Ravenswood School 4 unit 1999 3 hours.

Question 1

a) Reduce the complex expression $\frac{(2-i)(8+3i)}{(3+i)}$ to the form $a+ib$ where a and b are real numbers.

b) The complex number z is given by $z = -\sqrt{3}+i$

i) Write down the values of $\arg z$ and $|z|$

ii) Hence or otherwise show that $z^7 + 64z = 0$

c) Find the roots of the equation $(2+i)z^2 - 4z + (2-i) = 0$

expressing any complex roots in the form $a+ib$ where a and b are real.

Question 2 (begin a new page)

a) Given that $P(x) = (x^4-1)(x^2-2)$ factorise P(x) completely over:

i) the rational numbers.

ii) the real numbers.

iii) the complex numbers.

b) If the polynomial $P(x) = x^4 + x^2 + 6x + 4$ has a rational zero of multiplicity 2, find all the zeros of P(x) over the complex field.

c) Consider the polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$

i) If P(x) has roots $(a+bi)$, $(a-2bi)$ where a and b are real find the values of a and b.

ii) Hence find the zeros of P(x) over the complex field and express P(x) as the product of two quadratic factors.

Question 3 (begin a new page)

Pi

a) Let α, β, γ be the roots of the polynomial $x^3 + 4x^2 - 3x + 1 = 0$

Find the equations with roots:

i) $2\alpha, 2\beta, 2\gamma$ ii) $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$

b) Graph the function $f(x) = 1 - x^2$ for $-2 \le x \le 2$

Without using calculus, neatly sketch the following curves, clearly showing their main features. Use half a page for each graph.

i) $y = |f(x)|$ ii) $|y| = f(x)$

iii) $y = \{f(x)\}^2$ iv) $y = e^{f(x)}$

Question 4 (begin a new page)

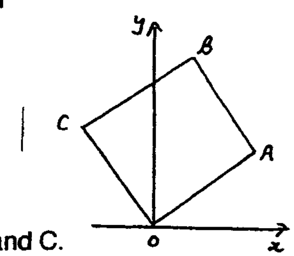
a) Given that 1, w and w^2 are the cube roots of unity, the roots of $z^3 = 1$

simplify $(1-w)(1-w^2)((1-w^4)(1-w^8)$

b) Sketch the following loci on separate Argand diagrams:

i) $\arg(z+1+i) = \frac{\pi}{4}$ ii) $|z-2| = |z+i|$

c) OABC is a square in the complex plane and the point A represents thecomplex number z.



i) State the complex numbers represented by B and C.

ii) Draw the square reflected in the x axis to become OA'B'C'

What complex numbers are represented by A',B',C'.

Question 5 (begin a new page)

a) Find the value of k such that the equation

$x^2 - 3x + k - 2 = 0$ has two distinct real roots.

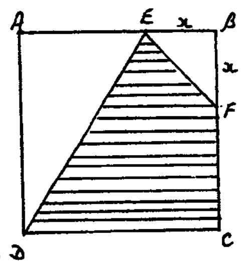
b) A Ravenswood old girl leaves a will in which she establishes a fund of $50 000 for the students of Ravenswood. This money is to be invested at 6% interest compounded annually. Under the conditions of the will no money is to be withdrawn from the fund during the first 20 years.

i) If these instructions were followed, what amount would be in the fund at the end of 20 years ?

ii) Suppose that at the beginning of each subsequent year after establishment the old girls union decides to add $1000 to the fund. This also earns 6% compounded annually.

How much money would now be in the fund at the end of 20 years.?

c) ABCD is a square of side 2 units E and F are chosen on AB and BC respectively such that BE = BF = x units. EF and ED are joined.



i) Show that the area of the quadrilateral EFCD is given by $A = \frac{1}{2}(4 + 2x - x^2)$

ii) Find the maximum area of this quadrilateral.

d) The continuous curve corresponding to the function $y = f(x)$ has the following properties in the closed interval $a \le x \le b$

$f(x) > 0, \quad f'(x) < 0, \quad f''(x) > 0$

i) Sketch a curve satisfying these conditions.

ii) State the least value of $f(x)$ in this interval.

