ST IGNATIUS COLLEGE RIVERVIEW

TASK 4

YEAR 12

2004

EXTENSION 2

TRIAL HSC EXAMINATION

Time allowed: 3 hours + 5 minutes reading time.

Instructions to Candidates

- Attempt **all** questions
- Show all necessary working.
- Marks may be deducted for missing or poorly arranged work.
- Board approved calculators may be used.
- Each question attempted must be returned in a ***separate*** writing booklet clearly marked Question 1, Question 2 etc, on the cover
- **Each booklet must have your name and the name of your mathematics teacher written on the cover.**

Question 1 (15 marks) Use a SEPARATE writing booklet. Marks

a

If $Z_1 = 1+2i$, $Z_2 = 2-i$ and $Z_3 = 1-\sqrt{3}\,i$,
Express in the form $(a+bi)$ where a and b are real.

(i) $Z_1 + Z_2$ 1

(ii) $\dfrac{1}{Z_2}$ 1

(iii) $(Z_3)^3$ 2

b

Express $\dfrac{4+3i}{3+i}$ in the form $(a+bi)$ where a and b are real numbers. 2

c

(i) Express $Z = \sqrt{3} + i$ in modulus- argument form. 1

(ii) Hence, show that $Z^7 + 64Z = 0$. 3

d

(i) Find the square root(s) of $(-8+6i)$. 3

(ii) Hence, solve the equation

$2Z^2 - (3+i)Z + 2 = 0$, expressing Z in the form $(a+bi)$ where a and b are real. 2

Question 2 (15 marks) Use a SEPARATE writing booklet. Marks

a Evaluate

(i) $\int_0^{\frac{\pi}{4}} x \sin 2x \, dx$. 3

(ii) $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$. 2

(iii) $\int_0^{\frac{\pi}{3}} \frac{\tan x}{1+\cos x} \, dx$. (using $t = \tan \frac{x}{2}$). 4

b Show that , if $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$. 6

Then $I_n + I_{n-2} = \frac{1}{n-1}$, where n is an integer and $n \geq 3$

Hence evaluate I_7 .

Question 3 (15 marks) Use a SEPARATE writing booklet. Marks

a The point A $(a \cos\alpha, b \sin\alpha)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

B is the foot of the perpendicular from A to the x-axis. The normal at A cuts the x-axis at C.

(i) Represent this information with a suitable diagram. 1

(ii) Derive the equation of the normal AC. 3

(iii) Show that the length of CB is $\left|\frac{b^2 \cos\alpha}{a}\right|$. 3

b Consider the hyperbola H with equation $4x^2-9y^2=36$. The point $R(x_1, y_1)$ is an arbitrary point on H.

(i) Prove that the equation of the tangent l at R is $4x_1x-9y_1y=36$. 3

(ii) Find the co-ordinates of the point K at which l cuts the x-axis. 1

(iii) Hence, prove that $\frac{SR}{PR}=\frac{SK}{PK}$ where S and P are the foci of H. 4

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

a The equation $x^3 - 3x + 3 = 0$ has roots which are α, β and γ. Find the equation in x where the roots are α^2, β^2 and γ^2. 4

b The base of a solid is a circle of radius 2 units. A diameter runs through the centre of the base. Any cross section of the solid formed by a plane perpendicular to the given diameter is an equilateral triangle. 5

Show that the volume of the solid is $\dfrac{32\sqrt{3}}{3}$ units^3.

c The region bounded by the curve $y = \log_e x$, the straight lines $y = 1$ and $x = 3$ is rotated about the y-axis. Find the volume of the resulting solid using the method of cylindrical shells. 6

Question 5 (15 marks) Use a SEPARATE writing booklet. Marks

a Find the four fourth roots of -16 in the form $(a+bi)$. 4

b A function is defined by $f(x)=\frac{\log_e x}{x}$ for $x>0$.

(i) Find the x intercept. 1

(ii) Find the turning point. 2

(iii) Find the point of inflection. 2

(iv) Sketch the graph of $y=f(x)$. 2

c Consider the function in part (b) sketch

i $y=|f(x)|$. 2

ii $y=\frac{1}{f(x)}$. 2

Question 6 (15 marks) Use a SEPARATE writing booklet. Marks

a Consider the polynomial $Q(x) = ax^4 + bx^3 + cx^2 + dx + e$ where a, b, c, d and e are integers. Suppose α is an integer such that $Q(\alpha) = 0$.

(i) Prove that α is a factor of e. 2

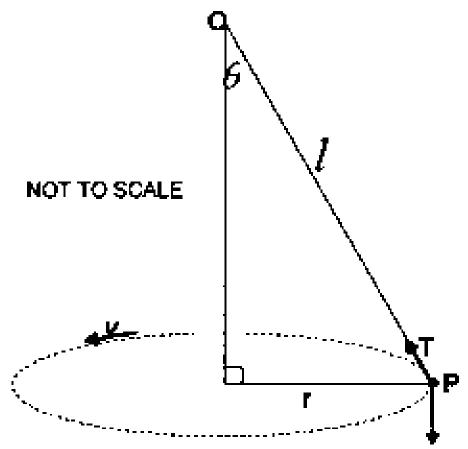
(ii) Prove that the polynomial equation $P(x) = 0$, where $P(x) = 4x^4 - x^3 + 3x^2 + 2x - 3$ does not have an integer root. 2

b It is estimated that the probability that a torpedo will hit its target is $\frac{1}{3}$.

(i) If 5 torpedoes are fired, what is the probability of 3 successes. 2

(ii) How many torpedoes must be fired so that the probability of at least one success should be greater than 0.9? 2

c



The above diagram shows a light string of length l, fixed at O, and making an angle 0 with the vertical as shown in the above diagram. A particle is attached at P. The particle moves with uniform speed v metres / second in a horizontal circle of radius r. The centre of the circle is directly below O.

