Student Number......................

SCECGS REDLANDS

TRIAL HIGHER SCHOOL CERTIFICATE

1999

MATHEMATICS- 4 UNIT

DIRECTIONS TO CANDIDATES

- Time allowed- Three hours (plus 5 minutes reading time).
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are given.
- *Board-approved* calculators may be used.

QUESTION ONE (15 Marks)

a) (i) Differentiate $x \ln x$. **2**

(ii) Hence, or otherwise, find $\int \ln x \, dx$ **2**

b) Using the substitution $u = e^x - 1$, or otherwise, evaluate **3**

$$\int_1^2 \frac{e^{2x}}{e^x - 1} dx$$

c) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate **4**

$$\int_0^{\frac{\pi}{2}} \frac{1}{5 + 3\cos x} dx$$

d) (i) Find real numbers a, b and c such that **2**

$$\frac{1}{(x+1)(x^2+4)} = \frac{a}{(x+1)} + \frac{bx+c}{(x^2+4)}$$

(ii) Hence find **2**

$$\int \frac{1}{(x+1)(x^2+4)} dx$$

QUESTION TWO (15 Marks)

a) Let $z = 1 + i$ and $u = \sqrt{3} - i$

(i) Find $\text{Im}(uz)$ **1**

(ii) Find $|u - z|$ **1**

(iii) Find $-i\overline{u}$ **1**

(iv) Express $\frac{u}{z}$ in the form $a + ib$, where a and b are real numbers. **2**

b) If $z = a + ib$ and $z = (1 + i)^4$, find the values of a and b. **3**

c) Find the locus of a point Z which moves so that **3**

$(z - \overline{z})^2 + 2(z + \overline{z}) + 1 = 0$

Show that this locus is a parabola.

d) Find the square roots of $8 + 15i$ **4**

QUESTION THREE (15 Marks)

a) (i) Solve the equation $Z^3 - 1 = 0$, over the field of Complex numbers, leaving your answer in mod-arg form. **2**

(ii) Let w be a non-real root of the equation $Z^3 - 1 = 0$. Show that w^2 is the other non-real root of the equation. **2**

(iii) Hence, or otherwise, form the cubic equation which has roots; $-1,\ 1 + w,\ 1 + w^2$. **3**

b) (i) Differentiate $\cos^{-1}(\sin x)$ **2**

(ii) Determine all critical points and intercepts with the co-ordinate axes for the function $y = \cos^{-1}(\sin x)$ over the domain $-2\pi \le x \le 2\pi$. **3**

(iii) Sketch the graph of $y = \cos^{-1}(\sin x)$ over $-2\pi \le x \le 2\pi$. Clearly label all critical points and intercepts with the co-ordinate axes. **3**

QUESTION FOUR (15 Marks)

a) Show that $1 - i$ is a zero of the polynomial
$P(x) = x^3 + x^2 - 4x + 6$
Hence resolve $P(x)$ into irreducible factors over:

(i) The field of Real Numbers **2**

(ii) The field of Complex Numbers **2**

b) Given that the equation $A(x) = 0$, where **3**
$A(x) = x^4 + x^3 - 3x^2 - 5x - 2$, has root of multiplicity 3,
find all roots of $A(x)$.

c) The polynomial $P(x) = x^4 + 4x^3 - 3x^2 - 4x - 2$ has zeros **4**
$\alpha, \beta, \gamma\, \delta$. Find the polynomial with zeros $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$.

d) The equation $2x^3 - 9x^2 + 7 = 0$ has roots α, β, γ. **4**
Evaluate $\alpha^3 + \beta^3 + \gamma^3$.

QUESTION FIVE (15 Marks)

a) A particle moves in a straight line with acceleration given by
$\ddot{x} = -9x + 36$.

(i) Show that the motion is simple harmonic and state the centre of motion. **2**

(ii) Find x at time t given that $x = 0$ and $\dot{x} = 9$ when $t = 0$. **3**

b) (i) On separate diagrams sketch the graphs of: **3**

$$a(x) = |4 - x^2|\ , \qquad b(x) = \frac{1}{4 - x^2} \quad \textit{and} \quad c(x) = \frac{x^2}{4 - x^2}$$

clearly showing all intercepts with the co-ordinate axes and any asymptotes.

