ST IGNATIUS COLLEGE RIVERVIEW

ASSESSMENT TASK NUMBER 4

TRIAL HSC EXAMINATION

YEAR 12

2006

EXTENSION 2

Time allowed: 3 hours (+ 5 minutes reading time)

Instructions to Candidates

- Attempt **all** questions
- Show all necessary working.
- Marks may be deducted for missing or poorly arranged work.
- Board approved calculators and templates may be used.
- Each question must be returned in a ***separate*** writing booklet marked Q1, Q2 etc
- **Each booklet must have your name**.

Question 1 [15 marks] Start a new answer booklet.

(a) Find $\int \frac{1}{\sqrt{2x-x^2}}\,dx$. [2]

(b) (i) Resolve $\frac{1}{(x-2)(x+1)}$ into partial fractions. [2]

(ii) Hence, find $\int \frac{1}{(x-2)(x+1)}\,dx$. [2]

(c) Evaluate $\int_0^m x\sqrt{m-x}\,\,dx\,;\ x \ge 0,\ m \ge x$. [3]

(d) Evaluate $\int_0^{\frac{\pi}{3}} \frac{\tan x}{1+\cos x}\,dx$, using the substitution $t = \tan\frac{x}{2}$. [3]

(e) Evaluate $\int_0^1 \sin^{-1} y\ dy$. [3]

Question 2 [15 marks] Start a new answer booklet.

(a) Consider the complex number $z = 1 + \sqrt{3}i$.

(i) Find $|z|$ [1]

(ii) Find $\arg\left(\bar{z}\right)$ [1]

(iii) Express $\frac{1}{z}$ in the form $a + bi$, where a and b are real. [1]

(b) (i) Find the three cube roots of $-i$ in modulus argument form. [3]

(ii) Represent these three cube roots of $-i$ on an Argand diagram. [1]

(c) Consider the locus in the complex plane defined by $\mathrm{Re}\left(z^2\right) = 4$ where $z = x + iy$.

(i) Find the Cartesian equation of this locus. [1]

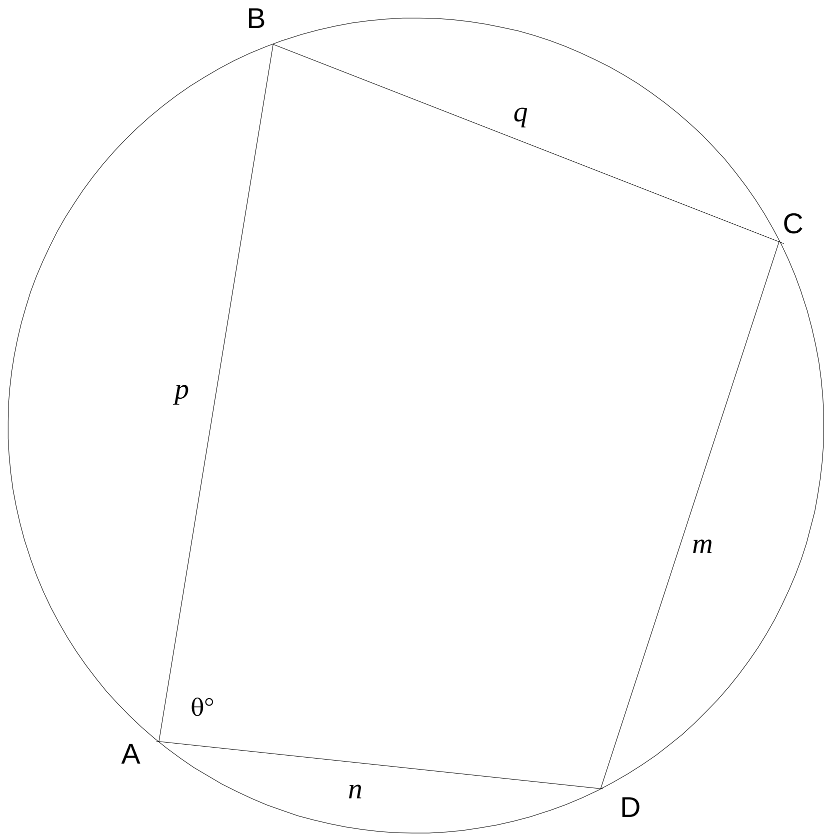
(ii) Describe this locus. [1]

(d) If $\arg\left(z - 3 - 2i\right) = \frac{3\pi}{4}$ where $z = x + iy$, sketch the locus of z on an Argand diagram. [2]

Question 2 is continued on the next page

Question 2 CONTINUED

(e) ABCD is a cyclic quadrilateral with sides p, q, m, n as shown in the diagram below.



If the angle DAB is θ degrees:

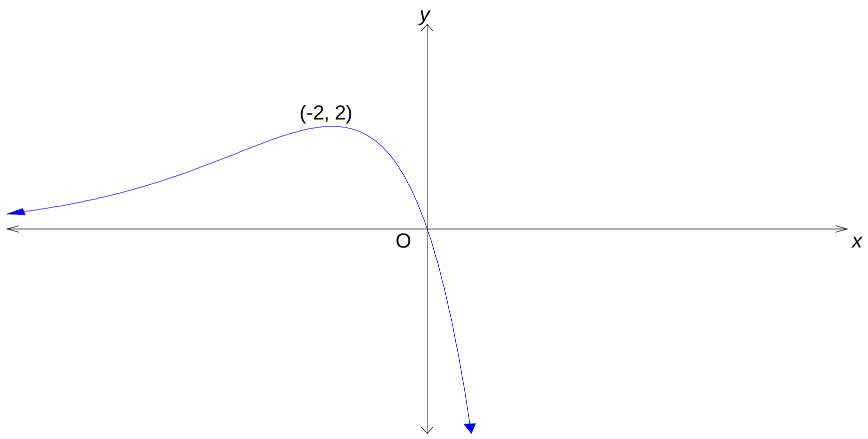
(i) Write down two expressions for the length of diagonal BD. [2]

(ii) Show that [2]

$$\cos\theta = \frac{p^2 - q^2 + n^2 - m^2}{2(pn + qm)}$$

Question 3 [15 marks] Start a new answer booklet.