Question 6 (begin a new page)

a) The base of a solid is in the circle $x^2 + y^2 = 16$ and every plane section perpendicular to the x axis is a rectangle whose hight is twice its base. Find the volume of the solid.

b) The region R in the first quadrant is such that $y \le 4x^2 - x^4$ is rotated about the y axis to form a solid of revolution. Use the method involving cylindricAl shells to find the volume of this solid.

c) Find the volume obtained when the area in the first quadrant enclosed be the curves $y = \sin x$ and $y = \cos x$ and the y axis is rotated about the x axis.

Question 7 (begin a new page)

a) In a class of 30 girls 25 study mathematics and 20 study History. If a girl is picked at random from this class find the probability that she studies both Mathematics and History.

b) An inspector selects 3 light bulbs from their daily production for testing. If a bulb does not last longer than 1000 hours it is called a "failure". If the probability of a failure is 0.0002 find the probability that he selects

i) 1 failure

ii) 2 failures

iii) no failures

iv) at least 1 failure.

c) A highway running West-East passes through town A and town B which are 130km apart. Another town C is N 43° E of town A and N 58° W of town B. A new road is to be built from town C running due South to the highway. How long will this road be?

d) Given

x	0	0.2	0.4	0.6	0.8
f(x)	0	0.20	0.39	0.56	0.71

Use Simpson's Rule with 5 function values to calculate $\int_0^{0.8} f(x)\,dx$

Question 8 (begin a new page)

a) Draw neat sketch graphs of the following showing their main features

i) $y = 2 - \sin x$ ii) $y = \ln(2x-6)$

b) Express $\dfrac{x^2-4x-1}{(1+x^2)(1+2x)}$ as the sum of two partial fractions.

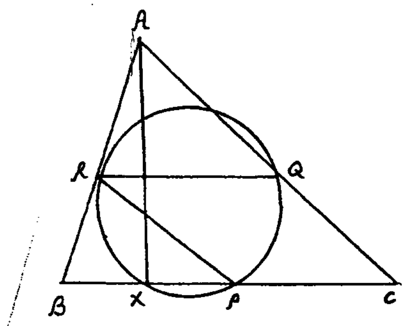
Hence find $\displaystyle\int \frac{x^2-4x-1}{(1+x^2)(1+2x)}\,dx$

c) P,Q,R are the mid-points of the sides BC, CA and AB of a triangle ABC. The circle through P, Q and R meets the three sides again at X, Y and Z respectively. show that:

i) RPCQ is a parallelogram.

ii) Triangle XQC is isosceles.

iii) AX is perpendicular to BC



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~~1999~~ 4 Unit ½ Yearly Solutions. Piii

Q.1. a) $\dfrac{(2-i)(8+3i)}{3+i} = \dfrac{16+6i-8i-3i^2}{3+i}$

$= \dfrac{19-2i}{3+i} \times \dfrac{3-i}{3-i}$

$= \dfrac{57-19i-6i+2i^2}{9-i^2}$

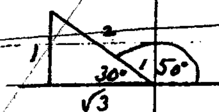
$= \dfrac{55-25i}{10}$

$= \dfrac{11}{2} - \dfrac{5i}{2}$

$\boxed{4}$

b) $z = -\sqrt{3} + i$

i) $\arg z = \dfrac{5\pi}{6}$ $|z| = 2$



ii) $z^7 + 64z = \left(2\,\text{cis}\,\dfrac{5\pi}{6}\right)^7 + 64 \cdot 2\,\text{cis}\,\dfrac{5\pi}{6}$

$= 128\,\text{cis}\,\dfrac{35\pi}{6} + 128\,\text{cis}\,\dfrac{5\pi}{6}$

$\boxed{5}$

$= 128\,\text{cis}\left(-\dfrac{\pi}{6}\right) + 128\,\text{cis}\,\dfrac{5\pi}{6}$

