STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x,\ x > 0$

SAINT IGNATIUS' COLLEGE RIVERVIEW

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1997

4 UNIT MATHEMATICS

Time Allowed: 3 Hours
(plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES:

1. This paper contains 8 questions.
2. ALL questions may be attempted.
3. ALL questions are of equal value.
4. The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.
5. All necessary workings should be shown in every question. Marks may not be awarded for careless or badly arranged work.
6. Standard integrals are supplied. Approved calculators may be used.
7. EACH question attempted is to be returned in a separate writing booklet clearly marked Question 1, Question 2, etc. on the cover. Each booklet must show your Candidate Number.
8. If required, additional writing booklets may be obtained from the Examination Supervisor upon request.

Students are advised that this is a trial examination only and cannot in any way guarantee the content or format of the 1997 4 Unit Mathematics Higher School Certificate Examination

a) If $z=\dfrac{2-4i}{1+i}$ find: (2)

i) $\bar{z}$

ii) iz

b) i) Find the roots of the equation $z^5+1=0$ (3)

ii) If z_1, z_2, z_3, z_4 and z_5 are the roots of $z^5+1=0$, draw the pentagon represented by these points on an Argand diagram. (3)

Show that the side of this regular pentagon is of length $2\sin\dfrac{\pi}{5}$.

c) If $z_1=1+i$, $z_2=2+6i$, $z_3=-1+7i$ find all possible complex numbers z_4, so that z_1, z_2, z_3 and z_4 form a parallelogram. (3)

d) On separate diagrams, draw a neat sketch of the locus defined by: (4)

i) $\arg\left(\dfrac{z-i}{z+i}\right)=\dfrac{\pi}{2}$

ii) $|z-2| = 3|z+2i|$

[QUE]STION 2 **USE A SEPARATE WRITING BOOKLET**

Reduce the polynomial $P(x)=x^6-2x^4-x^2+2$ into irreducible factors over: (3)

i) the rational field Q.

ii) the real field R.

iii) the complex field C.

b) Given that the polynomial $P(x)=x^4+x^2+6x+4$ has a rational zero of multiplicity 2, find all the zeros of $P(x)$ over the complex field. (5)

c) Consider the polynomial $P(x)=x^4-4x^3+11x^2-14x+10$ (7)

i) If $P(x)$ has roots $a+bi$, $a-2bi$ (where a, b are real) find the values of a and b.

ii) Hence, find the zeros of $P(x)$ over the complex field and express $P(x)$ as the product of two quadratic factors.

QUESTIONS

a) Evaluate and leave your answer in exact form (5)

i) $\int_{-\frac{1}{2}}^{0} \frac{dx}{2+4x+4x^2}$

ii) $\int_{0}^{3} \sqrt{\frac{5-x}{5+x}}\,dx$

b) By using the method of integration by parts, find: (5)

$\int \ln(3x-5)dx \ ; \ x > \frac{5}{3}$

c) Find $\int \frac{dx}{3\sin x - 4\cos x}$ by using the substitution $t = \tan\frac{x}{2}$ (5)

QUESTION 4 USE A SEPARATE WRITING BOOKLET

a) Consider the conic defined by the equation $\frac{x^2}{19-l}+\frac{y^2}{7-l}=1$ (7)

i) Determine the real values of l for which the equation defines and ellipse and a hyperbola.

ii) Sketch the curve corresponding to the value $l=3$.

iii) Describe how the shape of this curve changes as l increases from 3 towards 7.

iv) What is the limiting position of the curve as 7 is approached?

b) T is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with centre, O. (8)

A line drawn through O, parallel to the tangent to the ellipse at T, meets the ellipse at M and N. Prove that the area of the triangle TMN is independent of the position of T.

QUESTION 5

a) The equation $2x^3 - 9x^2 + 7 = 0$ has roots α, β and γ. (4)

Find the equation with roots α^3, β^3 and γ^3.

b) Sketch the region R for which $0 \le y \le x^2 - x^4$ and $0 \le x \le 1$ (11)

i) Calculate the maximum value of y in R.

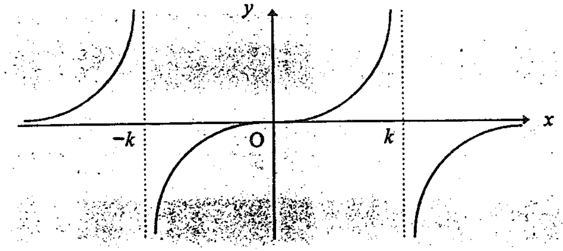
Find the volume generated when R is rotated about the y-axis, using the method of:

ii) Cylindrical shells ; and

iii) Slices.

QUESTION 6 USE A SEPARATE WRITING BOOKLET

a) The graph of $y=f(x)$ is shown below: (6)



Draw sketches of the following:

i) $y=f(x-k)$

ii) $y=[f(x)]^2$

iii) $y=f'(x)$

iv) $y=\dfrac{1}{f(x)}$

b) Consider the relation whose equation is $y^2=x^2(1-x^2)$ (5)

i) Draw a neat sketch of the graph representing this relation showing the intercepts on the co-ordinate axes.

ii) Find the area of the region enclosed by the graph of this relation.

c) i) Sketch on the same number plane: (4)

$y=|x|-2$; and

$y=4+3x-x^2$

ii) Hence, or otherwise, solve $\dfrac{|x|-2}{4+3x-x^2}>0$

a) *PQRS* is a cyclic quadrilateral whose diagonals intersect at *A*. A circle is drawn through *P*, *Q* and *A*. Prove that the tangent at *A* to this circle is parallel to *RS*. (4)

b) A sequence of numbers U_n is such that $U_1 = 3, U_2 = 21$ and $U_n = 7U_{n-1} - 10U_{n-2}$ for $n \geq 3$ (5)

Use the method of mathematical induction to show that $U_n = 5^n - 2^n$ for $n \geq 1$.

c) The railway line around a circular arc of radius $8u^2$ metres is banked by raising the outer rail to a level above the inner rail. When an Electric Parcel Van of mass m kg travels at u metres/second along this track, the lateral thrust on the inner rail is the same as the lateral thrust on the outer rail at a speed of $2u$ metres /second. (6)

(i) Calculate the angle of banking.