If the particle is to maintain its motion in a horizontal circle, show by resolving forces vertically and horizontally, that the particle's velocity is given by $v = \sqrt{rg \tan \theta}$. (Note: g is the acceleration due to gravity) 4

d When a polynomial $P(x)$ is divided by $(x-3)$ the remainder is 5 and when it is divided by $(x-4)$ the remainder is 9. Find the remainder when $P(x)$ is divided by $(x-4)(x-3)$. 3

Question 7 (15 marks) Use a SEPARATE writing booklet. Marks

a If $\sin^{-1} x$, $\cos^{-1} x$ and $\sin^{-1}(1-x)$ are acute, show that 6

$\sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1$.

Hence, solve the equation

$\sin^{-1} x - \cos^{-1} x = \sin^{-1}(1-x)$.

b Find the general solution of the equation $3\tan^2 x = 2\sin x$. 5

c Each of the following statements is either true or false. Write 'True' or 'False' for each statement giving a brief reason for your answers. (You are not required to evaluate the integrals).

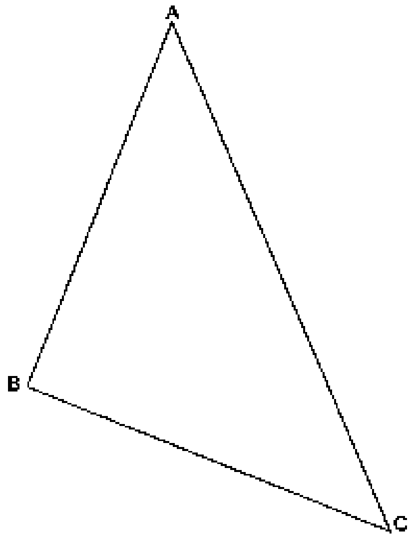
(i) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \, dx = 0$. 2

(ii) $\int_{-1}^{1} e^{-x^2} \cos^{-1} x \, dx = 0$. 2

Question 8 (15 marks) Use a SEPARATE writing booklet. Marks

a In the Argand diagram, the points A, B and C represent the complex numbers Z_1, Z_2 and Z_3 respectively.

What can you say about triangle ABC if $i(Z_3 - Z_2) = (Z_1 - Z_2)$. 2



b Solve for x if $|3x+3| + |x-1| \le 4x + 3$. 5

c A particle, projected vertically upward with initial speed u is subjected to forces which create a constant vertical downward acceleration of magnitude g and an acceleration, directed against the motion, of magnitude kv when the speed is v. 8

(i) Show that the acceleration function is given by $\ddot{x} = -g - kv$.

(ii) Prove that the maximum height reached by the particle after a time T is given by $T = \frac{1}{k}\log_e\left(\frac{g+ku}{g}\right)$.

(iii) Prove that the maximum height reached is $\frac{1}{k}(u - gT)$.

Aids to Solutions

SIC Ext. II Trial 2004

Question 1 (15 marks)

(a) $z_1 = 1 + 2i$, $z_2 = 2 - i$ and $z_3 = 1 - \sqrt{3}i$

(1) (i) $z_1 + z_2 = 3 + i$

(1) (ii) $\frac{1}{z_2} = \frac{1}{2-i} \frac{\times(2+i)}{\times(2+i)}$

$= \frac{2}{5} + \frac{1}{5}i$

(2) (iii) $(z_3)^3 = (1-\sqrt{3}i)^3$

$= 1 - 3\sqrt{3}i + 3(\sqrt{3}i)^2 - (\sqrt{3}i)^3$

$= 1 - 3\sqrt{3}i - 9 + 3\sqrt{3}i$

$= -8$

(2) (b) $\frac{4+3i}{3+i} \frac{\times(3-i)}{\times(3-i)} = \frac{12 - 4i + 9i + 3}{10}$

$= \frac{3}{2} + \frac{1}{2}i$

(1) (c) (i) $z = \sqrt{3} + i$

$|z| = \sqrt{3+1}$ $\quad$ $\arg z = \tan^{-1}\frac{1}{\sqrt{3}}$

$= 2$ $\quad$ $= \frac{\pi}{6}$

$\therefore \quad z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$\Big\}\ z = 2\,\text{cis}\,\frac{\pi}{6}$

(3) (ii) $z^7 = 2^7\,\text{cis}\,\frac{7\pi}{6}$ $\qquad$ $64z = 64\left(2\,\text{cis}\,\frac{\pi}{6}\right)$

$= 128\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$ $\qquad$ $= 128\,\text{cis}\,\frac{\pi}{6}$

$= -128\,\text{cis}\,\frac{\pi}{6}$

$z^7 + 64z = 0$ (as required)

Question 1 (Continued)

(3) (d) (i) let $a + bi = \sqrt{-8+6i}$ $(a, b \in \mathbb{R})$

$a^2 - b^2 + 2abi = -8 + 6i$

$\therefore a^2 - b^2 = -8$ — ①

and $2ab = 6$

$ab = 3$ — ②

from ② $a = \frac{3}{b}$ sub ① : $\left(\frac{3}{b}\right)^2 - b^2 = -8$

$9 - b^4 = -8b^2$

$b^4 - 8b^2 - 9 = 0$

$(b^2 - 9)(b^2 + 1) = 0$

$b^2 = -1$ No Real sol^n

$b = \pm 3$

when $b = 3$ $a = 1$

$b = -3$ $a = -1$

The square roots are

$1 + 3i$ and $-1 - 3i$

or $\pm(1 + 3i)$

(2) (ii) $2z^2 - (3+i)z + 2 = 0$

$z = \dfrac{(3+i) \pm \sqrt{(3+i)^2 - 4(2)(2)}}{4}$ $9 + 6i - 1 - 16$

$= \dfrac{(3+i) \pm \sqrt{-8+6i}}{4}$

$= \dfrac{3 + i + 1 + 3i}{4}$ or $\dfrac{3 + i - 1 - 3i}{4}$

$z = 1 + i$ or $z = \frac{1}{2} - \frac{1}{2}i$

Question 2 (15 marks)