(a) The curve shown in the diagram is the equation $y=f(x)$. There is a maximum turning point at (-2,2) and the curve crosses the x axis at (0,0).



Sketch the following curves on separate diagrams, showing all of the essential features.

(i) $y=f(x)-2$ [1]

(ii) $y=f(x-2)$ [1]

(iii) $y=f(2x)$ [1]

(iv) $y=f(-x)$ [1]

(v) $y=|f(x)|$ [1]

(vi) $y=\dfrac{1}{f(x)}$ [1]

(b) Show that the curves $x^2-y^2=c^2$ and $xy=c^2$, where 'c' is a constant, intersect at right angles. [4]

(c) (i) Factorise the expression $(1+x+x^2+x^3)$ over

(α) the real field [1]

(β) the complex field [1]

(ii) Hence, show that the equation $\dfrac{x^4}{4}+\dfrac{x^3}{3}+\dfrac{x^2}{2}+x+c=0$, where '$c$' is a constant, has no real roots if $c>\dfrac{7}{12}$ [3]

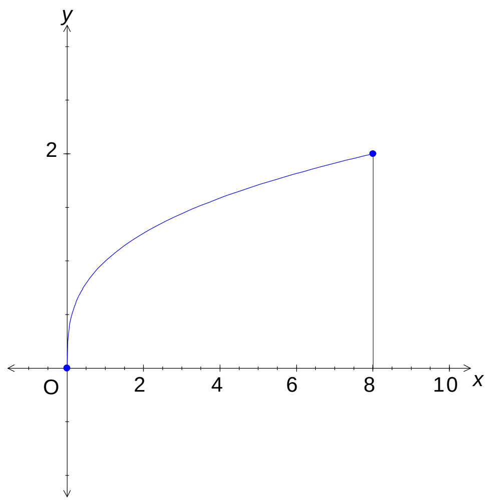
Question 4 [15 marks] Start a new answer booklet.

(a) (i) If $I_m = \int_0^1 x^m e^x \, dx$, where m is a positive integer, show that [3]

$$I_{m+1} = e - (m+1) I_m$$

(ii) Hence, evaluate: $\int_0^1 y^3 e^y \, dy$. [2]

(b) The sketch below shows the region enclosed by the curve $y = x^{\frac{1}{3}}$, the x axis and the ordinate $x = 8$. [4]



Find the volume generated when this region is rotated about the line $x = 8$, using the method of cylindrical shells.

(c) (i) Sketch on the number plane, the circle $(x-2)^2 + y^2 = 9$, stating the co-ordinates of all points of intersection with the axes. [2]

(ii) On this sketch, shade the region bounded by the circle and the y axis where $x \geq 0$. [1]

(iii) The shaded region forms the base of a solid with every cross-section perpendicular to the x-axis forming a square, one side of which lies in the base. Find the volume of the solid. [3]

Question 5 [15 marks] Start a new answer booklet.

(a) Solve the inequality: [3]

$$\frac{x+3}{x-2} > \frac{x+1}{x-3}$$

(b) Divide the polynomial $\left(x^3+5i\,x^2-7i\,x-3\right)$ by $\left(x-2i\right)$ using long division. [3]

(c) Show that $\left(2+\sqrt{3}i\right)$ is a zero of the polynomial $Q\left(x\right)=x^3-x^2-5x+21$. Hence, reduce $Q\left(x\right)$ to factors over the complex field. [3]

(d) The polynomial $P\left(x\right)=x^4+x^2+6x+4$ has a rational zero of multiplicity 2. Find this zero and show that there are no other real zeros. [3]

(e) If $\alpha,\ \beta,\gamma$ are the roots of the equation $x^3+lx+m=0$, where $m\neq 0$, obtain as functions of *l* and *m*, in their simplest form, the coefficients of the cubic equation whose roots are $\frac{1}{\alpha^2},\frac{1}{\beta^2},\frac{1}{\gamma^2}$. [3]

Question 6 [15 marks] Start a new answer booklet.

(a) An engine of mass *mkg* is travelling with a velocity of v metres / second around a curved banked track of radius R metres, inclined at α degrees to the horizontal plane.

(i) If there is no lateral thrust between the wheels and the rails and N is the normal reaction, prove that: [3]

$\tan\alpha = \frac{v^2}{Rg}$, where g is the acceleration due to gravity.

(ii) If the vintage steam engine "3801" negotiates the track at a speed of 72 kilometres per hour, without exerting lateral thrust on the rails, and the track has a radius of 200 metres and width 1.5 metres, calculate the distance to the nearest centimetre that the outer rail is raised above the inner rail. Take $g = 10$ metres per second squared [2]

(b) (i) A particle of mass m falls from rest, from a point O, in medium whose resistance is mkv, where k is a positive constant and v is the velocity after time t. Prove that the speed at time t is given by [3]

$\frac{g}{k}\left(1-e^{-kt}\right)$

(ii) An identical particle is projected upwards from O with initial velocity u in the same medium. If this particle is released simultaneously with the first, prove that the speed of the first particle when the second is momentarily at rest is given by: $\frac{wu}{w+u}$, where w is the terminal velocity of the first particle. [7]

Question 7 [15 marks] Start a new answer booklet.

