{1}{2}\int_0^1 e^{2x}\, dx$ ① correct int by parts

$= \left(\frac{1}{2}e^2 - 0\right) - \frac{1}{2}\left[\frac{1}{2}e^{2x}\right]_0^1$

$= \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4}e^0$

$= \frac{1}{4}e^2 + \frac{1}{4}$ ① Answer

c) $4x^3 + xy^2 = 5xy$

① $12x^2 + 1\cdot y^2 + 2yx\frac{dy}{dx} = 5\cdot y + 5x\cdot 1\frac{dy}{dx}$

$12x^2 + y^2 - 5y = (5x - 2xy)\frac{dy}{dx}$

① $\frac{dy}{dx} = \frac{12x^2 + y^2 - 5y}{5x - 2xy}$

When $x = 1$, $y = 4$:

$\frac{dy}{dx} = \frac{12 + 4^2 - 5(4)}{5 - 2(1)(4)}$

① $= \frac{-8}{3}$

d) $z = 4 + 2i \qquad \omega = -1 + 3i$

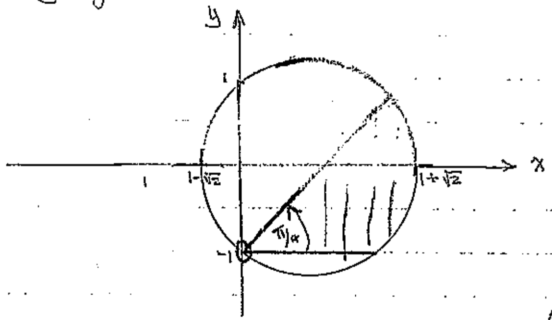
$z\omega = (4 + 2i)(-1 + 3i)$

$= -4 - 6 - 2i + 12i$

$= -10 + 10i$ — ① For product

$\arg z\omega = \frac{3\pi}{4}$ — ① For argument

e)



① for correctly placed circle

① Arc with open circle correctly placed

f) $x^3 + 6x + 1 = 0$

$\alpha\beta\gamma = -1$

Equation with roots $\alpha\beta$, $\beta\gamma$ and $\alpha\gamma$

$= \frac{\alpha\beta\gamma}{\gamma}, \frac{\alpha\beta\gamma}{\alpha}$ and $\frac{\alpha\beta\gamma}{\beta}$

$= \frac{-1}{\gamma}, \frac{-1}{\alpha}$ and $\frac{-1}{\beta}$

$\left(\frac{-1}{x}\right)^3 + 6\left(\frac{-1}{x}\right) + 1 = 0$

$\frac{-1}{x^3} - \frac{6}{x} + 1 = 0$ ① – Appropriate method.

$-1 - 6x^2 + x^3 = 0$

$x^3 - 6x^2 - 1 = 0$ ① correct eq^n

g) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \cos^3 x \, dx$

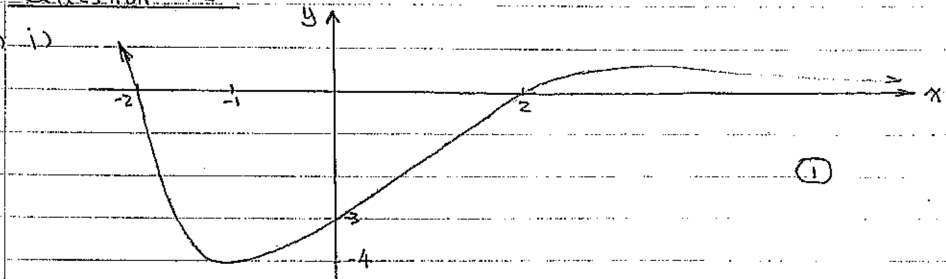
$= \frac{-1}{4} \cos^4 x \Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ — ① correct integral

$= -0 + \frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^4$

$= \frac{1}{16}$ — ① correct answer

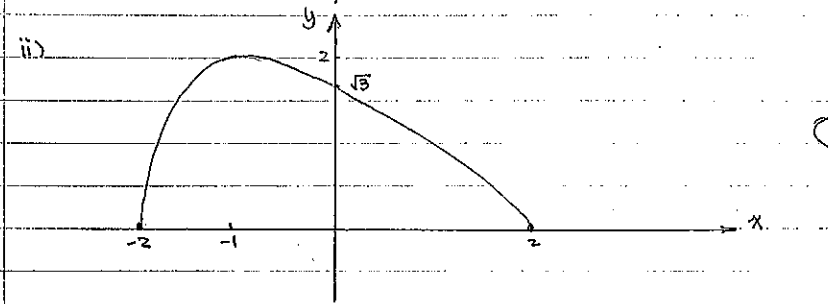
Question 12

a) i)