Show that the speed of the Electric Parcel Van is

$u\sqrt{\frac{5}{2}}$ metres / second when there is no lateral thrust exerted on the rails. Use $g = 10$ metres/second squared.

QUESTION 8 **USE A SEPARATE WRITING BOOKLET**

a) Use calculus to prove that the inequality $(1+x)^n > 1+nx$ is true whenever $x > 0$ and $n > 1$ (4)

b) Consider the two series, C and S, where: (11)

$C = 1 + \cos\theta + \cos 2\theta + \dots\dots\dots + \cos(n-1)\theta$; and

$S = \sin\theta + \sin 2\theta + \dots\dots\dots + \sin(n-1)\theta$

i) Multiply series S by i

ii) Write down $C + iS$

iii) If $z = \cos\theta + i\sin\theta$, use De Moivre's theorem to express $(C + iS)$ as a series in terms of z.

iv) Hence, show that $C + iS = \dfrac{1-z^n}{1-z}$ $(z \neq 1)$

v) Using the following results:-

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

show that:

$$C = \frac{\sin\frac{1}{2}n\theta\cos\frac{1}{2}(n-1)\theta}{\sin\frac{1}{2}\theta} \quad ; \text{ and}$$

$$S = \frac{\sin\frac{1}{2}n\theta\sin\frac{1}{2}(n-1)\theta}{\sin\frac{1}{2}\theta}$$

END OF EXAMINATION

Question 1

(a) $z = \frac{2-4i}{1+i}$

$= \frac{2-4i}{1+i} \times \frac{1-i}{1-i}$

$= \frac{2-2i-4i+4i^2}{1-i^2}$

$= \frac{2-4-6i}{1+1}$

$= \frac{-2-6i}{2}$

$= -1-3i$

(i) $\bar{z} = -1+3i$

(ii) $iz = -i-3i^2$
$= 3-i$

(b) $z^5+1=0$

$z^5=-1$

$|z| = \sqrt[5]{1} = 1$

$\arg(z) = \pi$

$z = r^{1/n} \operatorname{cis}\left(\frac{\theta+2k\pi}{n}\right)$

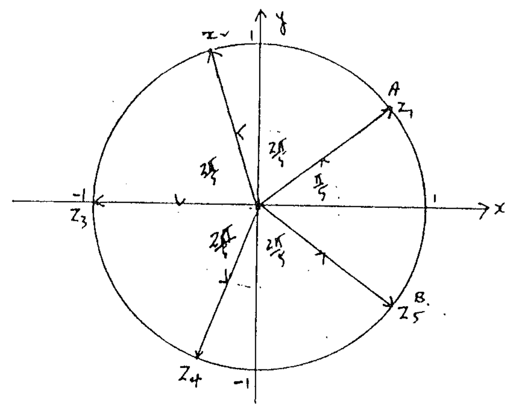
$z_1 = 1.\operatorname{cis}\frac{\pi}{5} = \cos\pi/5 + i\sin\pi/5$

$z_2 = 1.\operatorname{cis}\frac{3\pi}{5} = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$

$z_3 = 1.\operatorname{cis}\pi = \cos\pi + i\sin\pi$

$z_4 = 1.\operatorname{cis}\frac{7\pi}{5} = \cos-\frac{3\pi}{5} + i\sin-\frac{3\pi}{5}$

$z_5 = 1.\operatorname{cis}\frac{9\pi}{5} = \cos-\frac{\pi}{5} + i\sin\pi/5$.



consider points A and B as indicated.
using the cosine rule

$AB^2 = 1^2+1^2-2\times1\times1\cos\frac{2\pi}{5}$

$= 1+1-2\cos\frac{2\pi}{5}$

$= 2-2\cos\frac{2\pi}{5}$.

$= 2\left(1-\cos\frac{2\pi}{5}\right)$

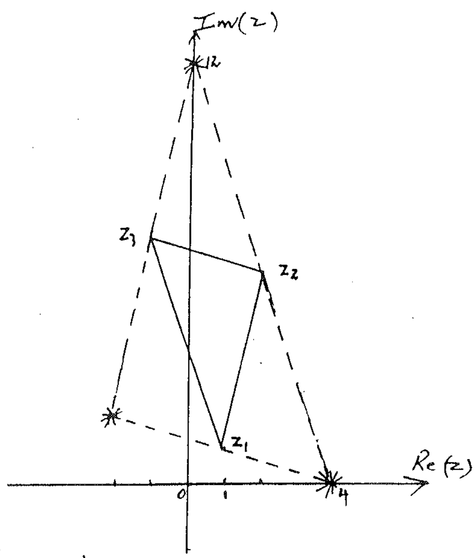
$= 2\times2\sin^2\pi/5$

$= 4\sin^2\pi/5$.