$= 128\left[\cos\dfrac{-\pi}{6} + i\sin\dfrac{-\pi}{6} + \cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right.$

$= 128\left[\dfrac{\sqrt{3}}{2} - \dfrac{i}{2} - \dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right]$

$z^7 + 64z = 0$

c) $(2+i)z^2 - 4z + (2-i) = 0$

$z = \dfrac{4 \pm \sqrt{16 - 4(2+i)(2-i)}}{2(2+i)}$

$= \dfrac{4 \pm \sqrt{16-20}}{2(2+i)}$ $\therefore z = \dfrac{2+i}{2+i}$ or $z = \dfrac{2-i}{2+i} \times \dfrac{2-i}{2-i}$

$\boxed{6}$

$= \dfrac{4 \pm \sqrt{-4}}{2(2+i)}$ $= 1$ $= \dfrac{4-4i+i^2}{4-i^2}$

$= \dfrac{4 \pm 2i}{2(2+i)}$ $= \dfrac{3-4i}{5}$

$= \dfrac{2 \pm i}{2+i}$ Sol' $z = 1$ or $z = \dfrac{3}{5} - \dfrac{4}{5}i$

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i) $y = 2 - \sin x$ ii) $y = \ln(2x - 6)$

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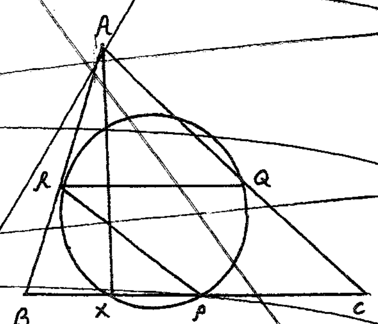

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Piii

1999 4 Unit ½ Yearly Solutions.

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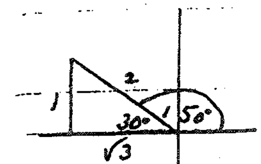

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$= 128\left[\cos -\frac{\pi}{6} + i\sin -\frac{\pi}{6} + \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$

$= 128\left[\frac{\sqrt{3}}{2} - \frac{1}{2}i - \frac{\sqrt{3}}{2} + \frac{1}{2}i\right]$

$z^7 + 64z = 0$

c) $(2+i)z^2 - 4z + (2-i) = 0$

$z = \dfrac{4 \pm \sqrt{16-4(2+i)(2-i)}}{2(2+i)}$

$= \dfrac{4 \pm \sqrt{16-20}}{2(2+i)}$

$\boxed{6}$

$= \dfrac{4 \pm \sqrt{-4}}{2(2+i)}$

$= \dfrac{4 \pm 2i}{2(2+i)}$

$= \dfrac{2 \pm i}{2+i}$

$\therefore z = \dfrac{2+i}{2+i} = 1$ or $z = \dfrac{2-i}{2+i} \times \dfrac{2-i}{2-i} = \dfrac{4-4i+i^2}{4-i^2} = \dfrac{3-4i}{5}$

Sol^n $z = 1$ or $z = \dfrac{3}{5} - \dfrac{4}{5}i$

$= (x-1)(x+1)(x \ldots)$

$= (x-1)(x+1)(x-i)(x+i)(x-\sqrt{2})(x+\sqrt{2})$ complex

$P(x) = x^4 + x^2 + 6x + 4$

$P'(x) = 4x^3 + 2x + 6$

when $P'(x) = 0$ $\quad 4x^3 + 2x + 6 = 0$

$P'(1) = 4 + 2 + 6 \neq 0$

$P'(-1) = -4 - 2 + 6 = 0$

$P(-1) = 1 + 1 - 6 + 4 = 0$

$P(-1) = 0$ + $P'(-1) = 0$ $\therefore x = -1$ is double root

$\therefore (x+1)^2$ is a factor of $P(x)$

$$\begin{array}{r} x^2 - 2x + 4 \\ x^2+2x+1 \overline{\big)\, x^4 + x^2 + 6x + 4} \\ \underline{x^4 + 2x^3 + x^2} \\ -2x^3 + 6x + 4 \\ \underline{-2x^3 - 4x^2 - 2x} \\ 4x^2 + 8x + 4 \\ \underline{4x^2 + 8x + 4} \\ 0 \end{array}$$