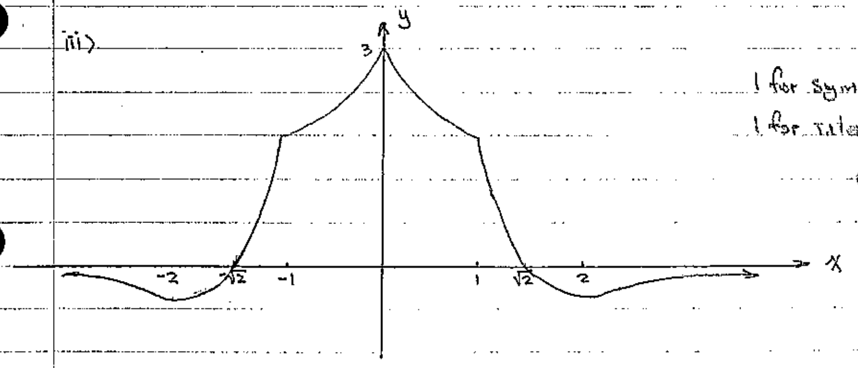
①

ii)



①

iii)



1 for symmetry

1 for intercepts

②

b) Let $P(x) = 2x^4 + 9x^3 + 6x^2 - 20x - 24$

$P'(x) = 8x^3 + 27x^2 + 12x - 20$

$P''(x) = 24x^2 + 54x + 12 = 0$

$4x^2 + 9x + 2 = 0$

$(4x + 1)(x + 2) = 0$

$x = -2$ OR $x = \frac{-1}{4}$

$P(\frac{-1}{4}) \neq 0 \qquad P(-2) = 0$

$P(x) = 2x^4 + 9x^3 + 6x^2 - 20x - 24$

②

c) $z^3 + 2z^2 + 3z - 4 = 0$

i) $\alpha + \beta + \gamma = -2$] (1)

$\alpha\beta + \beta\gamma + \alpha\gamma = 3$

$\alpha^2 + \beta^2 + \gamma^2$

$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= (-2)^2 - 2(3)$

$= -2$ (1)

ii) $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

$= 2\alpha\beta\gamma + \beta^2\gamma + \alpha\gamma^2 + \beta\gamma^2 + \alpha^2\beta + \beta^2\alpha + \alpha^2\gamma$

$= 2\alpha\beta\gamma + \alpha\beta(\alpha+\beta) + \alpha\gamma(\alpha+\gamma) + \beta\gamma(\beta+\gamma)$

$= 2\alpha\beta\gamma + \alpha\beta(-2-\gamma) + \alpha\gamma(-2-\beta) + \beta\gamma(-2-\alpha)$

$= 2\alpha\beta\gamma - 2(\alpha\beta + \beta\gamma + \alpha\gamma) - 3\alpha\beta\gamma$

$= -2(3) - 1(4)$

$= -10$ 1 (2)

iii) Equation with roots $\alpha+\beta$, $\beta+\gamma$ and $\alpha+\gamma$

$= \alpha+\beta+\gamma-\gamma$, $\alpha+\beta+\gamma-\alpha$ and $\alpha+\beta+\gamma-\beta$

$= -2-\alpha$, $-2-\beta$ and $-2-\gamma$

$(-2-z)^3 + 2(-2-z)^2 + 3(-2-z) - 4 = 0$ 1

$-(z^3 + 6z^2 + 12z + 8) + 2(z^2 + 4z + 4)$

$-6 - 3z - 4 = 0$

$z^3 + 4z^2 + 7z + 10 = 0$ 1 (2)

d) $\int_{-1}^{0} \frac{x}{(1-x)^4}\,dx$; $n = 1 - x \Rightarrow x = 1 - n$

$dn = -dx$

$x = 0,\ n = 1 - 0 = 1$

$x = -1,\ n = 1 - (-1) = 2$

$= \int_2^1 \frac{n-1}{n^4}\,dn$ 1

$= \int_2^1 (n^{-3} - n^{-4})\,dn$

$= \frac{-1}{2}n^{-2} + \frac{1}{3}n^{-3} \Big]_2^1$ 1

$= \left(\frac{-1}{2} + \frac{1}{3}\right) + \frac{1}{2}\times\frac{1}{4} - \frac{1}{3}\times\frac{1}{8}$

$= \frac{-1}{12}$ 1 (3)

