Question 1 Use a separate writing book.

Marks

a. Find the exact value of: 9

i. $\int_1^e \ln x\,dx$

ii. $\int_{-2}^{2} \frac{2dx}{x^2+4x+20}$

iii. $\int_0^1 \frac{x^2dx}{\sqrt{4-x^2}}$

iv. $\int_{-1}^{1} (\sin^{-1} x)^3\,dx$

b. If $U_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$ (where n is a positive integer), show that 6

$U_n = \left(\frac{n-1}{n}\right)U_{n-2}$.

Hence evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x\,dx$.

Question 2 Use a separate writing book.

Marks

a. Show that the pentagon formed by the roots of $z^5+1=0$ in the Argand plane has an area of A square units, where 3

$$A=\frac{5}{2}\sin\frac{2\pi}{5}$$

b. Describe and sketch the locus of the complex number z, satisfied by 3

$$1<|z|\le 4 \text{ and } |\arg z|\le\frac{3\pi}{4}$$

c. i. Find the square roots of the complex number $(-3-4i)$, expressing answer in the form $(a+ib)$. 6

ii. Hence solve the equation $z^2-(5-2i)z+6-4i=0$

d. A, B, C and D are four points in the Argand diagram representing the complex numbers Z_1, Z_2, Z_3 and Z_4 respectively. Given that Z_2 and Z_4 are purely imaginary and Z_1 and Z_3 are real; and $\arg(Z_2-Z_3)=\arg(Z_1-Z_4)$. Analyse this data with a suitable diagram and define clearly the quadrilateral ABCD. 3

Question 3 Use a separate writing book.

a. Consider the function $f(x)=(x-2)^2(x-1)$ 3

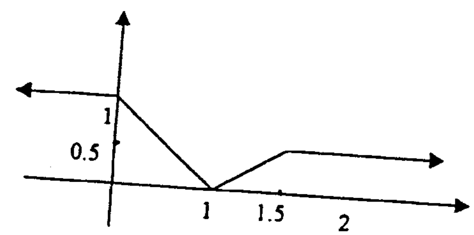
On separate diagrams draw neat sketches of:

i. $y=f(x)$

ii. $y=f(2x)$

iii. $y^2=f(2x)$

b. The diagram given is a sketch of the function $y=f(x)$. 5



On separate diagrams sketch:

i. $y=f(-x)$

ii. $y=[f(x)]^2$

iii. $y=\frac{1}{f(x)}$

iv. $|y|=f(x)$

v. $y=\log_e[f(x)]$

c. Solve graphically the inequation $2\cos|x|>1$ for $-\pi\le x\le 2\pi$ 4

d. Find the value of m, so that the equation $5x^5-3x^3+m=0$ has two equal positive roots. 3

P II

Question 4 Use a separate writing book. **Marks**

a. Prove that the equation of the tangent at the point $P(x_1,y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a>b$, is given by $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$. **9**

The tangent at P to this ellipse cuts the major axis at Q and the directrix $x=\frac{a}{e}$ at R. S is the corresponding focus. PT is the line from P perpendicular to the x-axis.

i. Draw a diagram to represent this information and label the co-ordinates of Q and R.

ii. Prove that $OT \times OQ = a^2$

iii. Prove that $\angle PSR = 90°$.

b. The tangents at $R(cr,\frac{c}{r})$ and $S(cs,\frac{c}{s})$ on the rectangular hyperbola $xy=c^2$ in the first quadrant, meet at T. Show that OT produced bisects RS, where O is the origin. **6**

Question 5 Use a separate writing book. **Marks**

a. i. Sketch the curve $y=\sin(\pi x^2)$ for $0\le x\le 2$, clearly showing all turning points and intercepts with the x-axis. **7**

ii. The area bounded by $y=\sin(\pi x^2)$ and the x-axis for $0\le x\le 1$ is rotated about the y-axis. Using the method of cylindrical shells, show that the volume of the solid formed is 2 cubic units.

b. i. Sketch $y=\frac{3-x^2}{1+x^2}$ clearly showing its intercepts with coordinate axes. **8**

ii. A solid is formed in which each cross-section perpendicular to the y-axis is a square. If one side of each square cross-section is parallel to the x-axis and its end points lie on the curve $y=\frac{3-x^2}{1+x^2}$, show that the volume of the solid whose base is bounded by the x-axis and the above curve is given by $V=4\int_0^3 \frac{3-y}{1+y}\,dy$

iii. Find the volume of the above solid.

Question 6 Use a separate writing book. **Marks**

a. How many different ways are there of seating four married couples at a circular table with men and women in alternate positions and no wife next to her husband? (Two seating arrangements are the same if each person has the same left and right hand neighbours.) **4**

b. Twelve pupils enter a quiz competition. From the twelve pupils, two teams of five pupils will be chosen to compete against each other. **4**

i. How many different competitions can be arranged?

ii. Jill, Grant and Robert are triplets amongst the twelve pupils. Find the probability that they will be chosen on the same team.

c. Given that $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$ where $C_r = {}^nC_r = \frac{n!}{r!(n-r)!}$. **7**

i. By investigating $\frac{d}{dx}\left[x(1+x)^n\right]$ show that

$$C_0 - 2C_1 + 3C_2 - \ldots + (-1)^n(n+1)C_n = 0$$

ii. By choosing the appropriate definite integral of the expression $(1+x)^n$, show

$$\frac{1}{2}C_0 + \frac{1}{3}C_1 + \frac{1}{4}C_2 + \ldots + \frac{1}{n+2}C_n = \frac{2^{n+2}-1}{n+2} - \frac{2^{n+2}-1}{n+1}$$

Question 7 Use a separate writing book.

Marks

a. A body of mass m is projected vertically upwards from the ground with speed u_0. The force due to gravity acting on the body is constant but there is a resisting force of magnitude mkv^2 at speed v, where k is a constant. **8**

Show that:

i. the maximum height H which the body reaches is given by $2kH = \ln\left(\dfrac{g+ku_0^2}{g}\right)$

ii. the speed v_0 with which the body reaches the ground is given by $2kH = \ln\left(\dfrac{g}{g-kv_0^2}\right)$

iii. Using the above results show that $\dfrac{1}{v_0^2} = \dfrac{1}{u_0^2} + \dfrac{k}{g}$

b. A mass m kg is suspended from the end of a light in elastic string of length l metres which is fixed to a point O. The mass is moving with constant angular velocity ω and describes a circle of radius R metres in the horizontal plane. **7**

i. Show, by considering the forces acting on the mass m, that the angle between the string and the vertical through O is given by $\tan\theta = \dfrac{R\omega^2}{g}$, where g is the acceleration due to gravity.

ii. Show that the period of rotation is given by $2\pi\sqrt{\dfrac{l\cos\theta}{g}}$.

iii. A light inelastic string is used to swing a mass of 600 grams in a circle with a period of 1.7 seconds. If the tension in the string is 20 newtons, find the length of the string.