(a) In the expansion of $(a+b)^{2m+1}$

(i) Find T_{m+1} and T_{m+2} (Note: T means term) [2]

(ii) Hence prove, that if $a+b=1$, then [2]

$$T_{m+1}+T_{m+2}=\binom{2m+1}{m}a^n\, b^m$$

(b) (i) P $(a\sec\alpha,\, b\tan\alpha)$ and Q $(a\sec\beta,\, b\tan\beta)$ are two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $x>0$.

(γ) Show that the gradient of PQ is : $\dfrac{b\cos\left(\frac{\alpha-\beta}{2}\right)}{a\sin\left(\frac{\alpha+\beta}{2}\right)}$ [3]

(θ) Show that he equation of PQ is : [3]

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right)-\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha+\beta}{2}\right)$$

Note: The following results may be used for parts (γ) and (θ) above :

$$\sin(A-B)=2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A-\cos B=-2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)$$

(ii) If the chord PQ subtends a right angle at the vertex A (a,0) show that $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=1-e^2$. [5]

Question 8 [15 marks] Start a new answer booklet.

(a) By using the substitution $y = \frac{\pi}{2} - x$, show that [7]

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot x} dx$$

Hence find the value of I .

(b) Let $z = x + iy$ be a complex number (x and y are real) satisfying

$$z\bar{z} + (1-2i)z + (1+2i)\bar{z} \leq 3$$

(i) Sketch the locus of z on an Argand diagram. [4]

(ii) Find the maximum and minimum values of $(x+y)$. [4]

End of the Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ \ n \neq -1;\ \ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x > 0$

B.L.C.

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Solutions

These solutions are not necessarily the best.

There may be suitable alternatives.

Please treat these solution(s) as an aid to understanding the Question(s) and as an extra resource for HSC preparation.

— Trial Higher School Certificate —

Question 1

(a) $I = \int \frac{1}{\sqrt{2x-x^2}}\,dx$

Now $2x - x^2 = -(x^2-2x)$

$= 1-(x^2-2x+1)$

$= 1-(x-1)^2$

$\therefore I = \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx$

$I = \sin^{-1}(x-1) + c$

(b) (i) Let $\frac{1}{(x-2)(x+1)} \equiv \frac{A}{x-2} + \frac{B}{x+1}$

$\equiv \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$

$\frac{1}{(x-2)(x+1)} \equiv \frac{(A+B)x+(A-2B)}{(x-2)(x+1)}$

Equate co-efficients

$A+B = 0$ --- ①

$A - 2B = 1$ --- ②

Solving: $B + 2B = -1$

$3B = -1$

$B = -\frac{1}{3}$

$\therefore A = \frac{1}{3}$

$\therefore \frac{1}{(x-2)(x+1)} = \frac{1}{3(x-2)} - \frac{1}{3(x+1)}$

(ii) $I = \int \frac{1}{(x-2)(x+1)}\,dx = \int\left[\frac{1}{3(x-2)} - \frac{1}{3(x+1)}\right]dx$

$= \frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + c$

$= \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + c$

Question 1 (continued)

(c)

$$I = \int_0^m x\sqrt{m-x}\,dx$$

Let $u = m - x$

$\therefore \frac{du}{dx} = -1$

$dx = -du$

when $\underline{x = m}$, $u = 0$

$\underline{x = 0}$, $u = m$

Now $x = m - u$

$$\therefore I = \int_m^0 (m-u)\sqrt{u}\,.-du$$

$$I = \int_0^m (mu^{\frac{1}{2}} - u^{\frac{3}{2}})\,du$$

Note: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$$I = \left[\frac{mu^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^m$$

$$I = 2\left[\frac{m}{3}u^{\frac{3}{2}} - \frac{1}{5}u^{\frac{5}{2}}\right]_0^m$$

$$I = 2\left[\left(\frac{m}{3}m^{\frac{3}{2}} - \frac{1}{5}m^{\frac{5}{2}}\right) - 0\right]$$

$$I = 2m^{\frac{5}{2}}\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{4}{15}m^{\frac{5}{2}}$$

$$I = \frac{4}{15}\sqrt{m^5}$$

Question 1 (continued)

(d) $I = \int_0^{\frac{\pi}{3}} \frac{\tan x}{1+\cos x}\,dx$

Let $t = \tan\frac{x}{2}$

$\therefore \quad \cos x = \frac{1-t^2}{1+t^2}$

$\tan x = \frac{2t}{1-t^2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$2dt = \left(1+\tan^2\frac{x}{2}\right)dx$

$2dt = (1+t^2)\,dx$

$dx = \frac{2\,dt}{1+t^2}$

when $x = \frac{\pi}{3}$; $t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$x = 0$; $t = \tan 0 = 0$

$$I = \int_0^{\frac{1}{\sqrt{3}}} \frac{\frac{2t}{1-t^2}}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2\,dt}{1+t^2}$$

$$I = \int_0^{\frac{1}{\sqrt{3}}} \frac{2t}{1-t^2}\,dt$$

$$I = \left[-\ln(1-t^2)\right]_0^{\frac{1}{\sqrt{3}}}$$

$I = -\ln\left(1-\frac{1}{3}\right) - \ln 1$

$I = -\ln\left(\frac{2}{3}\right) = \ln\frac{3}{2}$

Note $0.405465108\ldots$

$\doteqdot 0.405$ c.3.d.p.