(3) (a) (i) $I = \int_0^{\frac{\pi}{4}} x \sin 2x \, . dx$ let $u = x$, $\frac{du}{dx} = 1$, $\frac{dv}{dx} = \sin 2x$, $V = -\frac{1}{2}\cos 2x$

$= \left[-\frac{1}{2}x\cos 2x\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \frac{1}{2}\cos 2x \, . dx$

$= \left[-\frac{\pi}{8}\cos\frac{\pi}{2} - 0\right] + \frac{1}{4}\left[\sin 2x\right]_0^{\frac{\pi}{4}}$

$= 0 + \frac{1}{4}\left[\sin\frac{\pi}{2} - \sin 0\right]$

$= \frac{1}{4}$

(2) (ii) $I = \int_0^1 \frac{dx}{\sqrt{4-x^2}}$

$= \left[\sin^{-1}\frac{x}{2}\right]_0^1$

$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)$

$= \frac{\pi}{6}$

(4) (iii) $I = \int_0^{\frac{\pi}{3}} \frac{\tan x}{1+\cos x} \, . dx$

let $t = \tan\frac{x}{2}$

$x = 2\tan^{-1} t$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

when $x = 0$, $t = 0$

when $x = \frac{\pi}{3}$, $t = \frac{1}{\sqrt{3}}$

$I = \int_0^{\frac{1}{\sqrt{3}}} \frac{\frac{2t}{1-t^2}}{1 + \frac{1-t^2}{1+t^2}} \cdot \frac{2\,dt}{1+t^2}$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2t}{1-t^2} \times \frac{(1+t^2)}{2} \times \frac{2}{(1+t^2)} \, . dt$ (with $(1+t^2)$ and 2 cancelled)

Question 2 (Continued)

(a) (iii)

$$I = \int_0^{\frac{1}{\sqrt{3}}} \frac{2t}{1-t^2} \cdot dt$$

$$= -\left[\ln(1-t^2)\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= -\left[\ln\frac{2}{3} - \ln 1\right]$$

$$= \ln\frac{3}{2}$$

(b)

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \cdot dx$$

$$I_n + I_{n-2} = \int_0^{\frac{\pi}{4}} \tan^n x \cdot dx + \int_0^{\frac{\pi}{4}} \tan^{n-2} x \cdot dx$$

(3)

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \cdot (\tan^2 x + 1) \cdot dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \cdot \sec^2 x \; dx$$

$$= \frac{1}{n-1}\left[\tan^{n-1} x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{n-1}\left[\left(\tan\frac{\pi}{4}\right)^{n-1} - 0\right]$$

$$= \frac{1}{n-1} \quad \text{(as required)}$$

let $u = \tan x$

$$\frac{du}{dx} = \sec^2 x$$

$$du = \sec^2 x \cdot dx$$

$$\int u^{n-2} \cdot du$$

$$= \frac{1}{n-1} \cdot u^{n-1}$$

Question 2 (Continued)

(3) (b) Using $I_n + I_{n-2} = \frac{1}{n-1}$, $n \geq 3$

n=7 Then $I_7 + I_5 = \frac{1}{6}$

$I_7 = \frac{1}{6} - I_5$

n=5 Then $I_5 + I_3 = \frac{1}{4}$

$I_5 = \frac{1}{4} - I_3$

n=3 Then $I_3 + I_1 = \frac{1}{2}$

$I_1 = \int_0^{\frac{\pi}{4}} \tan x \, . dx$ $\qquad \tan x = \frac{\sin x}{\cos x}$

$= - \left[\ln \cos x \right]_0^{\frac{\pi}{4}}$ $\qquad = \frac{f'(x)}{f(x)}$

$= - \left[\ln \frac{1}{\sqrt{2}} - \ln 1 \right]$

$= \ln \sqrt{2}$ or $\frac{1}{2} \ln 2$

$I_7 = \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - \frac{1}{2} \ln 2$ $\qquad \frac{2}{12} - \frac{3}{12} + \frac{6}{12}$

$= \frac{5}{12} - \frac{1}{2} \ln 2$

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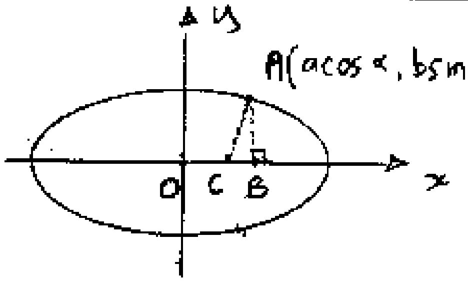
Question 3

(1) (a) (i)



(3) (ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by implicit diff^{n} we have

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{2x}{a^2} \cdot \frac{b^2}{2y}$$

$$m_T = -\frac{b^2 x}{a^2 y}$$

$$m_N = \frac{a^2 y}{b^2 x}$$

Equation of Normal at A $(a\cos\alpha, b\sin\alpha)$: $y - b\sin\alpha = \frac{a^2 b\sin\alpha}{b^2 a\cos\alpha}\left(x - a\cos\alpha\right)$

$$(b\cos\alpha)y - b^2\sin\alpha\cos\alpha = a\sin\alpha\,(x - a\cos\alpha)$$

$$= (a\sin\alpha)x - a^2\sin\alpha\cos\alpha$$

$\div$ by $\sin\alpha\cos\alpha$ $\quad \frac{by}{\sin\alpha} - b^2 = \frac{ax}{\cos\alpha} - a^2$

$$a^2 - b^2 = \frac{ax}{\cos\alpha} - \frac{by}{\sin\alpha}$$

(3) (iii) put $y = 0$: $\quad a^2 - b^2 = \frac{ax}{\cos\alpha}$

$$x = \frac{\cos\alpha\,(a^2 - b^2)}{a}$$

Question 3 (Continued)