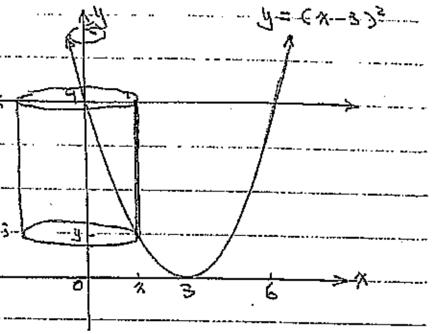
Q13

a) $\delta V = \pi(x+\delta x)^2(9-y) - \pi x^2(9-y)$

$= \pi(9-y)\left[(x+\delta x)^2 - x^2\right]$

$= \pi(9-y)(2x+\delta x)\delta x$

(1) $V \simeq \sum_{x=0}^{6} \pi\left[9-(x-3)^2\right]2x\,\delta x$, ignore δx^2



(1) $V = 2\pi\int_0^6 x^2(6-x)\,dx$

$= 2\pi\int_0^6 (6x^2 - x^3)\,dx$ (1) – correct limits and set up

$= 2\pi\left[2x^3 - \frac{1}{4}x^4\right]_0^6$

$= 2\pi\left[2\times 6^3 - \frac{1}{4}\times 6^4 - 0\right]$

(1) $= 216\pi$ cubic units

$dV = 2\pi x(9-y)\,dx$

$= 2\pi x(9-(x-3)^2)\,dx$

$= 2\pi x(9-(x^2-6x+9))\,dx$

$= 2\pi x(6x - x^2)\,dx$

$= 2\pi(6x^2 - x^3)\,dx$

b) $\frac{x^2}{25} - \frac{y^2}{9} = 1$

i) $e = \sqrt{\frac{9}{25}+1}$

$= \frac{\sqrt{34}}{5}$

$M_1 = \left(5\times\frac{5}{\sqrt{34}}, y_1\right) = \left(\frac{25}{\sqrt{34}}, y_1\right)$

$\frac{PS_1}{PM_1} = e$ (1) correctly finding e and the x coordinate of M_1 or the eqⁿ of d_1

$PS_1 = e \times PM_1$

$= \frac{\sqrt{34}}{5}\left(x_1 - \frac{25}{\sqrt{34}}\right)$

$= \frac{\sqrt{34}}{5}x_1 - \frac{\sqrt{34}}{5}\times\frac{25}{\sqrt{34}}$

$= \frac{\sqrt{34}}{5}x_1 - 5$ (1) correct demonstration

ii) $\frac{2x}{25} - \frac{2y}{9}\times\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{2x}{25}\times\frac{9}{2y}$

$= \frac{9x}{25y}$

$m_P = \frac{9x_1}{25y_1}$ (1) Derivative

Equation of tangent at P

$\Rightarrow \frac{9x_1}{25y_1} = \frac{y-y_1}{x-x_1}$ (1) obtaining eqⁿ using a correct method

$9x_1x - 9x_1^2 = 25y_1y - 25y_1^2$

$\frac{9x_1x}{9\times 25} - \frac{25y_1y}{9\times 25} = \frac{9x_1^2}{9\times 25} - \frac{25y_1^2}{9\times 25}$

$\frac{x_1x}{25} - \frac{y_1y}{9} = \frac{x_1^2}{25} - \frac{y_1^2}{9}$

$\frac{x_1x}{25} - \frac{y_1y}{9} = 1$; since $P(x_1, y_1)$ lies on the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$

So $\frac{x_1^2}{25} - \frac{y_1^2}{9} = 1$

iii) When $y = 0$,

$\frac{x_1x}{25} - 0 = 1$

$x_1x = 25$

$x = \frac{25}{x_1}$

$G = \left(\frac{25}{x_1}, 0\right)$

1/r/w

iv) (1) $GS_1 = \sqrt{34} - \frac{25}{x_1}$

In $\triangle GS_1P$,

$\frac{PS_1}{\sin\angle S_1GP} = \frac{GS_1}{\sin\angle S_1PG}$

$\frac{\frac{\sqrt{34}}{5}x_1 - 5}{\sin\alpha} = \frac{\sqrt{34} - \frac{25}{x_1}}{\sin\theta_1}$ (1) Correct substitution of relevant information into sine rule

$\sin\alpha = \frac{\left(\frac{\sqrt{34}}{5}x_1 - 5\right)\sin\theta_1}{\sqrt{34} - \frac{25}{x_1}}$