Question 1 (continued)

(e) $$I = \int_0^1 \sin^{-1}(y)\,dy.$$

$$I = \int_0^1 1 \times \sin^{-1}(y)\,dy$$

Let $u = \sin^{-1} y \Rightarrow u' = \frac{1}{\sqrt{1-y^2}}$

$v' = 1 \quad \Rightarrow \quad v = y$

$$\therefore I = \left[y \sin^{-1} y\right]_0^1 - \int_0^1 y \times \frac{1}{\sqrt{1-y^2}}\,dy$$

$$I = \left[1 \times \sin^{-1}(1) - 0 \times \sin^{-1}(0)\right] - \left[-\sqrt{1-y^2}\right]_0^1$$

#

$$I = \frac{\pi}{2} + \left[0 - \sqrt{1}\right]$$

$$I = \frac{\pi}{2} - 1$$

$$I = \frac{\pi - 2}{2}$$

$\int \frac{y}{\sqrt{1-y^2}}\,dy = -\sqrt{1-y^2} + c$ by substitution using $u = 1 - y^2$.

Note: $0.570796326\ldots\ldots$

$\doteqdot 0.57$ c. 2. d. p.

Question 2

(a) $z = 1+\sqrt{3}i$

(i) $|z| = \sqrt{1^2+(\sqrt{3})^2}$

$= \sqrt{1+3}$

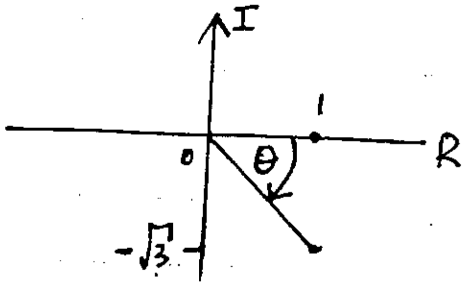
$= \sqrt{4} = 2.$

(ii) $\bar{z} = 1-\sqrt{3}i$

$\arg(\bar{z}) = \alpha$

$= -\tan^{-1}\left|-\frac{\sqrt{3}}{1}\right|$

$= -\frac{\pi}{3}$



(iii) $\frac{1}{z} = \frac{1}{1+\sqrt{3}i} = \frac{1}{1+\sqrt{3}i} \times \frac{1-\sqrt{3}i}{1-\sqrt{3}i}$

$= \frac{1-\sqrt{3}i}{1+3}$

$= \frac{1}{4} - \frac{\sqrt{3}}{4}i$

(b)(i) Solve $z^3 = -i$

$z^3 = 0-i$

$|a| = |0-i| = 1$

$\arg(a) = \arg(0-i) = -\frac{\pi}{2}$

Considering $z^n = a$

$$z = r^{\frac{1}{n}}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$$

$z_1 = 1\left(\cos\frac{-\frac{\pi}{2}}{3} + i\sin\frac{-\frac{\pi}{2}}{3}\right)$

$z_1 = \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}$

$z_1 = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

$z_2 = 1\left(\cos\frac{-\frac{\pi}{2}+2\pi}{3} + i\sin\frac{-\frac{\pi}{2}+2\pi}{3}\right)$

$z_2 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$z_2 = 0 + i$

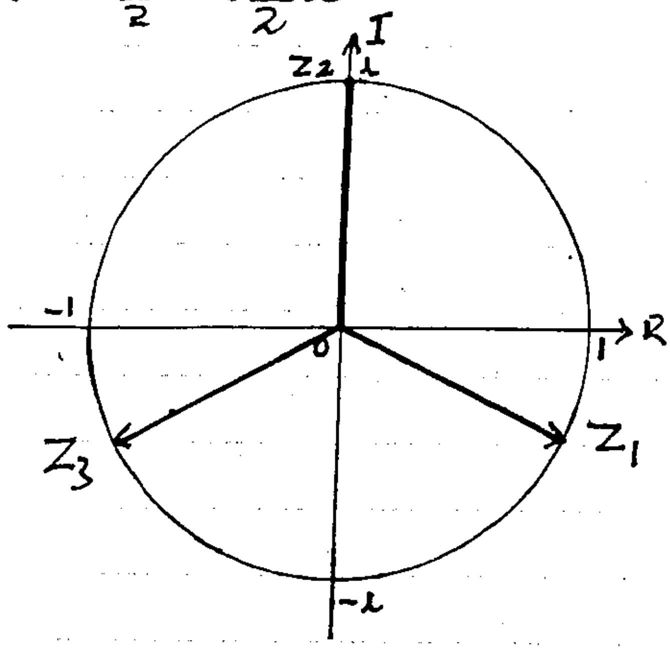
Question 2 (b) (continued)

$$Z_3 = 1\left(\cos\frac{-\frac{\pi}{2}+4\pi}{3} + i\sin\frac{-\frac{\pi}{2}+4\pi}{3}\right)$$

$$Z_3 = \cos-\frac{5\pi}{6} + i\sin-\frac{5\pi}{6}$$

$$Z_3 = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

(ii)