(a) (iii)

$$CB = |OB - OC|$$

$$= \left| a\cos\alpha - \frac{a^2 - b^2}{a}\cdot\cos\alpha \right|$$

$$= \cos\alpha \left[\frac{a^2 - a^2 + b^2}{a} \right]$$

$$= \left| \frac{b^2 \cos\alpha}{a} \right| \quad \text{(as required)}$$

(b) $4x^2 - 9y^2 = 36$ $\quad$ $R(x_1, y_1)$ on Hyperbola

(3) (i) By implicit differentiation

$$8x - 18y\,\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{8x}{18y}$$

$$= \frac{4x}{9y}$$

Equation of tangent at $R(x_1, y_1)$:

$$y - y_1 = \frac{4x_1}{9y_1}(x - x_1)$$

$$9y_1y - 9y_1^2 = 4x_1x - 4x_1^2$$

Since $R(x_1, y_1)$ lies on H Then $4x_1^2 - 9y_1^2 = 36$

$$36 = 4x_1x - 9y_1y \quad \text{(as required)}$$

(ii)

Question 3 (Continued)

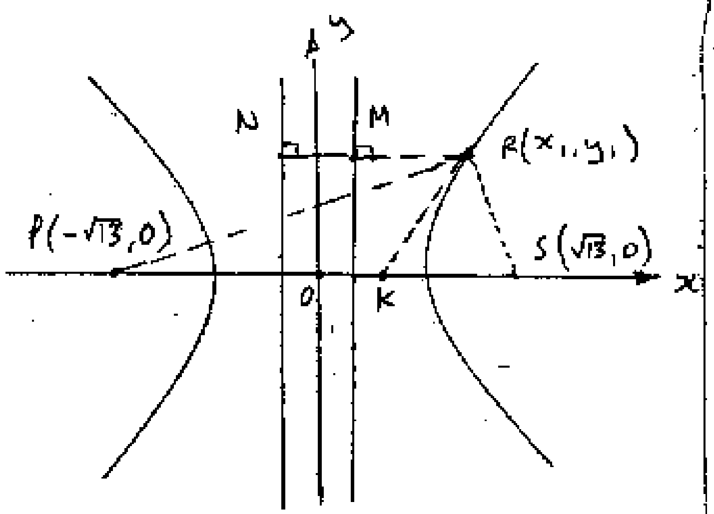
(1) (b) (ii) ℓ cuts x-axis when $y = 0$

$$36 = 4x_1 x - 0$$

$$x = \frac{9}{x_1}$$

$$\therefore K\left(\frac{9}{x_1}, 0\right)$$

(4) (iii) Prove $\frac{SR}{PR} = \frac{SK}{PK}$



$SK = OS - OK$

$= \sqrt{13} - \frac{9}{x_1}$

$PK = OP + OK$

$= \sqrt{13} + \frac{9}{x_1}$

$$\frac{SK}{PK} = \frac{\sqrt{13} - \frac{9}{x_1}}{\sqrt{13} + \frac{9}{x_1}}$$

$$= \frac{x_1\sqrt{13} - 9}{x_1\sqrt{13} + 9}$$

$\frac{x^2}{9} - \frac{y^2}{4} = 1$

$a = 3$

$b = 2$

$b^2 = a^2(e^2 - 1)$

$e^2 = 1 + \frac{b^2}{a^2}$

$= 1 + \frac{4}{9}$

$e = \frac{\sqrt{13}}{3}$

$S(\pm ae, 0)$

$S(\pm\sqrt{13}, 0)$

$x = \pm\frac{a}{e}$

$x = \pm\frac{9}{\sqrt{13}}$

Question 3 (Continued)

(b) (iii) By def^{n} of Hyperbola, we have

$$\frac{SR}{MR} = e \Rightarrow SR = eMR$$

and

$$\frac{PR}{NR} = e \Rightarrow PR = eNR$$

$$\frac{SR}{PR} = \frac{MR}{NR}$$

$MR = x_1 - \frac{9}{\sqrt{13}}$ and $NR = x_1 + \frac{9}{\sqrt{13}}$

$$\frac{MR}{NR} = \frac{x_1 - \frac{9}{\sqrt{13}}}{x_1 + \frac{9}{\sqrt{13}}}$$

$$\therefore \frac{SR}{PR} = \frac{x_1\sqrt{13} - 9}{x_1\sqrt{13} + 9}$$

$$= \frac{SK}{PK}$$ (as required)

Question 4 (15 marks)

(4) (a) $x^3 - 3x + 3 = 0$ roots α, β and γ.

α^2, β^2 and γ^2 will satisfy $x^{\frac{3}{2}} - 3x^{\frac{1}{2}} + 3 = 0$

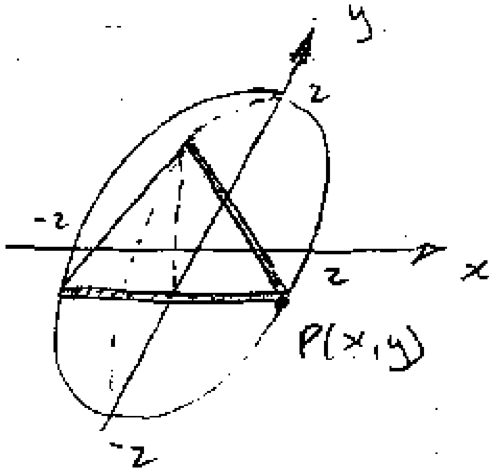
i.e. $x^{\frac{1}{2}}(x - 3) = -3$

Squaring: $x(x-3)^2 = 9$

$x(x^2 - 6x + 9) = 9$

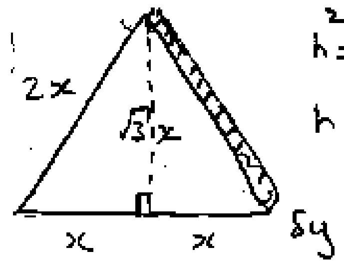
$x^3 - 6x^2 + 9x - 9 = 0$

(5) (b)



Base $x^2 + y^2 = 4$

Typical cross section.