7

$\therefore P(x) = (x-1)^2(x^2 - 2x + 4)$

when $x^2 - 2x + 4 = 0$

$x = \dfrac{2 \pm \sqrt{4 - 4.1.4}}{2}$

$= \dfrac{2 \pm \sqrt{-12}}{2}$

$= \dfrac{2 \pm 2\sqrt{3}i}{2} = \dfrac{2(1 \pm \sqrt{3}i)}{2}$

$\therefore$ zeros of $P(x)$ are $x = -1$ $x = -1$ $x = 1 \pm \sqrt{3}i$

$P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$

real coefficient $\therefore$ complex roots are conjugate pair

roots are $a \pm bi$ and $a \pm 2bi$

sum roots $(a+bi) + (a-bi) + (a+2bi) + (a-2bi) = -\frac{-}{}$

$\therefore 4a = 4 \qquad a = 1.$

product roots $(a+bi)(a-bi)(a+2bi)(a-2bi) = 10$

Piv

$(a^2+b^2)(a^2+4b^2) = 10$ but $a = 1$

$\therefore (1+b^2)(1+4b^2) = 10$

$1 + 5b^2 + 4b^4 = 10$

$4b^4 + 5b^2 - 9 = 0$

$(4b^2+9)(b^2-1) = 0$

$4b^2 + 9 = 0$ or $b^2 - 1 = 0$

no real sol^n $\qquad b = \pm 1$

$\boxed{6}$ $\qquad \therefore$ $a = 1 \quad b = \pm 1$

$\therefore$ zeros of $P(x)$ are $(1 \pm i)$ and $(1 \pm 2i)$

$(x - (1+i))(x - (1-i)) = ((x-1) - i)((x-1) + i)$

$= (x-1)^2 + 1$

$= (x^2 - 2x + 2)$

$(x - (1+2i))(x - (1-2i)) = ((x-1) - 2i)((x-1) + 2i)$

$= (x-1)^2 + 4$

$= (x^2 - 2x + 5)$

$\therefore P(x) = (x^2 - 2x + 2)(x^2 - 2x + 5)$

Question 3.

a) $\quad x^3 + 4x^2 - 3x + 1 = 0$

i) let new root $X = 2x$ or $x = \dfrac{X}{2}$

new eq^n $\left(\dfrac{X}{2}\right)^3 + 4\left(\dfrac{X}{2}\right)^2 - 3\left(\dfrac{X}{2}\right) + 1 = 0$

$\boxed{3}$ $\qquad \dfrac{X^3}{8} + X^2 - \dfrac{3X}{2} + 1 = 0$

$X^3 + 8X^2 - 12X + 8 = 0$

ii) let new root $X = \dfrac{1}{x}$ or $x = \dfrac{1}{X}$

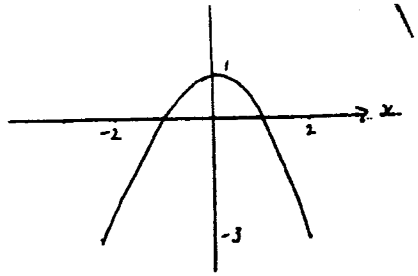
new eq^n $\left(\dfrac{1}{X}\right)^3 + 4\left(\dfrac{1}{X}\right)^2 - 3\left(\dfrac{1}{X}\right) + 1 = 0$