(c) $Re(z^2)=4$

(i) $z = x+iy$

$z^2 = (x+iy)^2$

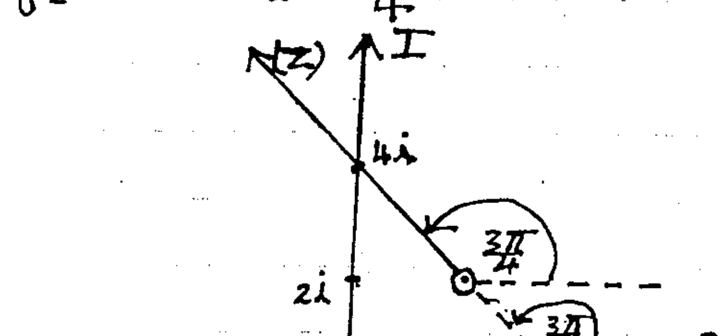
$= x^2+2xyi+i^2y^2$

$= x^2-y^2+2xyi$

$\therefore x^2-y^2=4$

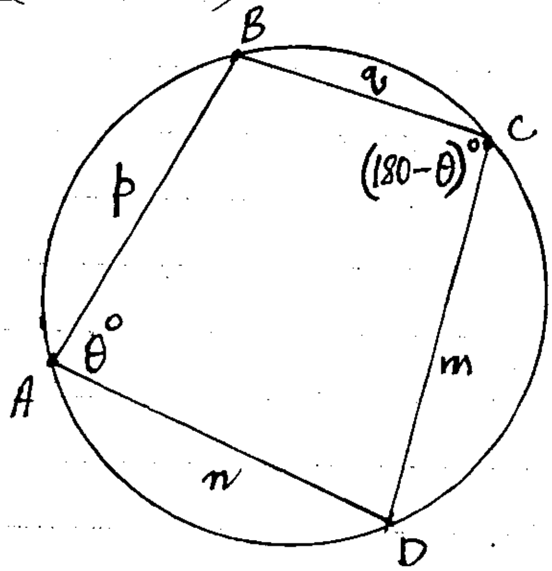
(ii) This is a rectangular Hyperbola

(d) $\arg[z-(3+2i)] = \frac{3\pi}{4}$



Question 2 (continued)

(e)



Note it is shown in the diagram That opposite angles of a cyclic quadrilateral are supplementary.

(i) In Δ DAB

$$BD^2 = n^2 + p^2 - 2np\cos\theta \quad --- \; (1)$$

In Δ BCD

$$BD^2 = q^2 + m^2 - 2qm\cos(180-\theta)^\circ \; ---- \; (2)$$

(ii) Equating (1) and (2)

$$n^2 + p^2 - 2np\cos\theta = q^2 + m^2 + 2qm\cos\theta$$

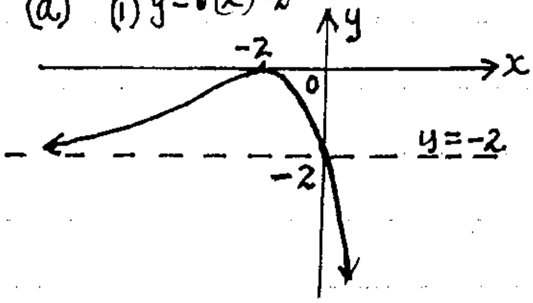
$$p^2 - q^2 + n^2 - m^2 = 2np\cos\theta + 2mq\cos\theta$$

$$2(np + mq)\cos\theta = p^2 - q^2 + n^2 - m^2$$

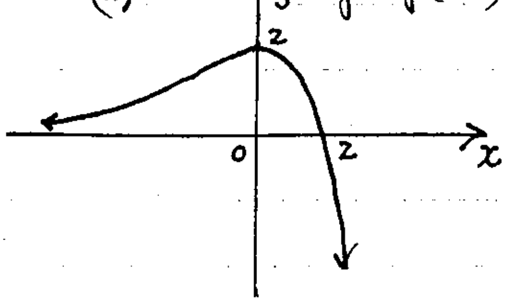
$$\cos\theta = \frac{p^2 - q^2 + n^2 - m^2}{2(np + mq)}$$