$AB = 2\sin\pi/5$

Page 1

(c)

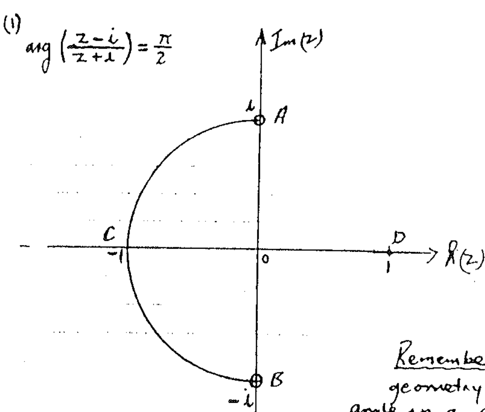


$z_1 = 1+i$
$z_2 = 2+6i$
$z_3 = -1+7i$

z_4 can be in any of the 3 positions indicated $*$
(Vertices of a triangle with z_1, z_2, z_3 as mid-points)

$\therefore z_4 = 4$ or $-2+2i$ or $12i$

(d)(i) $\arg\left(\frac{z-i}{z+i}\right) = \frac{\pi}{2}$



Remember Relevant geometry Theorem angle in a semicircle is a right angle.

The locus is a semi circle as shown to the left of the imaginary axis. Excluding the point A and B.

Note if $z = -1+0i$ (see point C)

Then $\frac{z-i}{z+i} = i$

ie $\arg\left(\frac{z-i}{z+i}\right) = \frac{\pi}{2}$

also if $z = 1+0i$ (see point D)

Then $\frac{z-i}{z+i} = -i$

$\arg\left(\frac{z-i}{z+i}\right) = -\frac{\pi}{2}$

ie C is part of locus while D is not

(ii) $|z-2| = 3|z+2i|$

If $z = x+iy$

then $z-2 = (x-2)+iy$

$z+2i = x+(y+2)i$

$\sqrt{(x-2)^2+y^2} = 3\sqrt{x^2+(y+2)^2}$

$x^2-4x+4+y^2 = 9(x^2+y^2+4y+4)$

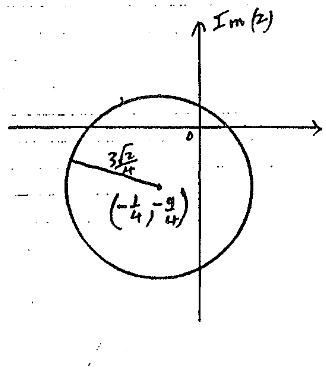
$x^2-4x+4+y^2 = 9x^2+9y^2+36y+36$

$0 = 8x^2+8y^2+36y+4x+32$

$x^2+\frac{1}{2}x+\left(\frac{1}{4}\right)^2+y^2+\frac{9}{4}y+\left(\frac{9}{4}\right)^2 = -4+\frac{1}{16}+\frac{81}{16}$

$\left(x+\frac{1}{4}\right)^2+\left(y+\frac{9}{4}\right)^2 = \frac{9}{8}$

circle : centre $\left(-\frac{1}{4}, -\frac{9}{4}\right)$ radius $\frac{3\sqrt{2}}{4}$



Locus of z is a circle

Question ②

(a) $P(x) = x^6-2x^4-x^2+2$

(i) $P(x) = x^4(x^2-2)-1(x^2-2)$

$= (x^4-1)(x^2-2)$

$= (x^2-1)(x^2+1)(x^2-2)$

$= (x+1)(x-1)(x^2-2)(x^2+1)$

(ii) $P(x) = (x+1)(x-1)(x+\sqrt{2})(x-\sqrt{2})(x^2+1)$

(iii) $P(x) = (x+1)(x-1)(x+\sqrt{2})(x-\sqrt{2})(x+i)(x-i)$

(b) $P(x) = x^4+x^2+6x+4$

$P'(x) = 4x^3+2x+6$

Now $P'(-1) = -4-2+6 = 0$

$\therefore$ (x+1) is a factor of $P'(x)$

$$\begin{array}{r l} & 4x^2-4x+6 \\ x+1 & \overline{)\,4x^3+0x^2+2x+6} \\ & \underline{4x^3+4x^2} \\ & -4x^2+2x \\ & \underline{-4x^2-4x} \\ & 6x+6 \\ & \underline{6x+6} \\ & 0+0 \end{array}$$

Now for $4x^2-4x+6$.

$\Delta = 16-4(4)(6) < 0$

so (x+1) is the only rational factor of $P'(x)$

hence $(x+1)^2$ must be a factor of Px

$$\begin{array}{r l} & x^2-2x+4 \\ x^2+2x+1 & \overline{)\,x^4+0x^3+x^2+6x+4} \\ & \underline{x^4+2x^3+x^2} \\ & -2x^3+0x^2+6x \\ & \underline{-2x^3-4x^2-2x} \\ & 4x^2+8x+4 \\ & \underline{4x^2+8x+4} \\ & 0+0+0 \end{array}$$

When $x^2-2x+4 = 0$

$x = \frac{2\pm\sqrt{4-4(1)(4)}}{2}$

$x = \frac{2\pm\sqrt{-12}}{2}$

$x = \frac{2\pm 2\sqrt{3}i}{2}$

$x = 1\pm i\sqrt{3}$

$\therefore$ Zeros of $P(x)$ are $-1,\ 1\pm i\sqrt{3}$.

(c) (i) $P(x) = x^4-4x^3+11x^2-14x+10$

Polynomial has real coefficients. Therefore the roots occur in conjugate pairs.

So $(a+bi)$ and $(a-bi)$ are roots.

If $v = a+bi$ then $\bar{v} = a-bi$

$\therefore$ $v+\bar{v} = 2a$ and $v\bar{v} = a^2+b^2$

and $(a+2bi)$ and $(a-2bi)$ are roots.