$\dfrac{1}{X^3} + \dfrac{4}{X^2} - \dfrac{3}{X} + 1 = 0$

$\boxed{3}$ $\qquad 1 + 4X - 3X^2 + X^3 = 0$

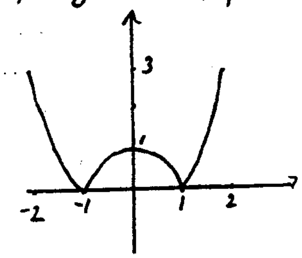
$X^3 - 3X^2 + 4X + 1 = 0$

$f(x) = 1 - x^2 \quad -2 \leq x \leq 2$



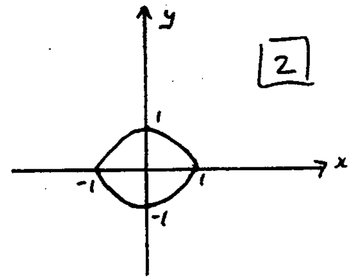
$\boxed{1}$

i) $y = |f(x)|$



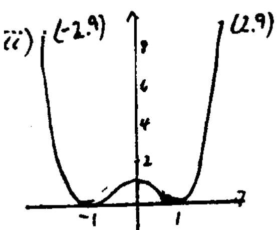
$\boxed{2}$

ii) $|y| = f(x)$



$\boxed{2}$

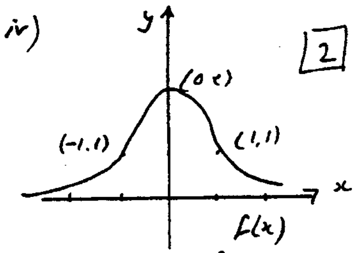
iii)



$y = [f(x)]^2$

$\boxed{2}$

iv)



$y = e^{f(x)}$

$\boxed{2}$

Question 4

i) $1, \omega + \omega^2$ are the roots of $z^3 = 1$

$\therefore \underline{\omega^3 = 1}$ and sum of roots of $z^3 - 1 = 0$

$\underline{1 + \omega + \omega^2 = 0}$

$\therefore (1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8) = (1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$

$= (1-\omega)^2(1-\omega^2)^2$

Q4 con(t)

$(1-\omega)^2(1-\omega^2)^2 = (1-2\omega+\omega^2)(1-2\omega^2+\omega^4)$

$= (1+\omega^2-2\omega)\ (1+\omega-2\omega^2)$

$= (-\omega-2\omega)\ (-\omega^2-2\omega^2)$ using $1+\omega+\omega^2=0$

$= (-3\omega)\ (-3\omega^2)$

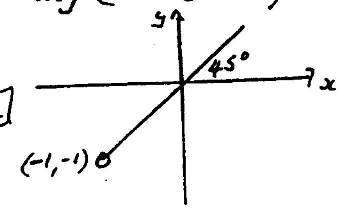
$= 9\omega^3$

$= \underline{9}$

$\boxed{6}$

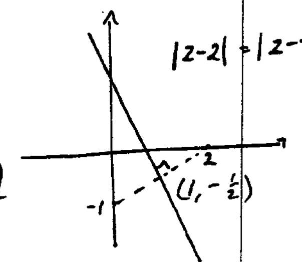
PV

b) $\arg(z - (-1-i)) = \frac{\pi}{4}$



$\boxed{2}$

$|z-2| = |z-i|$



$\boxed{2}$

c) A, is z

C is iz

B is $z + iz$

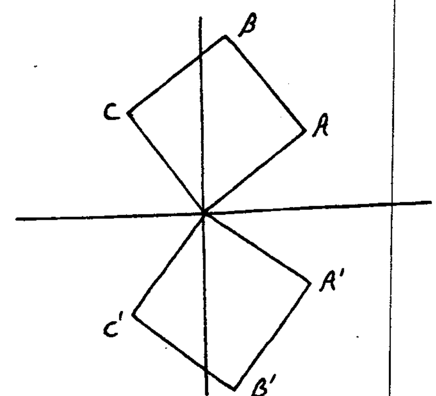
$\boxed{6}$

A' is $\overline{z}$

C' is $\overline{iz}$

B' is $\overline{z} + \overline{iz}$

or $\overline{z+iz}$



Question 5

a) $x^2 - 3x + (k-2) = 0$

For distinct real roots $\Delta > 0$

$(-3)^2 - 4 \times 1(k-2) > 0$

$9 - 4k + 8 > 0$

$-4k > -17$

$k < \frac{17}{4}$

$\boxed{2}$

(cont.)