Question 3

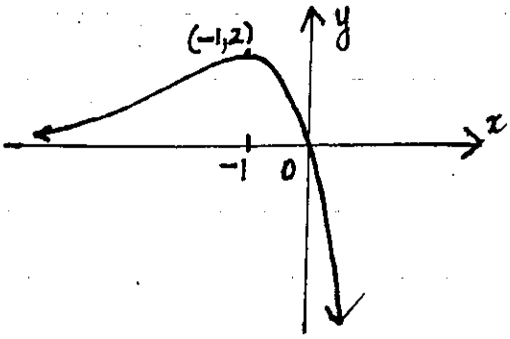
(a) (i) $y = f(x) - 2$



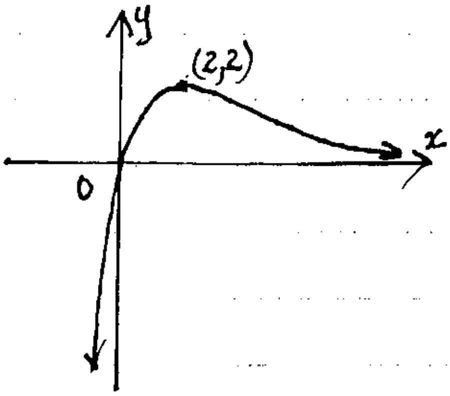
(ii) $y = f(x-2)$



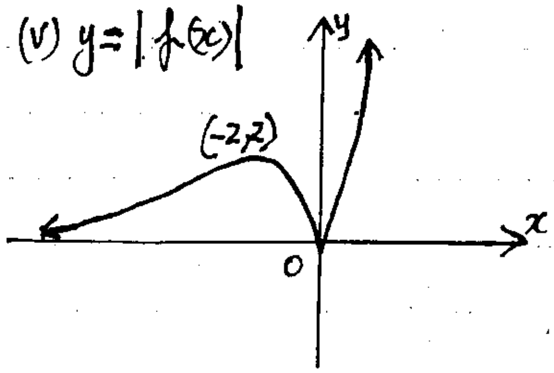
(iii) $y = f(2x)$



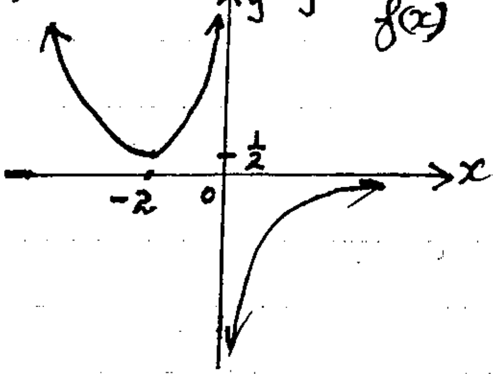
(iv) $y = f(-x)$



(v) $y = |f(x)|$



(vi) $y = \dfrac{1}{f(x)}$



Question 3 (continued)

(b) $x^2 - y^2 = c^2$ ---①

$xy = c^2$ ---②

Let (x_1, y_1) be the point of intersection of two two curves.

For ① $2x - 2y\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{x}{y}$$

Gradient of tangent at (x_1, y_1):

$$m_1 = \frac{x_1}{y_1}$$

For ② $1 \times y + \dfrac{dy}{dx} \times x = 0$

$$\frac{dy}{dx} = -\frac{y}{x}$$

Gradient of tangent at (x_1, y_1):

$$m_2 = -\frac{y_1}{x_1}$$

Now $m_1 m_2 = \dfrac{x_1}{y_1} \times -\dfrac{y_1}{x_1} = -1$

This is the condition for perpendicular lines

Therefore the two curves intersect at right angles.

Question 3 (continued)

(c)(i) $1+x+x^2+x^3$

(α) $1+x+x^2+x^3$

$= (1+x)+x^2(1+x)$

$= (1+x)(1+x^2)$

(β) $(x+1)(x^2-i^2)$

$= (x+1)(x+i)(x-i)$

(ii) Let $y = \frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x+c$

$\frac{dy}{dx} = x^3+x^2+x+1$

$= x^2(x+1)+1(x+1)$

$= (x+1)(x^2+1)$

Now for a stationary point $\frac{dy}{dx}=0$

$\therefore x=-1$

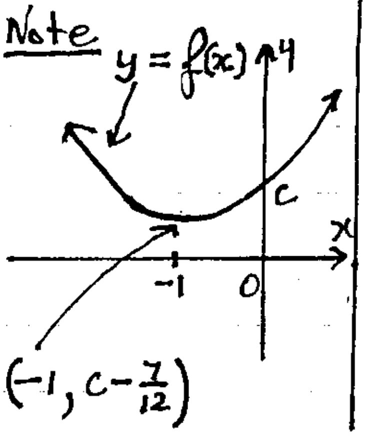
$\frac{d^2y}{dx^2} = 3x^2+2x+1$

at $x=-1$, $\frac{d^2y}{dx^2} = 3-2+1 = 2>0 \therefore \cup$

Noting That the "arrows" of The quartic "go up"!

$y_{min} = \frac{(-1)^4}{4}+\frac{(-1)^3}{3}+\frac{(-1)^2}{2}+(-1)+c$

Note



$= \frac{1}{4}-\frac{1}{3}+\frac{1}{2}-1+c$

$= \frac{3-4+6-12}{12}+c$

$= -\frac{7}{12}+c$

Now no real roots if

$-\frac{7}{12}+c>0$

$\therefore c>\frac{7}{12}$

Question 4

(a) (i) $I_m = \int_0^1 x^m e^x dx$

Let $v' = x^m \Rightarrow v = \frac{x^{m+1}}{m+1}$

Let $u = e^x \Rightarrow u' = e^x$

$I_m = \left[e^x \frac{x^{m+1}}{m+1}\right]_0^1 - \int_0^1 \frac{x^{m+1}}{m+1} e^x dx$

$I_m = \frac{1}{m+1}\left[e^x x^{m+1}\right]_0^1 - \frac{1}{m+1}\int_0^1 x^{m+1} e^x dx$