$= \frac{\frac{\sqrt{34}x_1 - 25}{5}}{\frac{\sqrt{34}x_1 - 25}{x_1}}\sin\theta_1$

$= \frac{x_1\sin\theta_1}{5}$ (1) Correct manipulation to achieve the show

(2) $M_2 = \left(\frac{-25}{\sqrt{34}}, y_1\right)$

$\frac{PS_2}{PM_2} = e$

$PS_2 = \frac{\sqrt{34}}{5}\left(x_1 + \frac{25}{\sqrt{34}}\right)$

$= \frac{\sqrt{34}}{5}x_1 + 5$

$GS_2 = \frac{25}{x_1} + \sqrt{34}$

② correct argument

① Heading toward correct argument

In $\triangle S_2PG$,

$$\frac{GS_2}{\sin\angle GPS_2} = \frac{PS_2}{\sin\angle S_2GP}$$

$$\frac{\frac{25}{x_1} + \sqrt{34}}{\sin\theta_2} = \frac{\frac{\sqrt{34}}{5}x_1 + 5}{\sin(180^\circ - \alpha)}$$

$$\sin(180^\circ - \alpha) = \frac{\frac{\sqrt{34}x_1 + 25}{5}}{\frac{25 + \sqrt{34}x_1}{x_1}} \times \sin\theta_2$$

$$\sin\alpha = \frac{x_1}{5}\sin\theta_2 \quad \text{—— } *$$

$$\frac{x_1}{5}\sin\theta_1 = \frac{x_1}{5}\sin\theta_2 \quad \text{; equating (1) and } *$$

$$\sin\theta_1 = \sin\theta_2$$

(3) From (2), $\sin\theta_1 = \sin\theta_2$,

then $\theta_1 = \theta_2$ OR $\theta_1 = 180^\circ - \theta_2$

If $\theta_1 = 180^\circ - \theta_2$,

then $\angle GPS_2 + \angle S_1PG = \angle S_1PS_2$

$= \theta_2 + \theta_1$

$= \theta_2 + 180^\circ - \theta_2$

$= 180^\circ$

$\angle S_1PS_2 = 180^\circ$ iff P lies on the x axis

But if P lies on the x axis, GP cannot bisect $\angle S_1PS_2$ since they all lie on the x axis.

So $\theta_1 \neq 180^\circ - \theta_2$,

$\theta_1 = \theta_2 \Rightarrow \angle S_1PG = \angle GPS_2$,

$\therefore$ GP bisects $\angle S_1PS_2$.