$h^2 = 4x^2 - x^2$

$h = \sqrt{3}\,x$

$\delta A = \frac{1}{2}bh$

$= \frac{1}{2} \cdot 2x \cdot \sqrt{3}\,x$

$= \sqrt{3}\,x^2$

Typical Volume

$\delta V \doteq \sqrt{3}\,x^2 . \delta y$

$V = \int_{-2}^{2} \sqrt{3}\,x^2 . dy$

Using $x^2 = 4 - y^2$

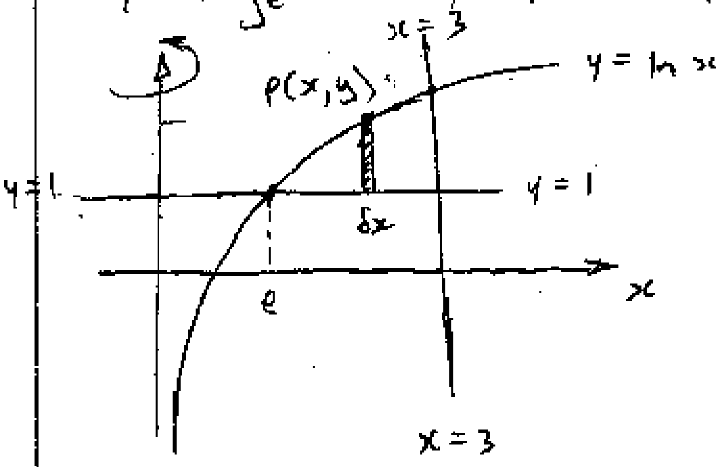
$= 2\sqrt{3}\int_0^2 (4 - y^2) . dy$

$= 2\sqrt{3}\left[4y - \frac{y^3}{3}\right]_0^2$

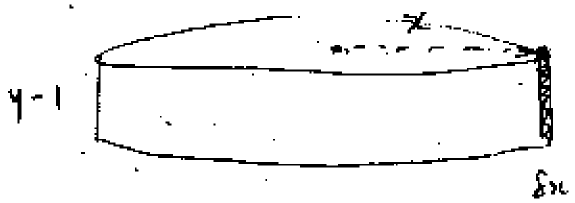
$= 2\sqrt{3}\left[8 - \frac{8}{3}\right] = \frac{32\sqrt{3}}{3}$ cubic units

Question 4 (Continued)

(6) (c) $1 = \log_e x \quad y = 1, \quad x = 3$



Typical Shell



$\delta V = 2\pi x (y-1) . \delta x$

$= 2\pi x (\ln x - 1) . \delta x$

$V = 2\pi \int_e^3 (x \ln x - x) . dx$ L.I.A.T.E.

Integ. By Parts

$u = \ln x \quad \frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x} \quad v = \frac{1}{2}x^2$

$\frac{V}{2\pi} = \int_e^3 x \ln x . dx - \left[\frac{x^2}{2}\right]_e^3$

$= \left[\frac{1}{2}x^2 \ln x\right]_e^3 - \int_e^3 \frac{1}{2}x^2 . \frac{1}{x} . dx - \left[\frac{x^2}{2}\right]_e^3$

$= \frac{9}{2}\ln 3 - \frac{1}{2}e^2 - \left[\frac{x^2}{4}\right]_e^3 - \left[\frac{x^2}{2}\right]_e^3$

$= \frac{9}{2}\ln 3 - \frac{1}{2}e^2 - \left[\frac{3x^2}{4}\right]_e^3$

$= \frac{9}{2}\ln 3 - \frac{1}{2}e^2 - \frac{27}{4} + \frac{3e^2}{4}$

$V = \pi\left[9\ln 3 - e^2 - \frac{27}{2} + \frac{3e^2}{2}\right]$

$= \pi\left[9\ln 3 - \frac{27}{2} + \frac{e^2}{2}\right]$ units^3.

Question 5 (15 marks)

(4) (a) let $z = r \operatorname{cis} \theta$

where $z^4 = -16$

$r^4 \operatorname{cis} 4\theta = 16 \operatorname{cis} \pi$

$\therefore r = 2$ and $4\theta = \pi + 2n\pi$, $n \in J$

$\theta = \dfrac{\pi + 2n\pi}{4}$

$z_1 = 2 \operatorname{cis} \frac{\pi}{4} = 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right) = \sqrt{2} + \sqrt{2}\, i$

$z_2 = 2 \operatorname{cis} \frac{3\pi}{4} = \qquad = -\sqrt{2} + \sqrt{2}\, i$

$z_3 = 2 \operatorname{cis} \frac{5\pi}{4} = \qquad = -\sqrt{2} - \sqrt{2}\, i$

$z_4 = 2 \operatorname{cis} \frac{7\pi}{4} = \qquad = \sqrt{2} - \sqrt{2}\, i$

(b) $f(x) = \dfrac{\ln x}{x}$, for $x > 0$

(1) (i) let $f(x) = 0$: $\dfrac{\ln x}{x} = 0$

$x = 1$

(2) (ii) $f'(x) = \dfrac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2}$

$= \dfrac{1 - \ln x}{x^2}$

x	2	e	3
$f'(x)$	+	0	−

put $f'(x) = 0$: $\dfrac{1 - \ln x}{x^2} = 0$

$\ln x = 1$

$x = e$

and $f(e) = \frac{1}{e}$

Max T.P. at $\left(e, \frac{1}{e}\right)$

Question 5 (Continued)