$\therefore$ If $w = a+2bi$ then $\bar{w} = a-2bi$

$\therefore$ $w+\bar{w} = 2a$ and $w\bar{w} = a^2+4b^2$

$\therefore$ sum of roots $= 2a+2a = 4a$

product of roots $= (a^2+b^2)(a^2+4b^2)$

So $4a = -(-4)$

$\therefore$ $a = 1$

and $(a^2+b^2)(a^2+4b^2) = 10$

$(1+b^2)(1+4b^2) = 10$

$1+4b^2+b^2+4b^4 = 10$

$4b^4+5b^2-9 = 0$

$(4b^2+9)(b^2-1) = 0$

Now b is real

$\therefore$ $b^2-1 = 0$

$b = \pm 1$

(ii) Zeros of $P(x)$ over the complex field are $1\pm i,\ 1\pm 2i$

$\therefore P(x) = (x-1-i)(x-1+i)(x-1-2i)(x-1+2i)$

$= (x^2-2x+2)(x^2-2x+5)$

Question (3)

(a) (i) $\displaystyle\int_{-\frac{1}{2}}^{0} \frac{dx}{2+4x+4x^2}$

Now $2+4x+4x^2$

$=4\left(\frac{1}{2}+x+x^2\right)$

$=4\left(\frac{1}{4}+\frac{1}{4}+x+x^2\right)$

$=4\left[\left(\frac{1}{2}\right)^2+\left(x+\frac{1}{2}\right)^2\right]$

$=\frac{1}{4}\displaystyle\int_{-\frac{1}{2}}^{0} \frac{dx}{\left(\frac{1}{2}\right)^2+\left(x+\frac{1}{2}\right)^2}$

$=\frac{1}{4}\left[\frac{1}{\frac{1}{2}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{1}{2}}\right)\right]_{-\frac{1}{2}}^{0}$

$=\frac{1}{2}\left[\tan^{-1}(2x+1)\right]_{-\frac{1}{2}}^{0}$

$=\frac{1}{2}\left[\tan^{-1}(1)-\tan^{-1}0\right]$

$=\frac{1}{2}\times\frac{\pi}{4} = \frac{\pi}{8}$

(ii) $I = \displaystyle\int_0^3 \sqrt{\frac{5-x}{5+x}}\,dx$

$I = \displaystyle\int_0^3 \frac{\sqrt{5-x}}{\sqrt{5+x}}\times\frac{\sqrt{5-x}}{\sqrt{5-x}}\,dx$

$I = \displaystyle\int_0^3 \frac{5-x}{\sqrt{25-x^2}}\,dx$

$I = \displaystyle\int_0^3 \left[\frac{5}{\sqrt{25-x^2}} - \underbrace{\frac{x}{\sqrt{25-x^2}}}_{\text{Substitution}}\right]dx$

consider $u = 25-x^2$

For $\displaystyle\int \frac{-x}{\sqrt{25-x^2}}\,dx$

$du = -2x\,dx$

$\frac{1}{2}\,du = -x\,dx$

$\frac{1}{2}\displaystyle\int u^{-\frac{1}{2}}\,du$

$=\frac{1}{2}\,\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+c$

$=\sqrt{u}+c$ etc

$I = \displaystyle\int_0^3 \left[\frac{5}{\sqrt{25-x^2}} - \frac{x}{\sqrt{25-x^2}}\right]dx$

$=\left[5\sin^{-1}\frac{x}{5} - \sqrt{25-x^2}\right]_0^3$

$=\left[\left(5\sin^{-1}\frac{3}{5}-4\right)-\left(0-5\right)\right]$

$=5\sin^{-1}\left(\frac{3}{5}\right)+1$

(b) $\displaystyle\int \ln(3x-5)\,dx\ ;\ x>\frac{5}{3}$

Let $u=\ln(3x-5)$

and $\frac{dv}{dx}=1$

$\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

$=\ln(3x-5)\times x - \displaystyle\int x\times\frac{3}{3x-5}\,dx$

$=x\ln(3x-5) - \displaystyle\int\frac{3x}{3x-5}\,dx$

$=x\ln(3x-5) - \displaystyle\int\left(1+\frac{5}{3x-5}\right)dx$

Note partial fractions : Long division

$=x\ln(3x-5) - \left(x+\frac{5}{3}\ln(3x-5)\right)+c$

$=x\ln(3x-5) - x - \frac{5}{3}\ln(3x-5)+c$

$=\left(x-\frac{5}{3}\right)\ln(3x-5) - x + c$

(c) $\displaystyle\int\frac{dx}{3\sin x-4\cos x}$

If $t=\tan\frac{x}{2}$

$\sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$\frac{1}{3\sin x-4\cos x}$

$=\dfrac{1}{\frac{6t}{1+t^2}-\frac{4(1-t^2)}{1+t^2}}$

$=\frac{1+t^2}{6t-4(1-t^2)}$

$=\frac{1+t^2}{6t-4+4t^2}$

$=\frac{1+t^2}{4t^2+6t-4}$

$\displaystyle\int\frac{1+t^2}{4t^2+6t-4}\times\frac{2\,dt}{1+t^2}$

$=\displaystyle\int\frac{dt}{2t^2+3t-2}$

$=\displaystyle\int\frac{dt}{(2t-1)(t+2)}$

considering partial fractions:

$\frac{1}{(2t-1)(t+2)} = \frac{A}{2t-1}+\frac{B}{t+2}$

$\frac{1}{D} = \frac{A(t+2)+B(2t-1)}{D}$

$\frac{1}{D} = \frac{(A+2B)t+(2A-B)}{D}$

$\left.\begin{array}{l}A+2B=0\\2A-B=1\end{array}\right\}$

Solving gives $A=\frac{2}{5}$; $B=-\frac{1}{5}$

$=\displaystyle\int\left[\frac{2}{5(2t-1)}-\frac{1}{5(t+2)}\right]dt$

$=\frac{1}{5}\ln|2t-1| - \frac{1}{5}\ln|t+2|+c$

$=\frac{1}{5}\ln\left|\left(\frac{2t-1}{t+2}\right)\right|+c$

$=\frac{1}{5}\ln\left|\left(\frac{2\tan\frac{x}{2}-1}{\tan\frac{x}{2}+2}\right)\right|+c$