② Correct argument

① Partially correct

Q14

a) $3\sqrt{x(1-x)} < |x-2|$

$3\sqrt{x(1-x)} < \sqrt{(x-2)^2}$

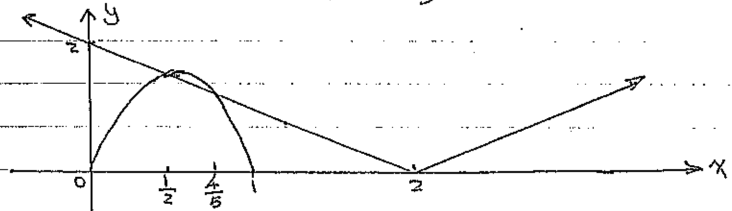
If $9x(1-x) = x^2 - 4x + 4$

$9x - 9x^2 = x^2 - 4x + 4$

$10x^2 - 13x + 4 = 0$

$(2x-1)(5x-4) = 0$

then $x = \frac{1}{2}$ OR $x = \frac{4}{5}$ 1



$0 < x < \frac{1}{2}$ OR $\frac{4}{5} < x < 1$ ③

1 1

b) i) $(x^2 - y^2)^2 \geq 0$ ①

$x^4 - 2x^2y^2 + y^4 \geq 0$

$x^4 + y^4 \geq 2x^2y^2$

ii) $OP = \sqrt{x^2 + y^2}$

$OP^4 = (x^2 + y^2)^2$ 1

$= x^4 + y^4 + 2x^2y^2$

$\leq x^4 + y^4 + x^4 + y^4$] 1

$= 1 + 1$; $x^4 + y^4 = 1$

$= 2$

$OP^4 \leq 2$ 1

$OP \leq 2^{\frac{1}{4}}$ ③

c) i) (1) $z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$

$z^n - \frac{1}{z^n} = z^n - z^{-n}$

$= (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)$

$= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$ ①

$= 2i\sin n\theta$

(2) $z^n + \frac{1}{z^n} = 2\cos n\theta$ 1

ii) (2) $(z - \frac{1}{z})^2 (z + \frac{1}{z})^2$

$= (2i\sin\theta)^2 (2\cos\theta)^2$

$= -4\sin^2\theta \cdot 4\cos^2\theta$

$= -16\sin^2\theta\cos^2\theta$ 1 —— *

$-16\sin^2\theta\cos^2\theta = z^4 + \frac{1}{z^4} - 2$, equating (1) and *

$-16\sin^2\theta\cos^2\theta = 2\cos 4\theta - 2$

$8\sin^2\theta\cos^2\theta = \frac{-1}{2}(2\cos 4\theta - 2)$

$= -\cos 4\theta + 1$

$= 1 - \cos 4\theta$ 1 ②

(1) $(z - \frac{1}{z})^2 (z + \frac{1}{z})^2$

$= [(z - \frac{1}{z})(z + \frac{1}{z})]^2$

$= (z^2 - \frac{1}{z^2})^2$

$= z^4 + \frac{1}{z^4} - 2$ ①

iii) $\int_1^2 x^2\sqrt{4 - x^2}\,dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\sin^2\theta\sqrt{4 - 4\sin^2\theta}\cdot 2\cos\theta\,d\theta$ | $x = 2\sin\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\sin^2\theta \cdot 2\cos\theta \cdot 2\cos\theta\,d\theta$ | $dx = 2\cos\theta\,d\theta$

$x = 1 = 2\sin\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 16\sin^2\theta\cos^2\theta\,d\theta$ 1 | $\Rightarrow \theta = \frac{\pi}{6}$

$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 - \cos 4\theta)\,d\theta$ | $x = 2 = 2\sin\theta$

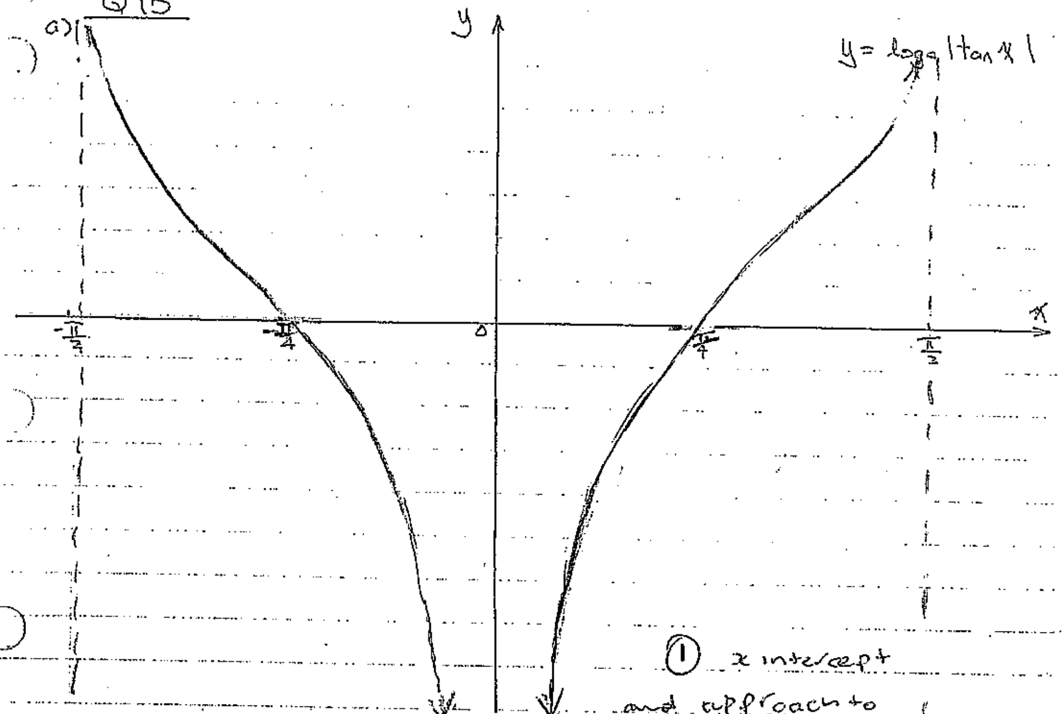
$= 2\left[\theta - \frac{1}{4}\sin 4\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ 1 | $\Rightarrow \theta = \frac{\pi}{2}$

$= 2\left[\frac{\pi}{2} - \frac{1}{4}\sin 2\pi - \frac{\pi}{6} + \frac{1}{4}\sin\frac{2\pi}{3}\right]$

$= \frac{2\pi}{3} + \frac{\sqrt{3}}{4}$ 1 ③

Q15

a)



① x intercept and approach to y axis.