$I_m = \frac{1}{m+1}(e-0) - \frac{1}{m+1} I_{m+1}$

$(m+1) I_m = e - I_{m+1}$

$I_{m+1} = e - (m+1) I_m$

(ii) $I_3 = \int_0^1 y^3 e^y dy.$

$I_3 = e - 3I_2$

$= e - 3[e - 2I_1]$

$= e - 3e + 6I_1$

$= -2e + 6[e - 1I_0]$

$= -2e + 6e - 6\int_0^1 e^y dy$

$= 4e - 6\left[e^y\right]_0^1$

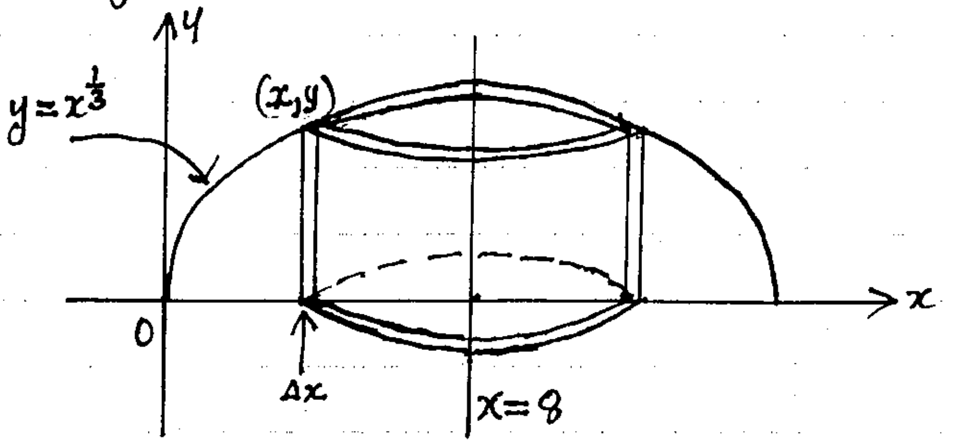
$= 4e - 6(e-1)$

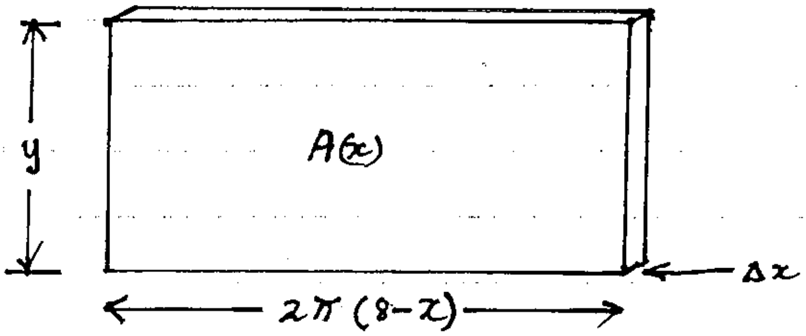
$= 6 - 2e.$

Note: 0.563436343......

$\doteqdot 0.56$ c. 2. d.p.

(b) $y = x^{\frac{1}{3}}$





$$A(x) = 2\pi(8-x)y.$$

$$\Delta V \doteqdot 2\pi(8-x)y\,\Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{x=8} 2\pi(8-x)x^{\frac{1}{3}}\Delta x$$

$$V = 2\pi \int_0^8 (8x^{\frac{1}{3}} - x^{\frac{4}{3}})\,dx$$

$$V = 2\pi \left[\frac{8x^{\frac{4}{3}}}{\frac{4}{3}} - \frac{x^{\frac{7}{3}}}{\frac{7}{3}} \right]_0^8$$

$$V = 2\pi \left[6x^{\frac{4}{3}} - \frac{3}{7}x^{\frac{7}{3}} \right]_0^8$$

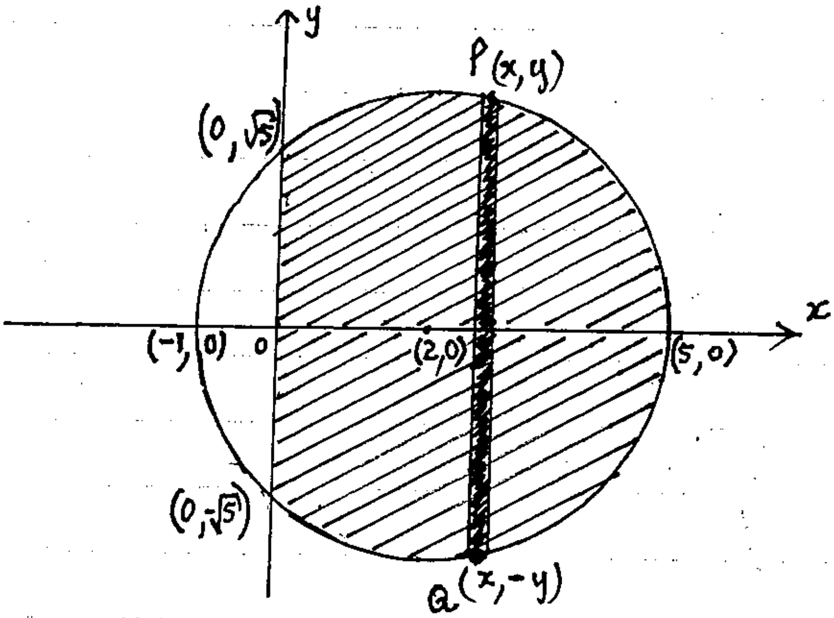
$$V = 2\pi\left(6 \times 16 - \frac{3}{7} \times 128 - 0\right)$$

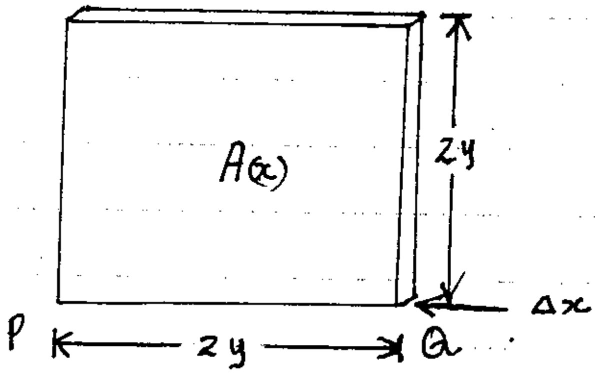
$V = \frac{576\pi}{7}$ units^3 Note $258.5081955\ldots\ldots$

$\doteqdot 258.51$ c. 2.d.p.