Question 4

(a) $\dfrac{x^2}{19-\ell} + \dfrac{y^2}{7-\ell} = 1$

(i) For ellipse $19-\ell > 0$ and $7-\ell > 0$

$\ell < 19 \quad \ell < 7$

$\therefore \ell < 7$

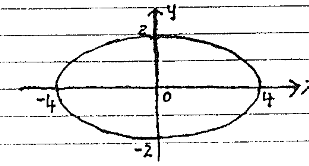
For a hyperbola $19-\ell > 0$ and $7-\ell < 0$

$\ell < 19$ and $\ell > 7$

$\therefore 7 < \ell < 19$

(ii) If $\ell = 3$, the equation becomes

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$



An ellipse

semi-major axis of 4

semi-minor axis of 2.

$b^2 = a^2(1-e^2)$

$e^2 = 1 - \dfrac{b^2}{a^2}$

$= 1 - \dfrac{4}{16}$

$= \dfrac{3}{4} \Rightarrow \dfrac{\sqrt{3}}{2}$

$ae = 4 \times \dfrac{\sqrt{3}}{2}$

$= 2\sqrt{3}$

Foci:

$S(2\sqrt{3}, 0)$ and $S'(-2\sqrt{3}, 0)$

As $\ell \to 7$, $a^2 \to 19-7$

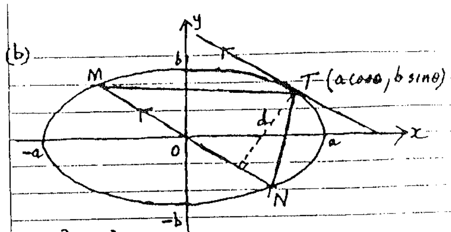
and $b^2 \to 7-7$

ie $a \to 2\sqrt{3}$ and $b \to 0$

$\therefore$ Curve approaches the line interval from $x = -2\sqrt{3}$ to $x = 2\sqrt{3}$

ie SS' of the above ellipse

(b)



$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ - - - - - (1)

Let P have co-ordinates $(a\cos\theta, b\sin\theta)$

Gradient of tangent at T

$\dfrac{2x}{a^2} + \dfrac{2y}{b^2}\cdot\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{b^2x}{a^2y}$

at T: $m = -\dfrac{b\cos\theta}{a\sin\theta}$

MN ∥ tangent at T.

Equation of MN:

$y = mx$

ie $y = -\dfrac{b\cos\theta}{a\sin\theta}.x$ - - - (2)

solving (1) and (2) for co-ordinates of M and N

$\dfrac{x^2}{a^2} + \dfrac{1}{b^2}\cdot\dfrac{b^2\cos^2\theta}{a^2\sin^2\theta}\cdot x^2 = 1$

$\dfrac{x^2}{a^2} + \dfrac{x^2\cos^2\theta}{a^2\sin^2\theta} = 1$

(b) (continued)

$\dfrac{x^2}{a^2}\left(\dfrac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\right) = 1$

$x^2 = a^2\sin^2\theta$

$x = \pm a\sin\theta$

sub in (2) gives

$y = -\dfrac{b\cos\theta}{a\sin\theta}\, x \pm a\sin\theta$

$y = \pm b\cos\theta$

ie M is $(-a\sin\theta, b\cos\theta)$

N is $(a\sin\theta, -b\cos\theta)$

Finding the distance MN

$MN = \sqrt{4a^2\sin^2\theta + 4b^2\cos^2\theta}$

$= 2\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$

Distance of MN from T

MN: $b\cos\theta x + a\sin\theta y = 0$

$d = \left|\dfrac{b\cos\theta(a\cos\theta) + a\sin\theta(b\sin\theta)}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right|$

$d = \left|\dfrac{ab(\sin^2\theta + \cos^2\theta)}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right|$

$d = \dfrac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$

Area of triangle TMN

$A = \frac{1}{2} \times MN \times d$

$A = \dfrac{1}{2} \times 2\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} \times \dfrac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$

$A = ab$

This result is independent of θ,

ie, independent of the position of T.

Question (5)

(a) $2x^3 - 9x^2 + 7 = 0$

α, β, γ.

$\alpha + \beta + \gamma = \frac{9}{2}$

$\alpha\beta + \alpha\gamma + \beta\gamma = 0$

$\alpha\beta\gamma = -\frac{7}{2}$

Now $x^3 = \frac{9}{2}x^2 - \frac{7}{2}$

α, β, γ each satisfy this equation

$$\left.\begin{aligned} \alpha^3 &= \tfrac{9}{2}\alpha^2 - \tfrac{7}{2} \\ \beta^3 &= \tfrac{9}{2}\beta^2 - \tfrac{7}{2} \\ \gamma^3 &= \tfrac{9}{2}\gamma^2 - \tfrac{7}{2} \end{aligned}\right\}$$

Add these three equations

$\alpha^3 + \beta^3 + \gamma^3 = \frac{9}{2}(\alpha^2 + \beta^2 + \gamma^2) - \frac{21}{2}$

Now $(\alpha + \beta + \gamma)^2 = (\alpha+\beta)^2 + 2(\alpha+\beta)\gamma + \gamma^2$

$(\alpha+\beta+\gamma)^2 = \alpha^2 + 2\alpha\beta + \beta^2 + 2(\alpha\gamma + \beta\gamma) + \gamma^2$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= \left(\frac{9}{2}\right)^2 - 2(0)$