(ii) In each case determine whether the function is ODD, EVEN or NEITHER ODD NOR EVEN. **3**

(iii) If $g(x) = a(x) + c(x)$, state the domain and range of $g(x)$. **2**

(iv) Find the x and y intercepts of $g(x)$. Leave your answer in exact form where necessary. **2**

QUESTION SIX (15 Marks)

a) Let $I_n = \int \sec^n x\,dx$, where n is a non-negative integer.

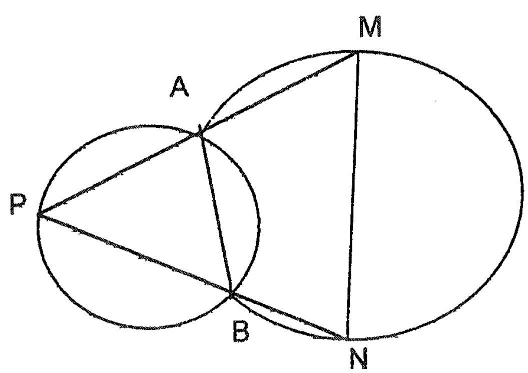
(i) Show that **4**

$$I_n = \frac{1}{n-1}\sec^{n-2} x.\tan x + \frac{n-2}{n-1} I_{n-2} \qquad n \geq 2$$

(ii) Hence or otherwise evaluate **3**

$$\int_0^{\frac{\pi}{4}} \sec^4 x\,dx$$

b) Two fixed circles intersect at **A** and **B**, as shown in the diagram. **P** is a variable point on the major arc of one circle. **PA** and **PB** are produced to meet the other circle at **M** and **N** respectively.



COPY THE DIAGRAM INTO YOUR ANSWER BOOKLET

(i) Prove that **MN** is of a constant length. **4**

(ii) Prove that the tangent at **P** is parallel to **MN**. **4**

QUESTION SEVEN (15 Marks)

a) Consider the ellipse $x^2 + 25y^2 = 25$.

(i) Determine the co-ordinates of the foci and the equations of the directrices. **2**

(iii) Sketch the ellipse $x^2 + 25y^2 = 25$, clearly showing the foci, directrices, and intercepts with the co-ordinate axes. **2**

b) Consider the locus of the point $P(x,y)$ whose co-ordinates satisfy the equations:

$$x = \cos t, \qquad y = 1 - \cos(2t)$$

(i) State the domain and range of the locus of P. **3**

(ii) Determine the Cartesian equation of the locus of P. **2**

(iii) Sketch the curve traced by the point $P(x,y)$. **2**

c) $S_n = \frac{1}{1\times 3} + \frac{1}{3\times 5} + \ldots\ldots\ldots\ldots + \frac{1}{(2n-1)(2n+1)}$, for all positive integers n. Prove by mathematical induction that **4**

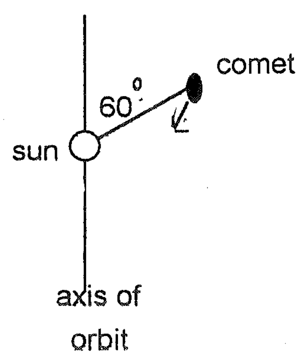
$$S_n = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$$

QUESTION EIGHT (15 Marks)

a) Show that the volume of a solid generated by revolving a circle of radius a about an axis in its plane at a distance of b units from the centre (when $b>a$) is $2\pi^2a^2b$. **6**

b) A particle moves around the circle $x^2 + y^2 = 3$. The velocity component parallel to the x-axis is $\frac{dx}{dt} = y$. Find $\frac{dy}{dt}$.
Does the particle travel clockwise or anti clockwise ? **4**

c) A comet moves in a parabolic orbit with the sun at the focus. When the comet is 8×10^7 kilometres from the sun, the line from the sun to it makes an angle of 60^0 with the axis of the orbit, (drawn in the direction in which the orbit opens), as shown in the diagram. What is the closest distance that the comet comes to the sun ? **5**



STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x,\ x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax},\ a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax,\ a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax,\ a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$