① Behaviour at $\frac{\pi}{2}$

b) let $P(x) = 8x^3 - 4x + 1$

i) $P(\frac{1}{2}) = 8(\frac{1}{2})^3 - 4(\frac{1}{2}) + 1$

$= 1 - 2 + 1$

$= 0$ ① - substitution of ½ to get 0

So $(2x - 1)$ is a factor of $P(x)$.

ii) $4\cos 2\theta\cos\theta + 1$

$= 4\cos\theta(2\cos^2\theta - 1) + 1$ ① Demonstration by substituting for $\cos 2\theta$.

$= 8\cos^3\theta - 4\cos\theta + 1$

$= 8x^3 - 4x + 1$, $x = \cos\theta$.

iii) $P(x) = 8x^3 - 4x + 1 = 0$

$(2x - 1)(4x^2 + 2x - 1) = 0$

$x = \frac{1}{2}$ or $x = \frac{-2 \pm \sqrt{4 - 4(4)(-1)}}{2(4)}$

$= \frac{-2 \pm 2\sqrt{5}}{8}$

$= \frac{1}{4}(-1 \pm \sqrt{5})$

① factorising and finding the other two roots

Since $\theta = 72^\circ$ is a solution of $4\cos 2\theta \cos\theta + 1 = 0$,
$x = \cos 72^\circ$ is a solution of $P(x) = 8x^3 - 4x + 1$
However $\cos 72^\circ \neq \frac{1}{2}$ and $\cos 72^\circ > 0$,
so $\cos 72^\circ = \frac{1}{4}(-1 + \sqrt{5})$

c) i) $(k+1)^2 + 5(k+1) + 8$
$= k^2 + 2k + 1 + 5k + 5 + 8$ 1 r/w
$= k^2 + 7k + 14$

ii) Let the statement be $\sum_{r=1}^{n} r(r+1)(\frac{1}{2})^{r-1} = 16 - (n^2 + 5n + 8)(\frac{1}{2})^{n-1}$.

Step 1: Show that the statement is true for n = 1.
LHS $= 1(1+1)(\frac{1}{2})^{1-1}$
① - correct setup showing link to part i) $= 2 \times \frac{1}{2}^0$
$= 2$
② - Body correct RHS w.m conclusion RHS $= 16 - (1^2 + 5\times 1 + 8)(\frac{1}{2})^{1-1}$
$= 16 - 14$
$= 2$
$= LHS$

Step 2: Assume statement is true for n = k;
i.e. assume $\sum_{r=1}^{k} r(r+1)(\frac{1}{2})^{k-1} = 16 - (k^2 + 5k + 8)(\frac{1}{2})^{k}$

Step 3: Prove statement is true for n = k+1;
i.e. prove $\sum_{r=1}^{k+1} r(r+1)(\frac{1}{2})^{k-1} = 16 - [(k+1)^2 + 5(k+1) + 8](\frac{1}{2})^k$

LHS $= \sum_{r=1}^{k+1} r(r+1)(\frac{1}{2})^{r-1}$
$= 1(2)(\frac{1}{2})^0 + 2(3)(\frac{1}{2})^1 + \dots + k(k+1)(\frac{1}{2})^{k-1} + (k+1)(k+2)(\frac{1}{2})^k$
$= 16 - (k^2 + 5k + 8)(\frac{1}{2})^{k-1} + (k+1)(k+2)(\frac{1}{2})^k$
$= (\frac{1}{2})^k [16(\frac{1}{2})^{-k} - (k^2 + 5k + 8)(\frac{1}{2})^{-1} + (k+1)(k+2)]$
$= (\frac{1}{2})^k [16 \times 2^k - 2(k^2 + 5k + 8) + k^2 + 3k + 2]$
$= (\frac{1}{2})^k [16 \times 2^k - 2k^2 - 10k - 16 + k^2 + 3k + 2]$
$= (\frac{1}{2})^k (16 \times 2^k - k^2 - 7k - 14)$
$= (\frac{1}{2})^k [16 \times 2^k - (k^2 + 7k + 14)]$
$= (\frac{1}{2})^k [16 \times 2^k - ((k+1)^2 + 5(k+1) + 8)]$; from (i)
$= 2^{-k} \times 16 \times 2^k - (\frac{1}{2})^k [(k+1)^2 + 5(k+1) + 8]$
$= 16 - [(k+1)^2 + 5(k+1) + 8](\frac{1}{2})^k$
$=$ RHS