(b) (iii) $f'(x) = \dfrac{1 - \ln x}{x^2}$

(2)

$$f''(x) = \frac{x^2 \cdot -\frac{1}{x} - (1 - \ln x) \cdot 2x}{x^4}$$

$$= \frac{-x - 2x + 2x \ln x}{x^4}$$

$$= \frac{2\ln x - 3}{x^3}$$

when $f''(x) = 0$: $2\ln x - 3 = 0$

$\ln x = \frac{3}{2}$

$x = e^{\frac{3}{2}}$

when $x = e^{\frac{3}{2}}$: $y = \dfrac{\frac{3}{2}}{e^{\frac{3}{2}}}$

$= \dfrac{3}{2e^{\frac{3}{2}}}$

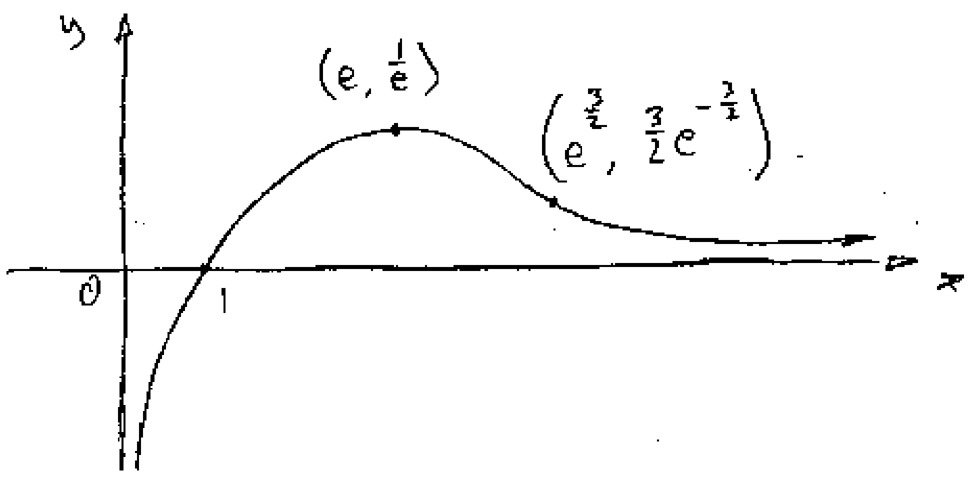
<u>Concavity Test</u>

x	e	$e^{\frac{3}{2}}$	e^2
$f''(x)$	$-$	0	$+$

$\therefore$ I.P. at $\left(e^{\frac{3}{2}}, \dfrac{3}{2e^{\frac{3}{2}}}\right)$

(iv) as $x \to \infty$ $\dfrac{\ln x}{x} \to 0$ (Since x dominates $\ln x$)

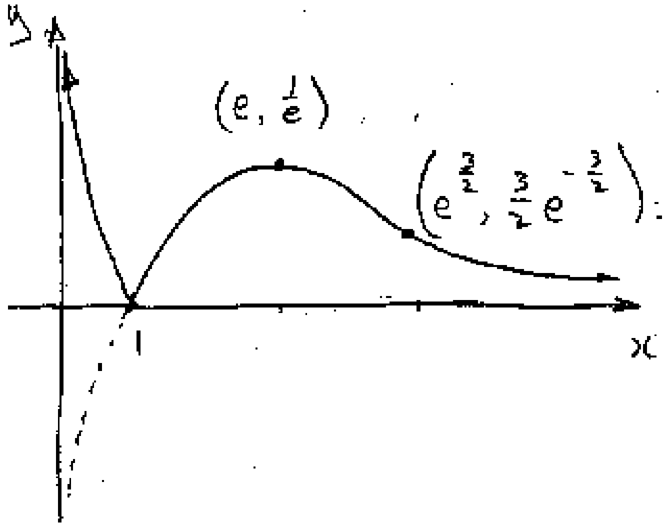
(2)



Question 5 (Continued)

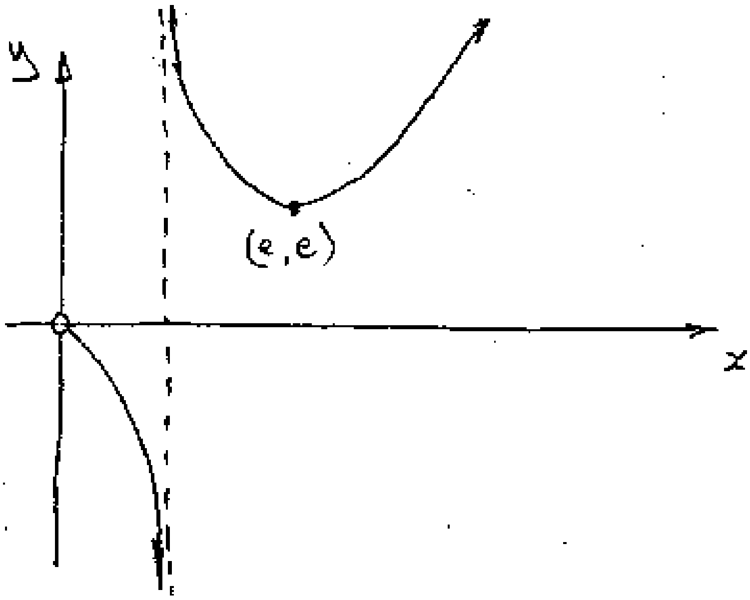
(c) (i) $y = |f(x)|$

(2)



(ii) $y = \dfrac{1}{f(x)}$

(2)



Question 6 (15 marks)

(a) $Q(x) = ax^4 + bx^3 + cx^2 + dx + e$

(2) (i) given α is an integer : $Q(\alpha) = 0$

$Q(\alpha) = a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e$

letting $Q(\alpha) = 0$: $a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha = -e$

$\alpha(a\alpha^3 + b\alpha^2 + c\alpha + d) = -e$

Since a, b, c, d are integers then $k\alpha = -e$ where k is also an integer

hence α is a factor of e.

(2) (ii) $P(x) = 4x^4 - x^3 + 3x^2 + 2x - 3$

The only possible integer roots are ± 1 and ± 3

$P(1) = 4 - 1 + 3 + 2 - 3 \neq 0$

$P(-1) = 4 + 1 + 3 - 2 - 3 \neq 0$

$P(3) = 324 - 27 + 27 + 6 - 3 \neq 0$

$P(-3) = 324 + 27 + 27 - 6 - 3 \neq 0$

$\therefore$ $P(x) = 0$ does not have an integer root.