Question 8 Use a separate writing book.

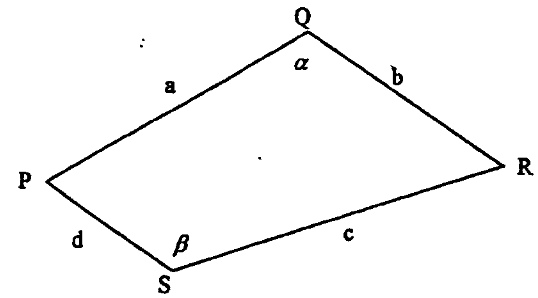
Marks

P III

a. Use Mathematical Induction to show that $3^n > n^3$ for integers $n>3$. **7**

Hence or otherwise prove that $\sqrt[3]{3} > \sqrt[n]{n}$ for $n>3$.

b. **8**



The sides of the quadrilateral PQRS have fixed lengths a, b, c and d, as shown in the diagram. The sizes of the angles PQR and PSR are α and β radians respectively, where $0<\alpha<\pi$, $0<\beta<\pi$.

i. Use the cosine rule to rule find two expressions for $(PR)^2$ and hence , by differentiating both expressions with respect to α, show that $\dfrac{d\beta}{d\alpha} = \dfrac{ab\sin\alpha}{cd\sin\beta}$

ii. Find an expression for the area of quadrilateral PQRS in terms of sin α and sin β, and hence prove that the area is a maximum when $\alpha+\beta=\pi$.

Question 1

(a) (i)
$$I=\int_1^e \ln x \, dx$$

If $\frac{dv}{dx}=1$ and $u=\ln x$

$$I=\left[x\ln x\right]_1^e-\int_1^e x\cdot\frac{1}{x}\,dx$$
$$=\left[x\ln x\right]_1^e-\int_1^e dx$$
$$=\left[x\ln x-x\right]_1^e$$
$$=\left[(e\ln e-e)-(1\ln 1-1)\right]$$
$$=e-e+1$$
$$I=1$$

(ii)
$$I=\int_{-2}^{2}\frac{2\,dx}{x^2+4x+20}\,dx$$
$$=\int_{-2}^{2}\frac{2\,dx}{(x+2)^2+4^2}$$

Note $x^2+4x+20$
$=x^2+4x+4+16$
$=(x+2)^2+4^2$

$$=\left[\frac{2}{4}\tan^{-1}\left(\frac{x+2}{4}\right)\right]_{-2}^{2}$$
$$=\frac{1}{2}\left[\tan^{-1}(1)-\tan^{-1}(0)\right]$$
$$=\frac{1}{2}\times\frac{\pi}{4}$$
$$I=\frac{\pi}{8}$$

(iii)
$$I=\int_0^1\frac{x^2\,dx}{\sqrt{4-x^2}}$$

Let $x=2\sin\theta$
$dx=2\cos\theta\,d\theta$
when $x=0,\ \theta=0$
$x=1\quad \theta=\frac{\pi}{6}$

$$=\int_0^{\pi/6}\frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\cdot 2\cos\theta\,d\theta$$
$$=\int_0^{\pi/6}\frac{4\sin^2\theta}{\sqrt{4\cos^2\theta}}\,2\cos\theta\,d\theta$$
$$=\int_0^{\pi/6}\frac{4\sin^2\theta\cdot 2\cos\theta}{2\cos\theta}\,d\theta$$
$$=4\int_0^{\pi/6}\sin^2\theta\,d\theta$$

Note $\cos 2\theta=1-2\sin^2\theta$
$2\sin^2\theta=1-\cos 2\theta$

$$=2\int_0^{\pi/6}(1-\cos 2\theta)\,d\theta$$
$$=2\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\pi/6}$$
$$=2\left[\left(\frac{\pi}{6}-\frac{1}{2}\sin\frac{\pi}{3}\right)-(0)\right]$$
$$=2\left[\frac{\pi}{6}-\frac{1}{2}\times\frac{\sqrt{3}}{2}\right]$$
$$=\frac{\pi}{3}-\frac{\sqrt{3}}{2}$$
$$I=\frac{2\pi-3\sqrt{3}}{6}$$

(iv) $I = \int (\sin^{-1}x)^3 dx$

If $f(x) = (\sin^{-1}x)^3$

then $f(-x) = (\sin^{-1}(-x))^3$

$= (-\sin^{-1}x)^3$

$= -(\sin^{-1}x)^3$

$= -f(x)$

$\therefore f(x)$ is an odd function

Hence $\int_{-1}^{1} (\sin^{-1}x)^3 dx = 0$

ie $I = 0$.

PV

(b) $U_n = \int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} (\sin^{n-1}x \cdot \sin x)\, dx$

If $\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

and $u = \sin^{n-1}x = (\sin x)^{n-1} \Rightarrow \frac{du}{dx} = (n-1)(\sin x)^{n-2}\cos x$

then $U_n = \left[(\sin^{n-1}x)(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} (-\cos x)(n-1)(\sin x)^{n-2}\cos x\, dx$

$U_n = \left[(-\cos x)(\sin^{n-1}x)\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\pi/2} \cos^2 x \sin^{n-2}x\, dx$

$= 0 + (n-1)\int_0^{\frac{\pi}{2}} (1-\sin^2 x)(\sin^{n-2}x)\, dx$

$U_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\, dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x\, dx$

$U_n = (n-1)U_{(n-2)} - (n-1)U_n$

$U_n + (n-1)U_n = (n-1)U_{n-2}$

$n U_n = (n-1)U_{n-2}$

$U_n = \left(\frac{n-1}{n}\right)U_{n-2}$.