Step 4 :

d) $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^n x \, dx$

i) $I_1 = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot x \, dx$
$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos x}{\sin x} \, dx$
$= [\ln(\sin x)]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$
$= \ln(\sin\frac{\pi}{4}) - \ln(\sin\frac{\pi}{6})$
$= \ln(\frac{1}{\sqrt{2}}) - \ln(\frac{1}{2})$
$= \ln 2^{-\frac{1}{2}} - \ln 2^{-1}$
$= -\frac{1}{2}\ln 2 + \ln 2$
$= \frac{1}{2}\ln 2$ 1 r/w

ii) $I_{n-2} + I_n$
$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^{n-2} x \, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^n x \, dx$
$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\cot^{n-2} x + \cot^n x) \, dx$
$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^n x (1 + \cot^{-2} x) \, dx$
$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^n x (1 + \tan^2 x) \, dx$ — ①
$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \tan^{-n} x \sec^2 x \, dx$
$= \frac{1}{1-n} \tan^{1-n} x \Big]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$ — ①
$= \frac{1}{1-n} [\tan^{1-n}\frac{\pi}{4} - \tan^{1-n}\frac{\pi}{6}]$
$= \frac{1}{1-n} [1 - (\frac{1}{\sqrt{3}})^{1-n}]$
$= \frac{1}{1-n} [1 - \sqrt{3}^{\,n-1}]$
$= \frac{-1}{1-n} (\sqrt{3}^{\,n-1} - 1)$
$= \frac{1}{n-1} (3^{\frac{n-1}{2}} - 1)$

① follow through

Q16

a) $z^4 + 4iz^3 - 4iz^2 + (8i+8)z - (32+32i) = 0$

i) Diagonals of a parallelogram bisect each other,
i.e midpoint of AC = midpoint of BD 1

$\frac{1}{2}(\alpha - 2\alpha) = \frac{1}{2}(\beta + \gamma)$]1

$-\alpha = \beta + \gamma$ —— *

$\alpha + \beta + \gamma = 0$ (2)

ii) Sum of roots $= \alpha + \beta - 2\alpha + \gamma = \frac{-4i}{1}$

$-\alpha + \beta + \gamma = -4i$ (1)

$-\alpha + (-\alpha) = -4i$; from *

$-2\alpha = -4i$

$\alpha = 2i$

iii) $\alpha = 2i$ $\qquad -2\alpha = -4i$

$\Rightarrow (z-2i)$ is a factor $\qquad \Rightarrow (z+4i)$ is a factor

So $(z-2i)(z+4i) = z^2 + 2iz + 8$ is a factor

$z^4 + 4iz^3 - 4iz^2 + (8i+8)z - (32+32i) = 0$

$(z^2 + 2iz + 8)(z^2 + 2iz - (4+4i)) = 0$ 1

Consider $z^2 + 2iz - (4+4i) = 0$

$$z = \frac{-2i \pm \sqrt{(2i)^2 - 4(1)(-4-4i)}}{2(1)}$$

$$= \frac{-2i \pm \sqrt{12+16i}}{2}$$ 1

$$= \frac{-2i \pm 2\sqrt{3+4i}}{2}$$

$= -i \pm (2+i)$

$= 2$ OR $-2-2i$

∴ Vertices are $2i$, $-4i$, 2 and $-2-2i$. 1

(3)

b) i) RHS $= 2\cos\theta(\cos\theta + i\sin\theta)$

$= 2\cos^2\theta + 2\sin\theta\cos\theta\, i$ (1)

$= 1 + \cos 2\theta + i\sin 2\theta$

$=$ LHS

ii) LHS $= \left(1 + \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n$

$= \left(2\cos\frac{\pi}{n}\right)^n \left(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n}\right)^n$; from (i)

$= 2^n\cos^n\frac{\pi}{n}\left[\cos\left(\frac{\pi}{n}\right)(n) + i\sin\left(\frac{\pi}{n}\right)(n)\right]$

$= 2^n\cos^n\frac{\pi}{n}(\cos\pi + i\sin\pi)$ (1)

$= 2^n\cos^n\left(\frac{\pi}{n}\right) \cdot (-1 + i0)$

$= -2^n\cos^n\frac{\pi}{n}$

$=$ RHS

iii) From (i), similarly $2\cos\theta(\cos\theta - i\sin\theta)$

$= 1 + \cos 2\theta - i\sin 2\theta$.