$= \frac{81}{4}$

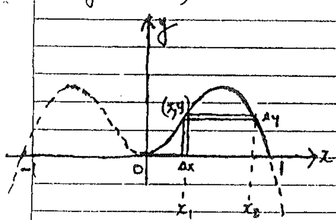
$\therefore \alpha^3 + \beta^3 + \gamma^3 = \frac{9}{2} \times \frac{81}{4} - \frac{21}{2}$

$= \frac{729 - 84}{8}$

$= \frac{645}{8}$

(b) $y = x^2 - x^4$

$y = x^2(1 - x^2)$



(i) $y' = 2x - 4x^3$

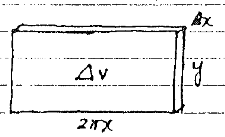
$y'' = 2 - 12x^2$

when $y' = 0$ $x = 0, \pm\frac{1}{\sqrt{2}}$

when $x = \frac{1}{\sqrt{2}}$, $y'' < 0$ ie $\cap$

$\therefore y_{max} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(ii)



$\Delta V = 2\pi x y \Delta x$

$\Delta V = 2\pi x(x^2 - x^4)\Delta x$

$V = 2\pi \int_0^1 (x^3 - x^5)\,dx$

$V = 2\pi \left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1$

$V = 2\pi\left[\frac{1}{4} - \frac{1}{6}\right]$

$V = 2\pi \times \frac{1}{12}$

$V = \frac{\pi}{6}$ cubic units

(iii) Take a strip as shown Rotating about y-axis – gives a washer.

Now $\Delta V = \pi(x_2^2 - x_1^2)\Delta y$

and $x^4 - x^2 + y = 0$

$\therefore x^2 = \frac{1 \pm \sqrt{1-4y}}{2}$

$x_2^2 = \frac{1+\sqrt{1-4y}}{2}$; $x_1^2 = \frac{1-\sqrt{1-4y}}{2}$

$x_2^2 - x_1^2 = \sqrt{1-4y}$

Hence $\Delta V = \pi(\sqrt{1-4y})\Delta y$

$V = \lim_{\Delta y \to 0} \sum_{y=0}^{1/4} \pi\sqrt{1-4y}\,\Delta y$

$V = \pi \int_0^{1/4} \sqrt{1-4y}\,dy$

$V = \pi \left[\frac{(1-4y)^{3/2}}{\frac{3}{2} \times -4}\right]_0^{1/4}$

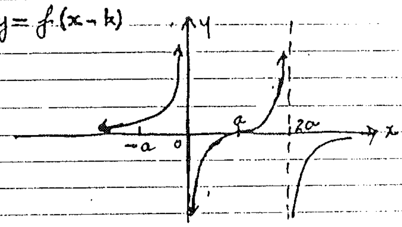
$V = -\frac{\pi}{6}[0 - 1]$

$V = \frac{\pi}{6}$ cubic units

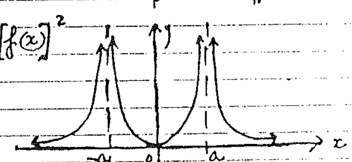
Question 6

(a)

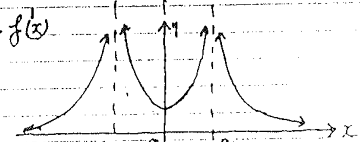
(i) $y = f(x - k)$



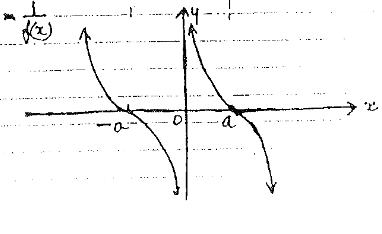
(ii) $y = [f(x)]^2$



(iii) $y = f'(x)$

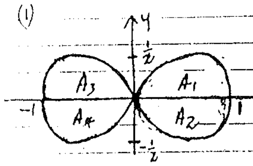


(iv) $y = \frac{1}{f(x)}$



(b) $y^2 = x^2(1-x^2)$

(i)



(ii) $y = \pm x\sqrt{1-x^2}$

considering $y = x\sqrt{1-x^2}$

$A_1 = \int_0^1 x\sqrt{1-x^2}\,dx$

using substitution

Let $u = 1 - u^2$

$-\frac{1}{2}\,du = x\,dx$

when $x = 0$, $u = 1$

$x = 1$, $u = 0$

$A_1 = -\frac{1}{2}\int_1^0 u^{\frac{1}{2}}\,du$

$A_1 = -\frac{1}{2}\left[\frac{u^{3/2}}{\frac{3}{2}}\right]_1^0$

$A_1 = -\frac{1}{3}\left[0-1\right]$

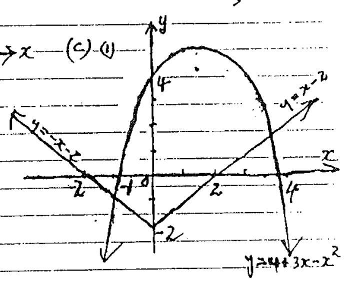
$A_1 = \frac{1}{3}$ unit^2

Now $A_1 = A_2 = A_3 = A_4$

$\therefore$ Required area $= 4 \times \frac{1}{3}$

$= 1\frac{1}{3}$ units^2

(c) (i)



Note $y = 4+3x-x^2$

$= -(x-4)(x+1)$

(ii) For $\dfrac{|x|-2}{4+3x-x^2} > 0$

Both $y = |x|-2$

and $y = 4+3x-x^2$

must have the same sign

Both are positive when

$2 < x < 4$

Both are negative when

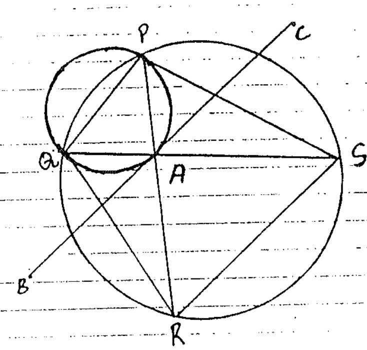
$-2 < x < -1$

$\therefore \dfrac{|x|-2}{4+3x-x^2} > 0$ when

$-2 < x < -1$ and $2 < x < 4$

Question 7

(a)



Let the tangent at A be BC.