i) \$50000 at 6% for 20 years amounts to $50000(1.06)^{20}$

$= \$160356.77$

$\boxed{2}$

ii) For 19 years \$1000 is invested annually.

1st \$1000 will amount to $1000(1.06)^{19}$

2nd $1000(1.06)^{18}$

final \$1000 $1000(1.06)^{1}$

$$\text{Total} = 1000(1.06)^1 + 1000(1.06)^2 + \ldots 1000(1.06)^{19}$$
$$= 1000[1.06 + 1.06^2 + \ldots 1.06^{19}]$$
$$= 1000 \cdot \frac{1.06(1.06^{19}-1)}{(1.06-1)}$$
$$= \$35785.59$$

Total money in fund $= \$160356.77 + \$35785.59$

$= \$196142.36$

$\boxed{4}$

i) i) Area quadrilateral = Square $- \triangle EBF - \triangle AED$

$$= 4 - \tfrac{1}{2}x^2 - \tfrac{1}{2}\cdot 2(2-x)$$
$$= 4 - \tfrac{1}{2}x^2 - 2 + x$$
$$= 2 + x - \tfrac{1}{2}x^2$$
$$A = \tfrac{1}{2}(4 + 2x - x^2)$$

$\boxed{4}$

ii) $\frac{dA}{dx} = \frac{1}{2}(2-2x)$

when $\frac{dA}{dx} = 0$ $2 - 2x = 0$

$x = 1$

x	1^-	1	1^+
$\frac{dA}{dx}$	$+$	0	$-$

Max

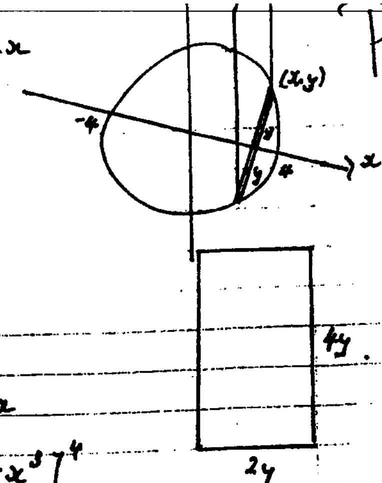
Maximum turning point at $x = 1$

$A = \frac{1}{2}(4 + 2 - 1)$

∴ area of quadrilateral is $2\frac{1}{2}$ sq units

Q.6 Vol rect. slice $= 2y \times 4y \cdot \Delta x$

$= 8y^2 \cdot \Delta x$



$$\therefore \text{Vol solid} = \lim_{\Delta x \to 0} \sum_{-4}^{4} 8y^2 \Delta x$$
$$= 2\int_0^4 8y^2\,dx$$
$$= 2\times 8\int_0^4 (16 - x^2)\,dx$$
$$= 16\left[16x - \tfrac{1}{3}x^3\right]_0^4$$
$$= 16\left[16\times 4 - \tfrac{1}{3}4^3\right]$$
$$= 16 \times \tfrac{2}{3} \times 64$$