$\int_0^{\frac{\pi}{2}} \sin^6 x\, dx = U_6$

$U_6 = \frac{5}{6}U_4$

$= \frac{5}{6} \times \frac{3}{4} U_2$

$= \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} U_0$

$U_6 = \frac{15}{48} \times \frac{\pi}{2} = \frac{5\pi}{32}$

Note $U_0 = \int_0^{\frac{\pi}{2}} dx$

$U_0 = [x]_0^{\frac{\pi}{2}}$

$U_0 = \frac{\pi}{2}$.

Question 2

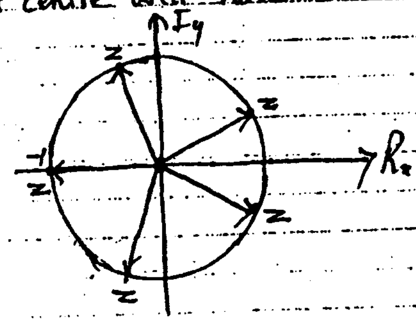
(a) $z^5+1=0$

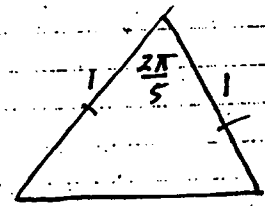
$z^5=-1$

$z=\sqrt[5]{-1}=-1$

5 roots equally spaced about a circle of radius 1 unit. One real root of -1; other 4 in conjugate pairs. Verticies are those of a regular pentagon.

$\therefore$ Pentagon is the sum of 5 congruent triangles. Each vertex at centre will have an angle of $\frac{2\pi}{5}$ radian





area of $\Delta = \frac{1}{2}ab\sin C$

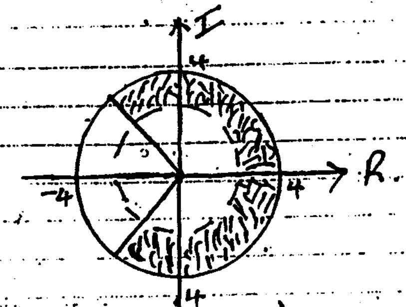
area of Pentagon $= 5\left(\frac{1}{2}\times1\times1\times\sin\frac{2\pi}{5}\right)$

$= \frac{5}{2}\sin\frac{2\pi}{5}$ units^2

(b) $1<|x|\leq 4$ $\qquad |\arg z|\leq\frac{3\pi}{4}$

$1<\sqrt{x^2+y^2}\leq 4$ $\qquad$ ie $-\frac{3\pi}{4}(\arg z)\leq\frac{3\pi}{4}$

$1<x^2+y^2\leq 16$



Annulus with circles of radii 1 and 4 units between lines $y=x; x<0$ and $y=-x; x<0$ as shown in the diagram.

(c) (i) Let $a+bi=\sqrt{-3-4i}$ where a, b are real

square both sides

$(a+bi)^2=-3-4i$

$a^2-b^2+2abi=-3-4i$

$a^2-b^2=-3$ ... ①

$2ab=-4$ ... ②

solving : gives $a=\pm 1$

when $a=1$, $b=-2$

and when $a=-1$, $b=2$

hence the square roots of $(-3-4i)$ are $(1-2i)$ and $(-1+2i)$

(ii) $z^2-(5-2i)z+(6-4i)=0$

$$z=\frac{(5-2i)\pm\sqrt{(5-2i)^2-4(6-4i)}}{2}$$

$$z=\frac{(5-2i)\pm\sqrt{-3-4i}}{2}$$

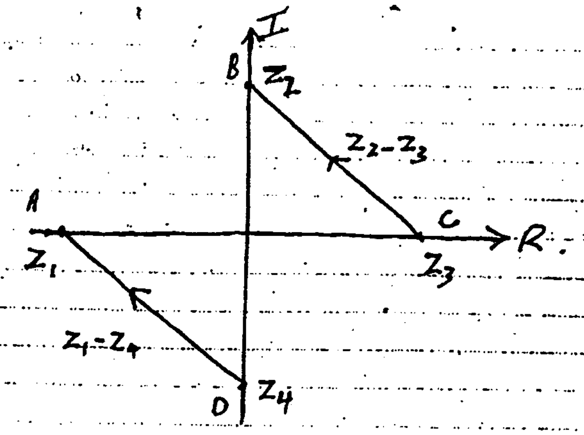
$$z=\frac{(5-2i)\pm(1-2i)}{2}$$

$$z=\frac{6-4i}{2} \text{ or } \frac{4}{2}$$

$$z=(3-2i) \text{ or } 2.$$

P VII

(d)