$\left(1 + \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}\right)^n$

$= \left(2\cos\frac{\pi}{n}\right)^n(\cos\pi - i\sin\pi)$

$= -2^n\cos^n\frac{\pi}{n}$ 1

$\left(1 + \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n - \left(1 + \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}\right)^n$

$= -2^n\cos^n\frac{\pi}{n} - \left(-2^n\cos^n\frac{\pi}{n}\right)$

$= -2^n\cos^n\frac{\pi}{n} + 2^n\cos^n\frac{\pi}{n}$

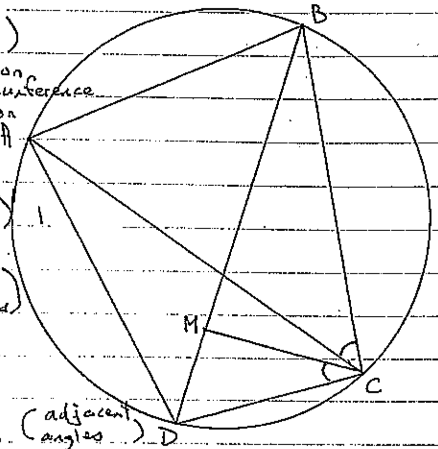
$= 0$ 1

(2)

c) i) In $\triangle ABC$ and $\triangle DMC$;

$\angle ACB = \angle DCM$ (given)

$\angle BAC = \angle MDC$ (angles on the circumference standing on the same arc BC)



$\triangle ABC \parallel\!| \triangle DMC$ (equiangular) 1

$\frac{AC}{CD} = \frac{AB}{MD}$ (corresponding sides of similar triangles)

$AB \times CD = AC \times MD$

In $\triangle ACD$ and $\triangle BCM$;

$\angle ACD = \angle ACM + \angle DCM$ (adjacent angles)

$= \angle ACM + \angle ACB$ ($\angle ACB = \angle DCM$, given)

$= \angle BCM$ (adjacent angles)

$\angle DAC = \angle MBC$ (angles on the circumference standing on the same arc CD)

$\triangle ACD \parallel\!| \triangle BCM$ (equiangular) 1

$\frac{AD}{BM} = \frac{AC}{BC}$ (corresponding sides of similar triangles)

$AD \times BC = AC \times BM$

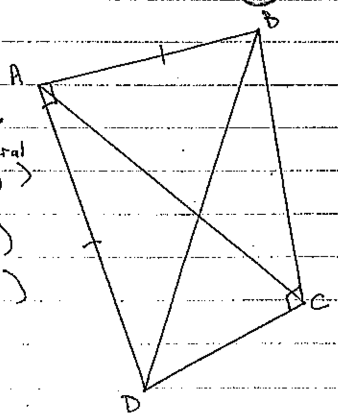
So $AB \times CD + BC \times AD$

$= AC \times MD + AC \times BM$ 1 (3)

$= AC(MD + BM)$

$= AC \times BD$

ii) $\angle BAD = \angle BCD$ ($\angle BCD = 90°$ given and opposite angles of cyclic quadrilateral are supplementary)



In $\triangle ABD$;

$BD^2 = AB^2 + AD^2$ (by Pythagoras')

$= AB^2 + AB^2$ (AD = AB given)

$= 2AB^2$

$BD = \sqrt{2}\, AB$ — ☆

$AC \times BD = AB \times CD + BC \times AD$

$\Rightarrow AC \times \sqrt{2}AB = AB \times CD + BC \times AB$

$= AB(CD + BC)$

$AC = \frac{1}{\sqrt{2}}(CD + BC)$

$AC^2 = \frac{1}{2}(CD + BC)^2$

$= \frac{1}{2}(CD^2 + BC^2 + 2(BC \times CD))$

$= \frac{1}{2}(2AB^2 + 2(BC \times CD))$; from ✻

$= AB^2 + BC \times CD$ — ①

Area of ABCD = Area of $\triangle ABD$ + Area of $\triangle BCD$

$18 = \frac{1}{2}(AB \times AD) + \frac{1}{2}(BC \times CD)$

$18 = \frac{1}{2}AB^2 + \frac{1}{2}(BC \times CD)$

$36 = AB^2 + (BC \times CD)$ — ②

① $\Rightarrow AC^2 = AB^2 + BC \times CD$

$= AB^2 + 36 - AB^2$; from ②

$= 36$

$AC = 6\text{cm}$ (2)

✻ In $\triangle BCD$;

$BD^2 = BC^2 + CD^2$ (by Pythagoras')

$2AB^2 = BC^2 + CD^2$ (from ☆ where $BD = \sqrt{2}AB$)