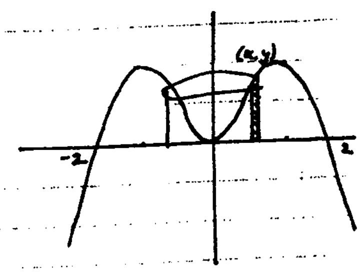
Vol solid $= \frac{2048}{3}$ cu units

b) $y = 4x^2 - x^4$

$4x^2 - x^4 = 0$

$x^2(4 - x^2) = 0$

$x^2(2-x)(2+x) = 0$



Vol shell $= 2\pi r h \times$ thickness

$= 2\pi x y \,\Delta x$

$$\text{Vol solid} = \lim_{\Delta x \to 0}\sum_0^2 2\pi x y\,\Delta x$$
$$= 2\pi\int_0^2 x(4x^2 - x^4)\,dx$$
$$= 2\pi\int_0^2 (4x^3 - x^5)\,dx$$
$$= 2\pi\left[x^4 - \tfrac{1}{6}x^6\right]_0^2$$
$$= 2\pi\left[16 - \tfrac{1}{6}\times 64 - 0\right]$$

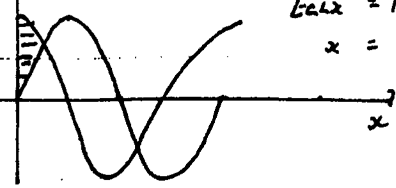
Vol $= \frac{32\pi}{3}$ cu units.

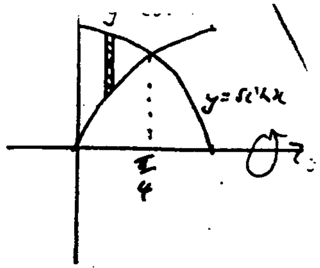
P vi

n. C. intersection $\sin x = \cos x$

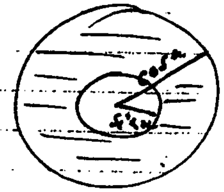
$\tan x = 1$

$x = \frac{\pi}{4}$





Vol disc $= (\pi\cos^2 x - \pi\sin^2 x)\Delta x$



Vol solid $= \pi \sum_{0}^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x)\Delta x$

$= \pi \int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x)\,dx$

$= \pi \int_0^{\frac{\pi}{4}} \cos 2x.\,dx$

$= \pi \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \pi\left[\frac{1}{2}\sin\frac{2\pi}{4} - \frac{1}{2}\sin 0\right]$

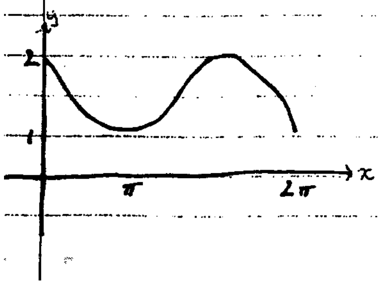
$= \frac{\pi}{2}\sin\frac{\pi}{2}$

$= \frac{\pi}{2}\times 1$

Volume $= \frac{\pi}{2}$ cu. units.

7. See 2 unit solutions.

8 $y = 2 - \sin x$



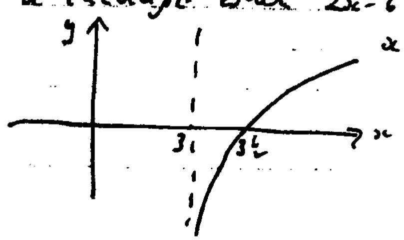
$y = \ln(2x-6)$

asymptote when $2x - 6 = 0$

$x = 3$

x intercept when $2x - 6 =$

$x =$



P VII

8 b) Let $\frac{x^2-4x-1}{(1+x^2)(1+2x)} = \frac{Ax+B}{1+x^2} + \frac{C}{1+2x}$

$\therefore x^2 - 4x - 1 = (Ax+B)(1+2x) + C(1+x^2)$

$x = -\frac{1}{2}$: $\frac{1}{4} + 2 - 1 = 0 + C(1+\frac{1}{4})$

$1\frac{1}{4} = 1\frac{1}{4}C \qquad C = 1$

$x = 0$: $-1 = B + C$ but $C = 1 \therefore B = -2$

$x = 1$: $1 - 4 - 1 = (A-2)(1+2) + 1(1+1)$

$-4 = 3A - 6 + 2$

$0 = 3A \qquad \therefore A = 0$

$\therefore \frac{x^2-4x-1}{(1+x^2)(1+2x)} = \frac{-2}{1+x^2} + \frac{1}{1+2x}$

$\int \frac{x^2-4x-1}{(1+x^2)(1+2x)}\,dx = \int \frac{-2}{1+x^2} + \frac{1}{1+2x}\,.\,dx$