B and D lie on the Imaginary axis

A and C lie on the Real axis

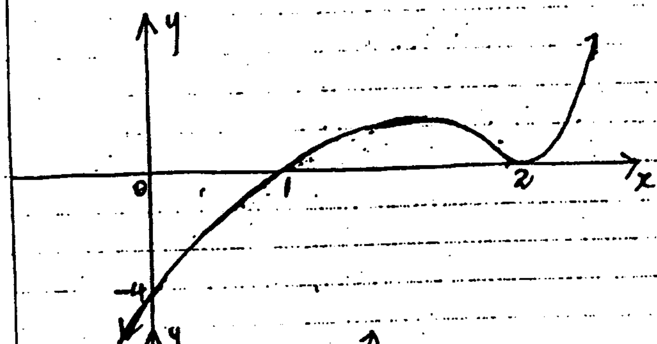
$\arg(z_2-z_3)=\arg(z_1-z_4)$

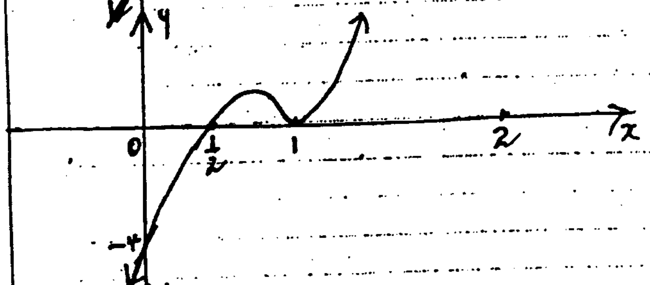
$\Rightarrow CB \parallel DA$

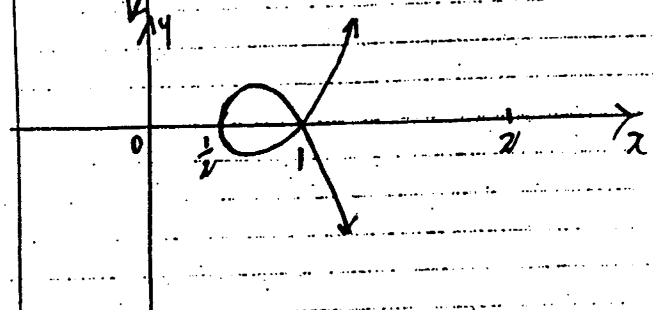
ABCD is a trapezium

Question 3

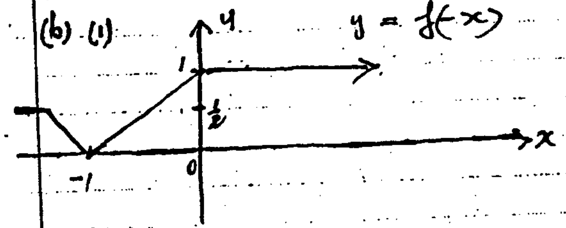
(a) $f(x) = (x-2)^2(x-1)$



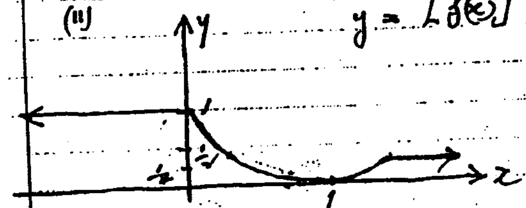




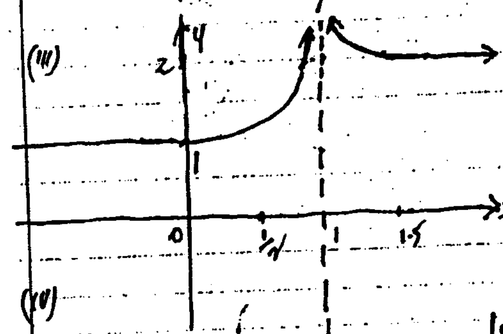
(b) (i) $y = f(-x)$



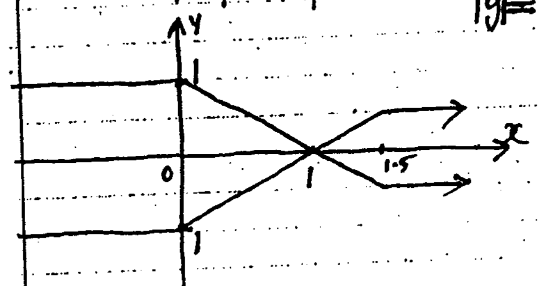
(ii) $y = [f(x)]^2$



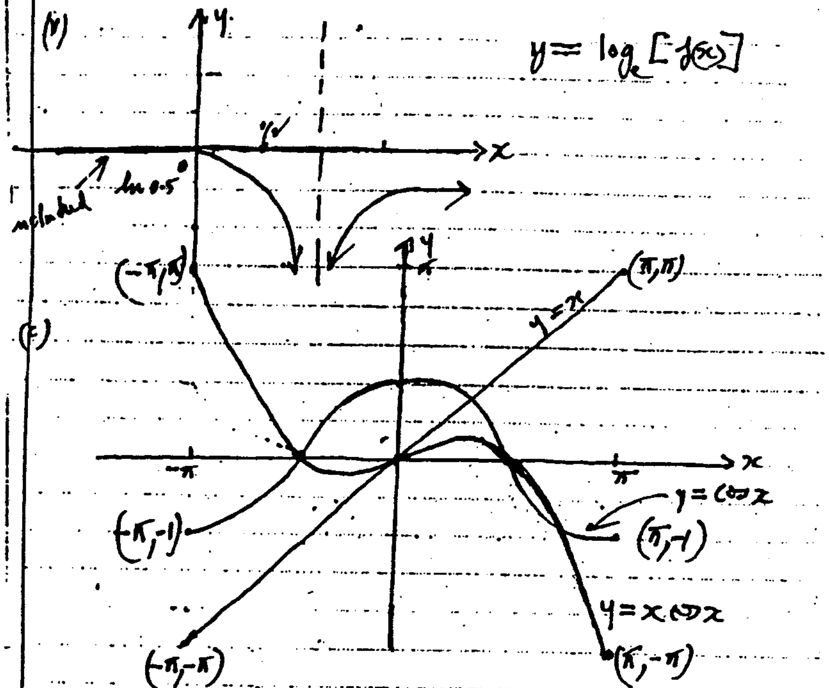
(iii) $y = \frac{1}{f(x)}$



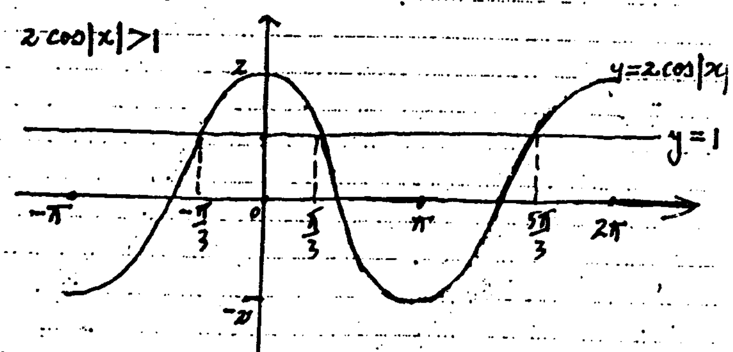
(iv) $|y| = f(x)$



(v)



(d) $2\cos|x| > 1$



Ans $\left\{-\frac{\pi}{3} < x < \frac{\pi}{3}\right\}$ or $\left\{\frac{5\pi}{3} < x \leq 2\pi\right\}$

(d) $5x^5 - 3x^3 + m = 0$

Let $P(x) = 5x^5 - 3x^3 + m$.