(b) (2) (i) $P(H) = \frac{1}{3}$, $P(M) = \frac{2}{3}$

Five Torpedoes fired

$P(3\text{ Hits}) = {}^5C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^2$

$= \frac{40}{243}$

(2) (ii) $P\left(\begin{matrix}\text{at least} \\ 1 \text{ hit}\end{matrix}\right) = 1 - P\left(\begin{matrix}\text{no} \\ \text{hits}\end{matrix}\right)$

$= 1 - \left(\frac{2}{3}\right)^n$

$\therefore 1 - \left(\frac{2}{3}\right)^n > 0.9$

$\left(\frac{2}{3}\right)^n < 0.1$

$\left(\frac{2}{3}\right)^5 = 0.132$

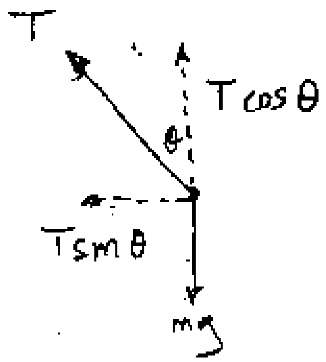
$\left(\frac{2}{3}\right)^6 = 0.088$

Requires 6 torpedoes

Question 6 (Continued)

(c) (4)

Forces on P



Vertically (zero net force)

$T\cos\theta = mg$ — ①

Radially

$T\sin\theta = \frac{mv^2}{r}$ — ②

② ÷ ① : $\tan\theta = \dfrac{\frac{mv^2}{r}}{mg}$

$= \dfrac{v^2}{gr}$

$v^2 = gr\tan\theta$

$v = \sqrt{gr\tan\theta}$ (as required)

(d) (3)

$P(x) = (x-3)(x-4) + R(x)$

Degree of $R(x) < 2$

let $R(x) = ax + b$

$P(3) = 5$: $5 = 3a + b$ — ①

$P(4) = 9$: $9 = 4a + b$ — ②

② − ① : $4 = a$

sub ② $9 = 16 + b$

$b = -7$

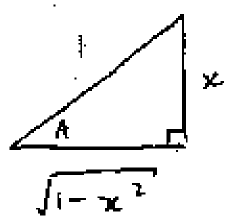
$\therefore R(x) = 4x - 7$

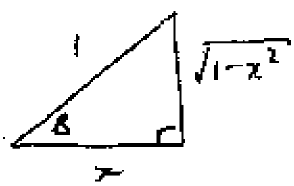
Question 7 (15 marks)

(a) $\sin^{-1} x$, $\cos^{-1} x$ and $\sin^{-1}(1-x)$ are acute

(3) let $A = \sin^{-1} x$ let $B = \cos^{-1} x$

$\sin A = x$ $\cos B = x$





$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$= x^2 - \sqrt{1-x^2}\,.\,\sqrt{1-x^2}$$
$$= x^2 - (1-x^2)$$
$$= 2x^2 - 1 \quad \left(\text{as required}\right)$$

(3) Solve $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(1-x)$

$$\sin(A-B) = \sin\left(\sin^{-1}(1-x)\right)$$
$$2x^2 - 1 = 1 - x$$
$$2x^2 + x - 2 = 0$$
$$x = \frac{-1 \pm \sqrt{1+4(2)(2)}}{4}$$
$$= \frac{-1 \pm \sqrt{17}}{4}$$

Since $x = \sin A$
then $-1 \le x \le 1$ $\Bigg]$ $x = \dfrac{-1+\sqrt{17}}{4}$

(≈ 0.78)

Question 7 (continued)

(b) (5)

$$3\tan^2 x = 2\sin x$$

$$3\frac{\sin^2 x}{\cos^2 x} = 2\sin x$$

$$3\sin^2 x = 2\sin x\,(1-\sin^2 x)$$

$$3\sin^2 x = 2\sin x - 2\sin^3 x$$

$$2\sin^3 x + 3\sin^2 x - 2\sin x = 0$$

$$\sin x\,(2\sin^2 x + 3\sin x - 2) = 0$$

$2\sin x \times -1$, $\sin x \times +2$

$$\sin x\,(2\sin x - 1)(\sin x + 2) = 0$$

either $\sin x = 0$ or $\sin x = \frac{1}{2}$ $\quad$ $\sin x \neq -2$.

$x = n\pi$ $\quad$ $x = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$

$$x = n\pi + (-1)^n \cdot \frac{\pi}{6}$$

(c) (2)

(i) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \,.\, dx = 0$ $\quad$ True

x^3 – is an odd function
$\cos x$ – is an even function
] odd × even = odd

(2)

(ii) $\displaystyle\int_{-1}^{1} e^{-x^2} \cdot \cos^{-1} x \, dx = 0$ $\quad$ False.

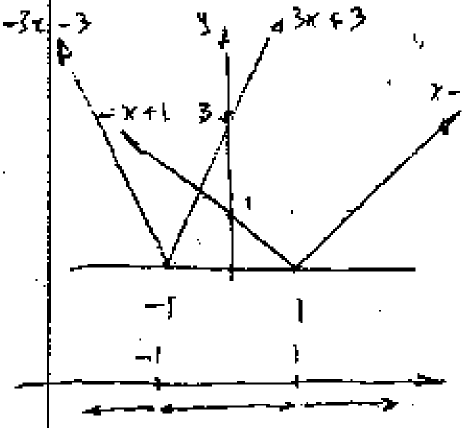
$\cos^{-1} x > 0$ for $-1 < x < 1$
$e^{-x^2} > 0$ for all x

$\therefore$ product can never equal zero

Question 8 (15 marks)