$\widehat{BAQ} = \widehat{QPA}$ ($\angle$ between tangent and chord $= \angle$ in alternate segment)

$\widehat{QPA} = \widehat{ASR}$ ($\angle$'s in same segment)

$\therefore \widehat{BAQ} = \widehat{ASR}$

$\therefore BC \parallel RS$ ($=$ corresponding $\angle$'s)

(b)

Define the statement $S(n)$: $U_n = 5^n - 2^n$ for $n \geq 1$

Consider $S(1) = 5^1 - 2^1 = 3 = u_1 \Rightarrow S(1)$ true

$S(2) = 5^2 - 2^2 = 21 = u_2 \Rightarrow S(2)$ true

$S(3) = 5^3 - 2^3 = 117 = u_3 \Rightarrow S(3)$ true

Let k be a positive integer, $k \geq 3$.

If $S(n)$ is true for all integers $n \leq k$, Then $U_n = 5^k - 2^k$, for $n = 1, 2, 3, ---, k$.

Consider $S(k+1)$

$U_{k+1} = 7u_k - 10u_{k-1}$ (since $k+1 \geq 4$)

$$U_{k+1} = 7(5^k - 2^k) - 10(5^{k-1} - 2^{k-1})$$

$$= 7(5^k - 2^k) - 10\left(\frac{5^k}{5} - \frac{2^k}{2}\right)$$

$$= 7.5^k - 7.2^k - 2.5^k + 5.2^k$$

$$= 5.5^k - 2.2^k$$

$$= 5^{k+1} - 5^{k+1},$$

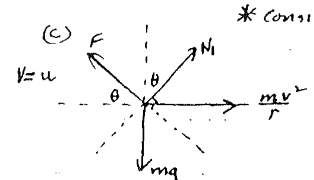
if $S(n)$ is true for $n = 1, 2, ---, k$.

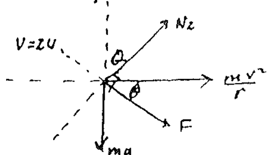
For $k = 3, 4, ---$, $S(n)$ true for all positive integers $n \leq k$ implies $S(k+1)$ is true.

Hence by induction, $S(n)$ is true for all positive integers n.

$\therefore U_n = 5^n - 2^n$ for $n \geq 1$

(c) * Consider two separate cases.





Vert

$N_1 \cos\theta + F\sin\theta = mg$ — ①

Hor

$N_1 \sin\theta - F\cos\theta = \frac{m \times u^2}{8u^2}$

ie $N_1 \sin\theta - F\cos\theta = \frac{m}{8}$ — ②

Vert

$N_2 \cos\theta - F\sin\theta = mg$ — ③

Hor

$N_2 \sin\theta + F\cos\theta = \frac{m \times 4u^2}{8u^2}$

ie $N_2 \sin\theta + F\cos\theta = \frac{m}{2}$ — ④

$\left.\begin{array}{l} ① \times \sin\theta : \\ ② \times \cos\theta : \end{array}\right\}$ then subtract.

$$F\sin^2\theta + F\cos^2\theta = mg\sin\theta - \frac{m}{8}\cos\theta$$

$$F(\sin^2\theta + \cos^2\theta) = m(g\sin\theta - \frac{1}{8}\cos\theta)$$

$$\therefore F = m(g\sin\theta - \frac{1}{8}\cos\theta) \quad -- ⑤$$

$\left.\begin{array}{l} ③ \times \sin\theta : \\ ④ \times \cos\theta : \end{array}\right\}$ Then subtract.

$$-F\sin^2\theta - F\cos^2\theta = mg\sin\theta - \frac{m}{2}\cos\theta$$

$$\therefore F = m(\frac{1}{2}\cos\theta - g\sin\theta) \quad -- ⑥$$

Equate ⑤ and ⑥

$$g\sin\theta - \frac{1}{8}\cos\theta = \frac{1}{2}\cos\theta - g\sin\theta$$

$$2g\sin\theta = \frac{5}{8}\cos\theta$$

$$\tan\theta = \frac{5}{16 \times 10} \quad (g = 10\, m/s^2)$$

$\therefore \theta = 1°47'$

or $F = 0$

$$N\cos\theta = mg \quad \text{— (1)}$$

$$N\sin\theta = \frac{mV^2}{8u^2} \quad \text{— (2)}$$

(2) $\div$ (1) $\quad \frac{\sin\theta}{\cos\theta} = \frac{mV^2}{8u^2} \times \frac{1}{mg}$

$$\tan\theta = \frac{V^2}{8 \times u^2 \times 10} = \frac{V^2}{80u^2} \quad [g = 10\,m/sec^2]$$

But from (7) above

$$\tan\theta = \frac{1}{32} \quad \cancel{7t\theta}$$

Equate these results for $\tan\theta$

$$\frac{V^2}{80u^2} = \frac{1}{32}$$

$$V^2 = \frac{80u^2}{32}$$

$$V^2 = \frac{5u^2}{2}$$

$$V = u\sqrt{\tfrac{5}{2}} \quad \text{or} \quad \sqrt{\tfrac{5u^2}{2}}\ m/sec$$

Question (8)

(a) $(1+x)^n > 1 + nx$

$x > 0$ and $n > 1$

Soln

Consider the difference

$$f(x) = (1+x)^n - (1+nx)$$

then $f'(x) = n(1+x)^{n-1} - n$

$$f'(x) = n\left[(1+x)^{n-1} - 1\right]$$

since $x > 0$ and $n - 1 > 0$

we have $(1+x)^{n-1} > 1$

so $f'(x) > 0$

therefore the function ($f(x)$) is increasing (from 0 to ∞)

In particular $f(0) < f(x)$ when $0 < x$.