$P'(x) = 25x^4 - 9x^2$

$= x^2(25x^2 - 9)$

$= x^2(5x-3)(5x+3)$

Note condition given: For double root at $x = \alpha$ we have: $P(\alpha) = 0$ and $P'(\alpha) = 0$

Now when $P'(x) = 0$

$x = 0$ or $x = \frac{3}{5}$ or $x = -\frac{3}{5}$

If $x > 0$. Then consider $x = \frac{3}{5}$.

when $x = \frac{3}{5}$

when $P(x) = 0$

$5\left(\frac{3}{5}\right)^5 - 3\left(\frac{3}{5}\right)^3 + m = 0.$

$\left(\frac{3}{5}\right)^3\left(5 \times \frac{9}{25} - 3\right) + m = 0$

$\frac{27}{125}\left(\frac{45}{25} - 3\right) + m = 0$

$\frac{27}{125}\left(-\frac{30}{25}\right) + m = 0$

$m = \frac{27}{125} \times \frac{6}{5}$

$m = \frac{162}{625}$

Question 4

(a) $P(x_1, y_1)$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{y}{b^2}\frac{dy}{dx} = -\frac{x}{a^2}$$

$$\frac{dy}{dx} = -\frac{x}{a^2} \times \frac{b^2}{y}$$

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

at P gradient of tangent.

$$m = -\frac{b^2x_1}{a^2y_1}$$

Equation of tangent.

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$$

$$a^2y_1y - a^2y_1^2 = -b^2x_1x + b^2x_1^2$$

$$b^2x_1x + a^2y_1y = b^2x_1^2 + a^2x_1^2$$

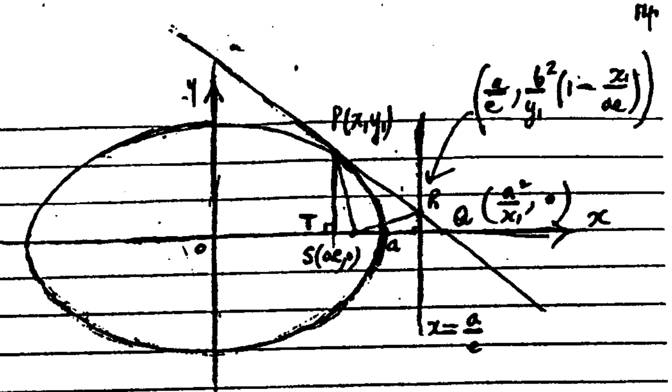
Divide throughout by a^2b^2

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

Now $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$ because P lies on ellipse.

$$\therefore \frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$$

(i)



tangent PQ: $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$ ——— (1)

Point Q: Let $y = 0$ in (1)

$$\therefore x = \frac{a^2}{x_1}$$

directrix: $x = \frac{a}{e}$ ——— (2)

Sub (2) in (1) $\frac{x_1}{a^2} \cdot \frac{a}{e} + \frac{y_1y}{b^2} = 1$

$$\frac{x_1}{ae} + \frac{y_1y}{b^2} = 1$$

$$\frac{y_1y}{b^2} = 1 - \frac{x_1}{ae}$$

$$y = \frac{b^2}{y_1}\left(1 - \frac{x_1}{ae}\right)$$

(ii) $OT = x_1$; $OQ = \frac{a^2}{x_1}$

$$OT \times OQ = x_1 \times \frac{a^2}{x_1} = a^2$$

(iii) Gradient of PS : $m_1 = \frac{y_1 - 0}{x_1 - ae}$

$m_1 = \frac{y_1}{x_1 - ae}$

Gradient of SR : $m_2 = \frac{\frac{b^2}{y_1}\left(1 - \frac{x_1}{ae}\right) - 0}{\frac{a}{e} - ae}$

$m_1 m_2 = \frac{y_1}{x_1 - ae} \times \frac{\frac{b^2}{y_1}\left(1 - \frac{x_1}{ae}\right)}{\frac{a}{e} - ae}$

$= \frac{b^2(ae - x_1)e}{(x_1 - ae)(a - ae^2)ae}$

$= -\frac{b^2}{a^2 - a^2e^2}$

$m_1 m_2 = -\frac{b^2}{a^2(1-e^2)}$

$m_1 m_2 = -\frac{b^2}{b^2}$ Note for an ellipse $b^2 = a^2(1-e^2)$

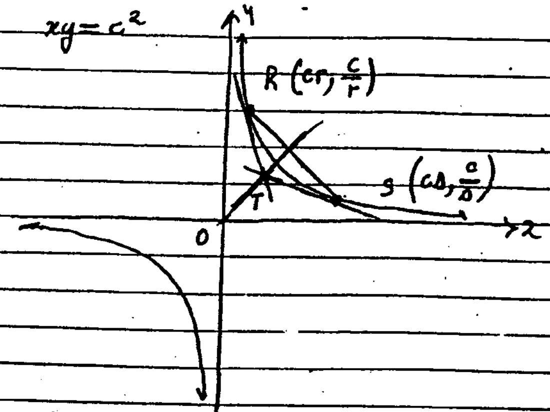
$m_1 m_2 = -1$

Now this is the condition for perpendicular lines

Hence $\hat{PSR} = 90^\circ$

P XI

(b) $xy = c^2$



Tangent at S.

$y = c^2 x^{-1}$

$y' = -\frac{c^2}{x^2}$

$m = -\frac{c^2}{(cs)^2}$

$m = -\frac{1}{s^2}$

Equation: $y - \frac{c}{s} = -\frac{1}{s^2}(x - cs)$

$s^2 y - cs = -x + cs$

$x + s^2 y = 2cs$ ---- (1)

Similarly equation of tangent at R:

$x + r^2 y = 2cr$ ---- (2)

Solve (1) and (2) for co-ordinates of T

(1) − (2) $y(s^2 - r^2) = 2c(s - r)$

$y(s + r) = 2c$

$y_T = \frac{2c}{s+r}$

sub in (1)

$$x + s^2\left(\frac{2c}{s+r}\right) = 2cs$$

$$x = 2cs - s^2\left(\frac{2c}{s+r}\right)$$

$$x = 2cs\left(1 - \frac{s}{s+r}\right)$$

$$x = 2cs\left(\frac{s+r-s}{s+r}\right)$$

$$x_T = \frac{2crs}{s+r}.$$

$\therefore$ Co-ordinates of $T = \left(\frac{2crs}{s+r}, \frac{2c}{s+r}\right)$

Gradient of OT : $m = \frac{2c/s+r}{2crs/s+r}$

$$m = \frac{1}{rs}$$

$\therefore$ Equation of OT is of the form $y = mx$

ie $y = \frac{1}{rs}x$ --- (3)

mid-point of RS: $M = \left(\frac{c(r+s)}{2}, \frac{c(r+s)}{2rs}\right)$

Now OT passes through the mid point of RS if the mid point satisfies the equation of OT.