$= -2\tan^{-1}x + \frac{1}{2}\ln(1+2x) + c$

c) In Δs ARQ + BAC

angle A is common

$\frac{AR}{AB} = \frac{AQ}{AC} = \frac{1}{2}$ (R + Q are midpts AB + AC)

$\therefore \Delta ARQ \,|||\, \Delta BAC$ [2 pairs sides in same ratio and same included angle]

$\therefore \angle AQR = \angle ACR$ (corresponding angles sim Δs.

$\therefore RQ \parallel BC$ (corresponding angles equal)

Similarly $RP \parallel AC$

$\therefore$ RPCQ is a parallelogram (2 pairs opp sides parallel)

$\angle QRP = \angle QXP$ (angles in same segment on arc QP)

$\angle QRP = \angle QCP$ (opp angles $\parallel^m$)

$\therefore \angle QXP = \angle QCP$

$\therefore \Delta XQC$ is isosceles.

8 (a) (i) $\lim_{n\to\infty} \frac{\log n}{n} = 0$

(ii) $\lim_{n\to -\infty} ne^{n} = 0$ (3)

(iii) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e$

(b) $\tan^{-1}(n+1) - \tan^{-1} n = \alpha - \beta$ say

$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$= \frac{n+1-n}{1+n(n+1)}$

$= \frac{1}{1+n+n^2}$

$\therefore \cot(\alpha-\beta) = n^2+n+1$ (2)

so $\tan^{-1}(n+1) - \tan^{-1}(n) = \cot^{-1}(n^2+n+1)$

find $\cot^{-1}1 + \cot^{-1}3 + \cdots + \cot^{-1}31$

when $n=0$, $\tan^{-1}1 - \tan^{-1}0 = \cot^{-1}1$

$n=1$, $\tan^{-1}2 - \tan^{-1}1 = \cot^{-1}3$

$n=5$, $\tan^{-1}6 - \tan^{-1}5 = \cot^{-1}31$

$\cot^{-1}1 + \cot^{-1}3 + \cdots + \cot^{-1}31$

$(\tan^{-1}1 - \tan^{-1}0) + (\tan^{-1}2 - \tan^{-1}1) + \cdots + (\tan^{-1}6 - \tan^{-1}5)$ (2)

$\tan^{-1}6 - \tan^{-1}0$

$= \tan^{-1}6$. (1)

(i) $f(x) = \sin x$ $\quad f(0) = 0$

$f'(x) = \cos x$ $\quad f'(0) = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$

$f^3(x) = -\cos x$ $\quad f^3(0) = -1$

$f^4(x) = \sin x$ $\quad f^4(x) = 0$ 2

$\therefore \sin x = 1x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

(ii) $f(x) = \cos x$ $\quad f(0) = 1$

$f'(x) = -\sin x$ $\quad f'(0) = 0$ 1

$f''(x) = -\cos x$ $\quad f''(0) = -1$

$f^3(x) = \sin x$ $\quad f^3(0) = 0$

$f^4(x) = +\cos x$ $\quad f^4(0) = 1$ (7)

$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

(iii) $f(x) = e^{ix}$ $\quad f(0) = 1$

$f'(x) = ie^{ix}$ $\quad f'(0) = i$ 2

$f''(x) = -e^{ix}$ $\quad f''(0) = -1$

$f^3(x) = -ie^{ix}$ $\quad f^3(0) = -i$

$f^4(x) = e^{ix}$ $\quad f^4(0) = 1$

$\therefore e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \cdots$ 1

$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$

$e^{ix} = \cos x + i\sin x$

$e^{i\pi} = \cos\pi + i\sin\pi$ 1