But $f(0) = 0$,

so $0 < (1+x)^n - (1+nx)$

and therefore $(1+x)^n > 1 + nx$ when $x > 0$

(b)

Solution

$$C = 1 + \cos\theta + \cos 2\theta + \dots + \cos(n-1)\theta$$

$$S = \sin\theta + \sin 2\theta + \dots + \sin(n-1)\theta$$

(I) $$iS = i\sin\theta + i\sin 2\theta + \dots + i\sin(n-1)\theta$$

(II) $$C + iS = 1 + (\cos\theta + i\sin\theta) + (\cos 2\theta + i\sin 2\theta) + \dots + (\cos(n-1)\theta + i\sin(n-1)$$

Let $z = \cos\theta + i\sin\theta$

using De Moivre's Theorem

(III) $$C + iS = 1 + z + z^2 + \dots + z^{n-1}$$

This series is a GP with $a = 1$, $r = z$ and number of terms n

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$C + iS = \frac{1(1-z^n)}{1-z}$$

(IV) $$C + iS = \frac{1-z^n}{1-z}$$

(V) $$\therefore C + iS = \frac{1-(\cos\theta + i\sin\theta)^n}{1-(\cos\theta + i\sin\theta)}$$

$$= \frac{1-(\cos n\theta + i\sin n\theta)}{1-(\cos\theta + i\sin\theta)}$$

realising denominator $$= \frac{1-(\cos n\theta + i\sin n\theta)}{1-(\cos\theta + i\sin\theta)} \times \frac{1+(\cos\theta + i\sin\theta)}{1+(\cos\theta + i\sin\theta)}$$

$$C + iS = \frac{(1-\cos n\theta)(1-\cos\theta) - i^2\sin n\theta\sin\theta + i(1-\cos n\theta)\sin\theta - i\sin n\theta(1-\cos}{(1-\cos\theta)^2 - i^2\sin^2\theta}$$

$$C + iS = \frac{(1-\cos n\theta)(1-\cos\theta) + \sin n\theta\sin\theta + i\{(1-\cos n\theta)\sin\theta - (1-\cos\theta)\sin n\theta}{(1-\cos\theta)^2 + \sin^2\theta}$$

Equating real parts.

$$C = \frac{(1-\cos n\theta)(1-\cos\theta) + \sin n\theta\sin\theta}{(1-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{1-\cos\theta - \cos n\theta + \cos n\theta\cos\theta + \sin n\theta\sin\theta}{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{1-\cos\theta - \cos n\theta + \cos(n\theta - \theta)}{2 - 2\cos\theta}$$

$$= \frac{1-\cos\theta - (\cos n\theta - \cos(n-1)\theta)}{2(1-\cos\theta)}$$

$$= \frac{2\sin^2\frac{1}{2}\theta - 2\sin\frac{1}{2}(2n-1)\theta\sin\frac{1}{2}\theta}{2 \times 2\sin^2\frac{1}{2}\theta}$$

$$= \frac{1}{2}\,\frac{\sin\frac{1}{2}\theta - \sin\frac{1}{2}(2n-1)\theta}{\sin\frac{1}{2}\theta}$$

$$= \frac{1}{2}\,\frac{2\sin\frac{1}{2}n\theta\cos\frac{1}{2}(1-n)\theta}{\sin\frac{1}{2}\theta} = \frac{\sin\frac{1}{2}n\theta\cos\frac{1}{2}(n-1)\theta}{\sin\frac{1}{2}\theta}$$

Equating imaginary parts.

$$S = \frac{(1-\cos n\theta)\sin\theta - (1-\cos\theta)\sin n\theta}{(1-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{\sin\theta - \sin\theta\cos n\theta - \sin n\theta + \sin n\theta\cos\theta}{2(1-\cos\theta)}$$

$$= \frac{\sin\theta - \sin n\theta + (\sin n\theta\cos\theta - \cos n\theta\sin\theta)}{\sin^2\frac{1}{2}\theta}$$

$$= \frac{\sin\theta - \sin n\theta + \sin(n\theta - \theta)}{\sin^2\frac{1}{2}\theta}$$

$$= \frac{\sin\theta + \sin(n-1)\theta - \sin n\theta}{\sin^2\frac{1}{2}\theta}$$

$$= \frac{\sin\theta + 2\cos\frac{(2n-1)\theta}{2}\sin -\frac{\theta}{2}}{\sin^2\frac{1}{2}\theta}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2} - 2\cos\frac{1}{2}(2n-1)\theta\sin\frac{\theta}{2}}{\sin^2\frac{1}{2}\theta}$$

$$= \frac{2\{\cos\frac{1}{2}\theta - \cos\frac{1}{2}(2n-1)\theta\}}{\sin\frac{1}{2}\theta}$$

$$= \frac{-2\sin\frac{1}{2}n\theta\sin\frac{1}{2}(1-n)\theta}{\sin\frac{1}{2}\theta}$$

$$= \frac{2\sin\frac{1}{2}n\theta\sin\frac{1}{2}(n-1)\theta}{\sin\frac{1}{2}\theta}$$