Sub M in (3)

$RHS = \frac{x}{rs} = \frac{c(r+s)}{2rs}$

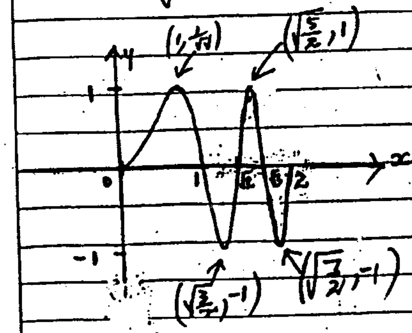
$LHS = y = \frac{c(r+s)}{2rs}$

$\therefore LHS = RHS.$

Hence OT produced bisects RS, where O is the origin

Question 5

(a) (i) $y = \sin(\pi x^2)$ $\quad 0 \le x \le 2$



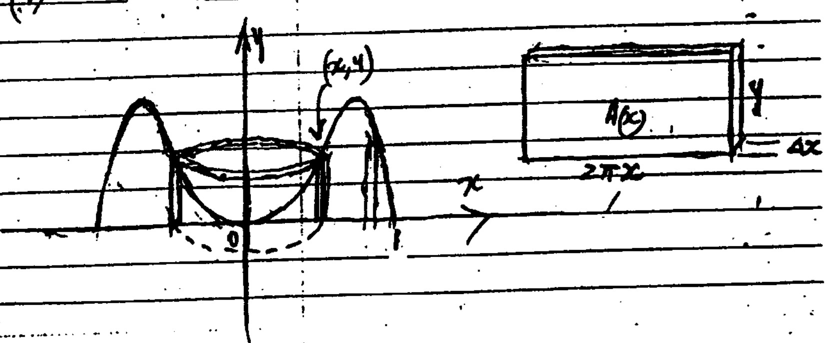
cuts the x axis when $\pi x^2 = 0, \pi, 2\pi, 3\pi, 4\pi$, etc

$x = 0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}, \pm 2.$

turning points when $\pi x^2 = 0, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$ etc

$x = 0, \pm\frac{1}{\sqrt{2}}, \pm\sqrt{\frac{3}{2}}, \pm\sqrt{\frac{5}{2}}, \pm\sqrt{\frac{7}{2}}$

(ii)



$A(x) = 2\pi x y$

$A(x) = 2\pi x\left[\sin(\pi x^2)\right]$

$\Delta V = 2\pi x \sin(\pi x^2)\,\Delta x$

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{1} 2\pi x \sin(\pi x^2)\,\Delta x$$

$$V = \int_0^1 2\pi x \sin(\pi x^2)\,dx$$

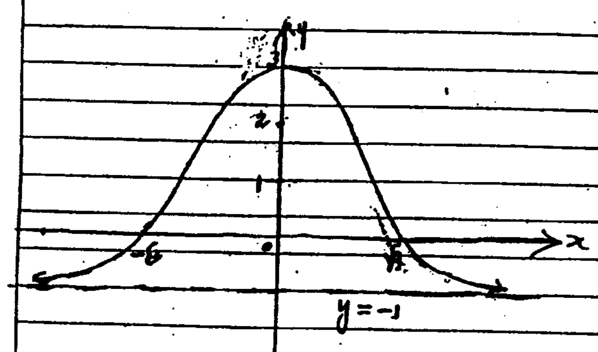
$$= -\left[\cos(\pi x^2)\right]_0^1$$

$$= -\left[\cos\pi - \cos 0\right]$$

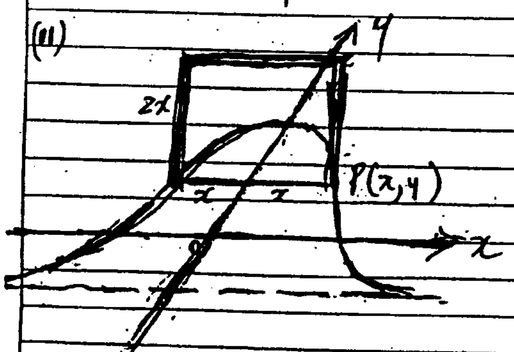
$$= -(-1-1)$$

$$= 2 \text{ units}^3.$$

(b) (i) $y = \dfrac{3-x^2}{1+x^2}$



(ii)



$$V = \lim_{\Delta y \to 0} \sum_{y=0}^{3} \Delta A\,\Delta y$$

$$= \lim_{\Delta y \to 0} \sum_{y=0}^{3} (2x)^2\,\Delta y$$

But $y = \dfrac{3-x^2}{1+x^2}$

$y + yx^2 = 3 - x^2$

$x^2(y+1) = 3-y$

$x^2 = \dfrac{3-y}{y+1}$

$$V = 4\int_0^3 \left(\frac{3-y}{y+1}\right)dy$$

$$V = 4\int_0^3 \left(-1 + \frac{4}{y+1}\right)dy$$

$$= 4\left[-y + 4\ln(y+1)\right]_0^3$$

$$V = 4\,(4\ln 4 - 3) \text{ cubic units.}$$

P XIV

Question 6

(a) No ways of seating men in alternate seats at a round table $= \frac{4!}{4}$

No of ways of seating women in remaining positions $= 4!$

$\therefore$ Total number of arrangements $= 3! \times 4!$
$= 144$

By considering each husband and wife as a pair, the number of arrangements of 4 pairs at a round table $= \frac{4!}{4}$
$= 6$

Each pair can be arranged in two ways

$\therefore$ number of ways with husband and wifes together $= 6 \times 2 = 12$.

$\therefore$ Number of ways with husband and wife not together $= 144 - 12$
$= 132$

(b) (i) Number of different competitions
= number of ways of choosing 5 from 12 × number of ways of choosing a second 5 from the remaining 7 ÷ number of ways of arranging the two groups

$= ({}^{12}C_5 \times {}^{7}C_5) \div 2$

$= 8316$.

(ii) Prob that Jill, Grant, Robert are amongst the group = no of ways of choosing J, G, R × no of ways of choosing the other 2 members of their team from the remaining 9 × the no of ways of choosing the second team of 5 ÷ 8316.