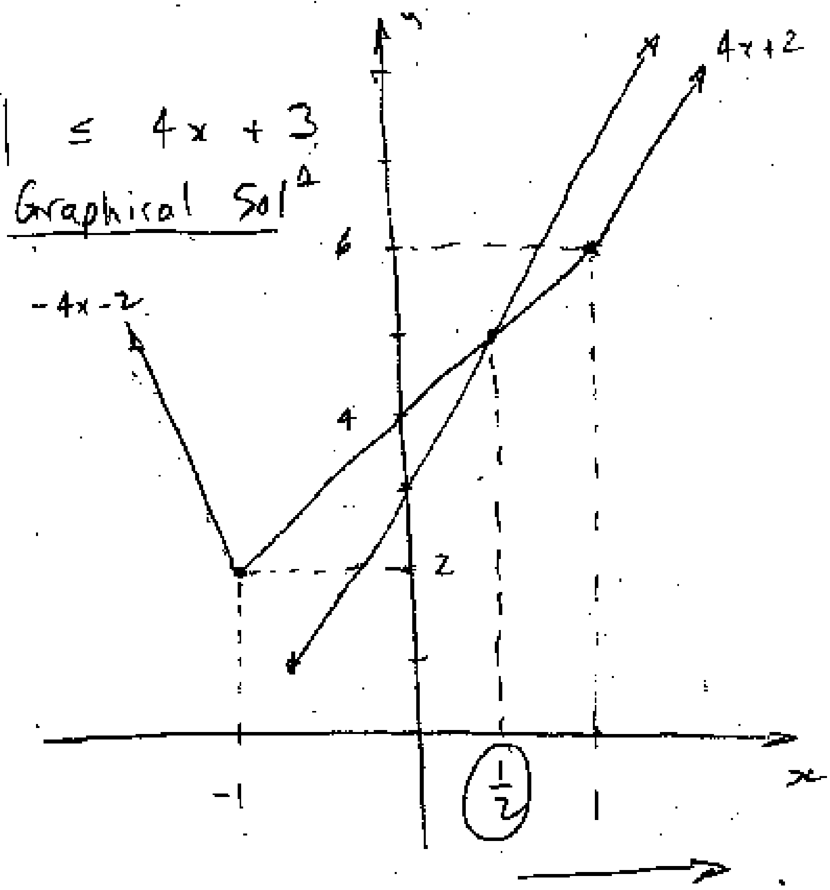
(b) $|3x+3| + |x-1| \le 4x+3$

(5)



$-4x-2 \quad 2x+4 \quad 4x+2$

Graphical Solⁿ



Graphical Solⁿ

$x \ge \frac{1}{2}$

Algebraic Solⁿ

for $x \le -1$	for $-1 \le x \le 1$	for $x \ge 1$
$-4x-2 \le 4x+3$	$2x+4 \le 4x+3$	$4x+2 \le 4x+3$
$-8x \le 5$	$1 \le 2x$	$2 \le 3$
$x \ge -\frac{5}{8}$	$x \ge \frac{1}{2}$	True for all $x \ge 1$
No Solⁿ	$\therefore \frac{1}{2} \le x \le 1$	

Final Solⁿ : $x \ge \frac{1}{2}$

(c)

(1) (i)

(3) (ii)

$t = 0$
$V = u$
$x = 0$

(i) Forces on P

$$m\ddot{x} = -mg - mkV$$

$$\ddot{x} = -g - kV$$

(ii)

$$\ddot{x} = -(g+kV)$$

$$\frac{dV}{dt} = -(g+kV)$$

$$\frac{dt}{dV} = \frac{-1}{g+kV}$$

Question 8 (continued)

(b) $t = \int \frac{-1}{g+kV} \cdot dV$

$= -\frac{1}{k}\ln(g+kV) + c$

when	$t=0$, $V=u$

$\therefore \quad c = \frac{1}{k}\ln(g+ku)$

$t = \frac{1}{k}\ln\left(\frac{g+ku}{g+kV}\right)$

max height	when $V=0$

$T = \frac{1}{k}\ln\left(\frac{g+ku}{g}\right)$ (as required)

(iii) Using $\ddot{x} = -(g+kV)$

(4)

$V.\frac{dV}{dx} = -(g+kV)$

$\frac{dV}{dx} = -\left(\frac{g+kV}{V}\right)$

$\frac{dx}{dV} = -\frac{V}{g+kV}$

$= -\frac{1}{k}\left(\frac{kV+g-g}{g+kV}\right)$

$= -\frac{1}{k}\left(1 - \frac{g}{g+kV}\right)$

$x = -\frac{1}{k}\int\left(1 - \frac{g}{g+kV}\right) \cdot dV$

$= -\frac{1}{k}\left[V - \frac{g}{k}\ln(g+kV)\right] + C$

when	$x=0$, $V=u$

$C = \frac{1}{k}\left[u - \frac{g}{k}\ln(g+ku)\right]$

Question 8 (Continued)

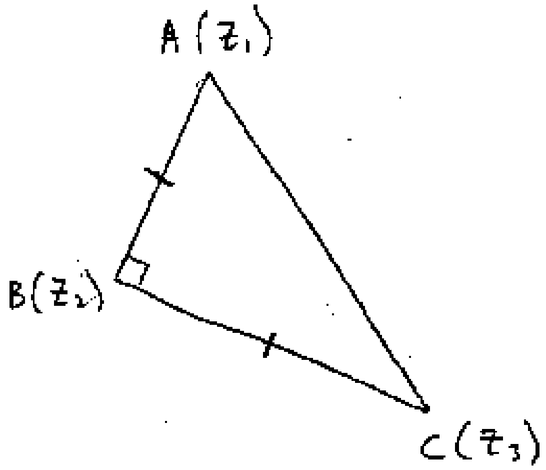
(b) $x = \frac{1}{k}\left[(u-v) - \frac{g}{k}\ln\left(\frac{g+ku}{g+kv}\right)\right]$

Max height when $v = 0$

$$x = \frac{1}{k}\left[u - \frac{g}{k}\ln\left(\frac{g+ku}{g}\right)\right]$$

$$= \frac{1}{k}\left[u - gT\right] \quad \text{(as required)}$$

(c)
(2)



Note $i(z_3 - z_2)$ rotates the vector by 90° (anti-clockwise).

ABC is a right angled triangle with AB = BC

$|z_3 - z_2| = BC$ and $|z_1 - z_2| = AB$.