$= \frac{{}^{3}C_3 \times {}^{9}C_2 \times {}^{7}C_2}{8316}$

$= \frac{72}{792}$

$= \frac{1}{11}$

(c)(i)

Given $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$ ---- (1)

$$\frac{d}{dx}\left[x(1+x)^n\right] = x \times n(1+x)^{n-1} + (1+x)^n \times 1$$

$$= xn(1+x)^{n-1} + (1+x)^n \text{ ---- (2)}$$

(1) $\times x$: $x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + \ldots + C_nx^{n+1}$ ---- (3)

Differentiating RHS of (3) and equating to (2) gives

$$xn(1+x)^n + (1+x)^n = C_0 + 2C_1x + 3C_2x^2 + \ldots (n+1)C_nx^n$$

When $x = -1$

$$0 = C_0 - 2C_1 + 3C_2 - \ldots + (-1)^n(n+1)C_n$$

(ii)
$$\int_0^1 (1+x)^n dx = \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1$$

$$= \left[\frac{2^{n+1}}{n+1} - \frac{1}{n+1}\right]$$

$$= \frac{2^{n+1}-1}{n+1} \text{ ------ (1)}$$

Now
$$\int_0^1 x(1+x)^n dx = \int_0^1 \left[(1+x)-1\right](1+x)^n dx$$

$$= \int_0^1 (1+x)^{n+1} - \int (1+x)^n dx$$

(using (1))
$$= \frac{2^{n+2}-1}{n+1+1} - \frac{2^{n+1}-1}{n+1} \text{ ---- (2)}$$

And from eq(1)

$$\int_0^1 x(1+x)^n dx$$

$$= \int_0^1 \left[C_0x + C_1x^2 + C_2x^3 + \ldots C_nx^{n+1}\right] dx$$

$$= \left[\frac{C_0x^2}{2} + \frac{C_1x^3}{3} + \frac{C_2x^4}{4} + \ldots \frac{C_nx^{n+2}}{n+2}\right]_0^1$$

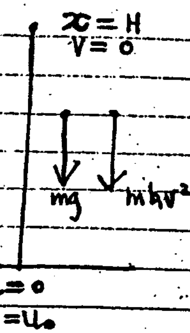
$$= \frac{C_0}{2} + \frac{C_1}{3} + \frac{C_2}{4} + \ldots + \frac{C_n}{n+2} \text{ ---- (3)}$$

equating (2) and (3) gives:

$$\frac{1}{2}C_0 + \frac{1}{3}C_1 + \frac{1}{4}C_2 + \ldots + \frac{1}{n+2}C_n = \frac{2^{n+2}-1}{n+2} - \frac{2^{n+1}-1}{n+1}$$

Question 7

(a) upwards



$$m\ddot{x} = -mg - mkv^2$$

$$\ddot{x} = -g - kv^2$$

$$v\frac{dv}{dx} = -(g + kv^2)$$

$$\frac{dv}{dx} = -\left(\frac{g+kv^2}{v}\right)$$

$$\frac{dx}{dv} = -\left(\frac{v}{g+kv^2}\right)$$

$$\int_0^H dx = \int_{u_0}^0 \frac{-v}{g+kv^2}\,dv$$

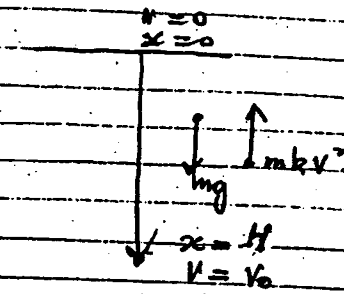
$$\left[x\right]_0^H = -\frac{1}{2k}\left[\ln(g+kv^2)\right]_{u_0}^0$$

$$H = -\frac{1}{2k}\left[\ln(g) - \ln(g+ku_0^2)\right]$$

$$H = \frac{1}{2k}\ln\left(\frac{g+ku_0^2}{g}\right)$$

$$\therefore\ 2kH = \ln\left(\frac{g+ku_0^2}{g}\right)$$

(ii) downwards



$$m\ddot{x} = mg - mkv^2$$

$$\ddot{x} = g - kv^2$$

$$v\frac{dv}{dx} = g - kv^2$$

$$\frac{dv}{dx} = \frac{g-kv^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{g-kv^2}$$

$$\int_0^H dx = \int_0^{V_0} \frac{v}{g-kv^2}\,dv$$

$$\left[x\right]_0^H = -\frac{1}{2k}\left[\ln(g-kv^2)\right]_0^{V_0}$$

$$H = -\frac{1}{2k}\left[\ln(g-kV_0^2) - \ln g\right]$$

$$2kH = \ln g - \ln(g-kV_0^2)$$

$$2kH = \ln\left(\frac{g}{g-kV_0^2}\right)$$

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(iii) $\ln\left(\frac{g+ku_0^2}{g}\right) = \ln\left(\frac{g}{g-kV_0^2}\right)$

$$\frac{g+ku_0^2}{g} = \frac{g}{g-kV_0^2}$$

$$g^2 = (g+ku_0^2)(g-kV_0^2)$$

$$g^2 = g^2 + g(ku_0^2 - kV_0^2) - k^2u_0^2V_0^2$$

$$k^2u_0^2V_0^2 = g(ku_0^2 - kV_0^2)$$

$$k^2u_0^2V_0^2 = gk(u_0^2 - V_0^2)$$

$$\frac{k}{g} = \frac{u_0^2 - V_0^2}{u_0^2V_0^2}$$

$$\frac{k}{g} = \frac{1}{V_0^2} - \frac{1}{u_0^2}$$

$$\frac{1}{V_0^2} = \frac{1}{u_0^2} + \frac{k}{g}.$$

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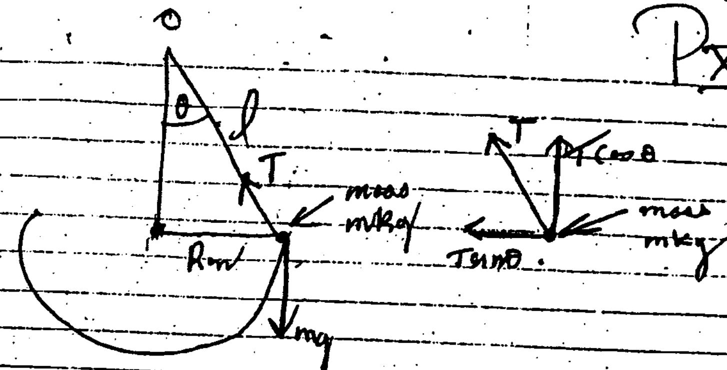
P XVII

(b)



(i) $T\cos\theta = mg$ Vertical ------ (1)

$T\sin\theta = mR\omega^2$ Horizontal ---- (2)

(2) ÷ (1) $\tan\theta = \frac{R\omega^2}{g}$ ------ (3)

(ii) Period $= \frac{2\pi}{\omega}$

and $\omega^2 = \frac{g\tan\theta}{R}$ using (3)

also $\frac{R}{l} = \sin\theta \Rightarrow R = l\sin\theta$.

$$\therefore \omega^2 = \frac{g\tan\theta}{l\sin\theta},$$

$$= \frac{g}{l\cos\theta}$$

$$\text{Period} = \frac{2\pi}{\sqrt{\frac{g}{l\cos\theta}}} = 2\pi\times\sqrt{\frac{l\cos\theta}{g}}$$

(iii) $T = 20N$

$m = 0.6kg$

Period $= 1.7sec.$

$g = 9.8 m/sec^2$

$T \cos\theta = mg$

$\cos\theta = \frac{mg}{T}$

$= \frac{9.8 \times 0.6}{20}$

$\cos\theta \doteqdot 0.294$

$1.7 = 2\pi\sqrt{\frac{l(0.294)}{9.8}}$

$\frac{l(0.294)}{9.8} = \left(\frac{1.7}{2\pi}\right)^2$

$l = \left(\frac{1.7}{2\pi}\right)^2 \times \frac{9.8}{0.294}$

$l = 2.4$ metres

PXVIII

Question 8

(a) For $n = 4$ investigate $3^n > n^3$

LHS $= 3^4 = 81$

RHS $= 4^3 = 64$

$\therefore$ statement $3^n > n^3$ is true for $n = 4$.

Assume the statement is true for $n = k$.

ie $3^k > k^3$

ie $3^k - k^3 > 0$

The aim is to show that the statement is true for $n = k+1$

ie. $3^{k+1} > (k+1)^3$

Consider $3^{k+1} - (k+1)^3$

$= 3.3^k - (k^3 + 3k^2 + 3k + 1)$

$= 3.3^k - k^3 - 3k^2 - 3k - 1$

$= 3.3^k - 3k^3 + 2k^3 - 3k^2 - 3k - 1$

$= 3(3^k - k^3) + k^3 - 3k^2 + 3k - 1 + k^3 - 6k$

$= 3(3^k - k^3) + (k-1)^3 + k^3 - 6k$

$= 3(3^k - k^3) + (k-1)^3 + k(k^2 - 6) > 0$,

since (a) $3^k - k^3 > 0$ by assumption

(b) $(k-1)^3 >$ since $k > 3$ and integral

(c) $k(k^2-6) > 0$ since $k > 3$ and integral

Thus $3^{k+1} > (k+1)^3$.

The statement is thus true for n = k+1 if it is true for n = k

But it is true for n = 4 and thus it is true for n = 5 and so on for all integral n > 3

now n is positive and integral

since $3^n > n^3$

take the (3n)th root of both sides

$$(3^n)^{\frac{1}{3n}} > (n^3)^{\frac{1}{3n}}$$

ie $3^{\frac{n}{3n}} > n^{\frac{3}{3n}}$

$$3^{1/3} > n^{1/n}$$

ie $\sqrt[3]{3} > \sqrt[n]{n}$

P XIX

(b) (i) a, b, c, d are constants

By the Cosine Rule:

$$(PR)^2 = a^2+b^2-2ab\cos\alpha$$
$$(PR)^2 = c^2+d^2-2cd\cos\beta$$

~~Equating these results~~

$$a^2+b^2-2ab\cos\alpha = c^2+d^2-2cd\cos\beta$$

differentiating both sides w.r.t. α

$$-2ab(-\sin\alpha) = -2cd\left(-\sin\beta\frac{d\beta}{d\alpha}\right)$$

$$2ab\sin\alpha = 2cd(\sin\beta)\frac{d\beta}{d\alpha}$$

$$\frac{d\beta}{d\alpha} = \frac{ab\sin\alpha}{cd\sin\beta}$$

(ii) Let A be the area of the quadrilateral.

$$A = \frac{1}{2}ab\sin\alpha + \frac{1}{2}cd\sin\beta$$

$$\frac{dA}{d\alpha} = \frac{1}{2}ab\cos\alpha + \frac{1}{2}cd\cos\beta\frac{d\beta}{d\alpha}$$

$$= \frac{1}{2}ab\cos\alpha + \frac{1}{2}cd\cos\beta \times \left(\frac{ab\sin\alpha}{cd\sin\beta}\right)$$

$$= \frac{1}{2}ab\left(\cos\alpha + \frac{\cos\beta\times\sin\alpha}{\sin\beta}\right)$$

$$\frac{dA}{d\alpha} = \frac{1}{2}ab\left(\frac{\sin\beta\cos\alpha + \cos\beta\sin\alpha}{\sin\beta}\right)$$

$$\frac{dA}{d\alpha} = \frac{1}{2}ab\,\frac{\sin(\alpha+\beta)}{\sin\beta}$$

Now $a \neq$ and $b \neq 0$

So if $\frac{dA}{d\alpha} = 0$

$\sin(\alpha+\beta) = 0$

ie $\alpha+\beta = 0$; $\alpha+\beta = \pi$, $\alpha+\beta = 2\pi$, etc.

But since $0 < \alpha < \pi$

$0 < \beta < \pi$

Then $0 < \alpha+\beta < 2\pi$

$\therefore \frac{dA}{d\alpha} = 0$ gives $\alpha+\beta = \pi$

sign change test: of $\frac{dA}{d\alpha}$ for $0 < (\alpha+\beta) < 2\pi$

First note for $0 < \beta < \pi$, $\sin\beta > 0$

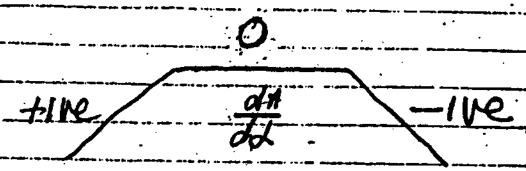
for $(\alpha+\beta) < \pi$ $\sin(\alpha+\beta) > 0$

$\therefore$ for $(\alpha+\beta) < \pi$ $\frac{dA}{d\alpha} > 0$

for $(\alpha+\beta) > \pi$ $\sin(\alpha+\beta) < 0$

$\therefore$ for $(\alpha+\beta) > \pi$ $\frac{dA}{d\alpha} < 0$

1 XX



$\therefore A$ is a maximum for $\alpha+\beta = \pi$