Question 4 (continued)

(c) (i) $(x-2)^2 + y^2 = 9$; circle: centre $(2,0)$; radius 3





$$A(x) = 2y \times 2y = 4y^2$$

$$\Delta V \doteqdot 4y^2 \Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{x=5} 4\left[9-(x-2)^2\right]\Delta x$$

$$V = 4\int_0^5 (5+4x-x^2)\,dx$$

$$V = 4\left[5x + 2x^2 - \frac{x^3}{3}\right]_0^5$$

$$V = 4\left[\left(25+50-\frac{125}{3}\right)-0\right]$$

$$V = \frac{400\pi}{3} \text{ cubic units}$$

Note: 418.8790205....
$\doteqdot$ 418.88 c. 2 d.p.

Question 5 (continued)

(a) $\dfrac{x+3}{x-2} > \dfrac{x+1}{x-3}$

$x \neq 2,\ x \neq 3$

$\dfrac{x+3}{x-2} \times (x-2)^2(x-3)^2 > \dfrac{x+1}{x-3} \times (x-2)^2(x-3)^2$

$(x+3)(x-2)(x-3)^2 > (x+1)(x-3)(x-2)^2$

$(x+3)(x-2)(x-3)^2 - (x+1)(x-3)(x-2)^2 > 0$

$(x-2)(x-3)\left[(x+3)(x-3) - (x+1)(x-2)\right] > 0$

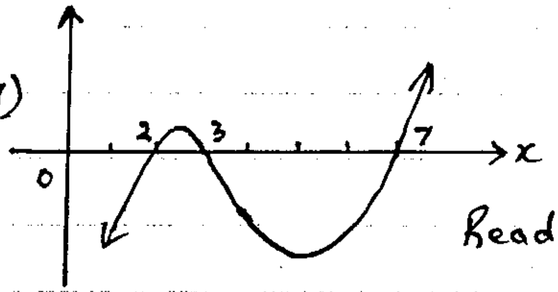
$(x-2)(x-3)\left[x^2-9-(x^2-x-2)\right] > 0$

$(x-2)(x-3)(x^2-9-x^2+x+2) > 0$

$(x-2)(x-3)(x-7) > 0$

consider

$y=(x-2)(x-3)(x-7)$



Read from Graph $y>0$

Solution : $2<x<3,\ x>7$

(b)

$$\begin{array}{r} x^2 + 7ix - (14+7i) \\ x-2i \,\overline{)\, x^3 + 5ix^2 - 7ix - 3} \\ \underline{x^3 - 2ix^2 \qquad\qquad\quad} \\ 7ix^2 - 7ix \qquad \\ \underline{7ix^2 - 14i^2x \qquad} \\ -(14+7i)x - 3 \\ \underline{-(14+7i)x + (-14+28i)} \\ (11-28i) \end{array}$$

$i^2=-1$

Remainder

Questions (continued)

(c) $Q(x) = x^3 - x^2 - 5x + 21$

$Q(2+\sqrt{3}i) = (2+\sqrt{3}i)^3 - (2+\sqrt{3}i)^2 - 5(2+\sqrt{3}i) + 21$

$Q(2+\sqrt{3}i) = 8 + 12\sqrt{3}i - 18 - 3\sqrt{3}i - 4 - 4\sqrt{3}i + 3 - 10 - 5\sqrt{3}i + 21$

$= 0$

$\therefore (2+\sqrt{3}i)$ is a zero of $Q(x)$

Now the co-efficients of $Q(x)$ are real hence the conjugate $(2-\sqrt{3}i)$ of $(2+\sqrt{3}i)$ is also a zero of $Q(x)$.

Now the product of the zeros: $\alpha\beta\gamma = -\frac{d}{a}$.

$(2+\sqrt{3}i)(2-\sqrt{3}i)\gamma = -21$

$7\gamma = -21$

$\gamma = -3$.

$\therefore -3$ is also a zero of $Q(x)$

$\therefore Q(x) = [x - (2+\sqrt{3}i)][x - (2-\sqrt{3}i)](x+3)$

(d) $P(x) = x^4 + x^2 + 6x + 4$ —— ①

$P'(x) = 4x^3 + 2x + 6$

Now $P(-1) = (-1)^4 + (-1)^2 + 6(-1) + 4 = 1 + 1 - 6 + 4 = 0$

And $P'(-1) = 4(-1)^3 + 2(-1) + 6 = -4 - 2 + 6 = 0$

$\therefore (x+1)$ is a factor of both $P(x)$ and $P'(x)$ so $x = -1$ is a zero of multiplicity 2 of $P(x)$

$\therefore P(x) = (x+1)^2(x^2+bx+c) = (x^2+2x+1)(x^2+bx+c)$

$P(x) = x^4 + (b+2)x^3 + (2b+c+1)x^2 + (b+2c)x + c$. —— ②

Equate co-efficients with ①

$b + 2 = 0 \Rightarrow b = -2$

$c = 4$

$\therefore (x^2 - 2x + 4)$ is a factor and $\Delta < 0$

Hence there are no other real zeros.

Question 5 (continued)

(e) $x^3 + lx + m = 0$, roots α, β, γ, $m \neq 0$

Cubic equation whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$

Let $y = \frac{1}{x^2}$

$\therefore x^2 = \frac{1}{y}$

so $x = \frac{1}{\sqrt{y}}$

$$\left(\frac{1}{\sqrt{y}}\right)^3 + l\left(\frac{1}{\sqrt{y}}\right) + m = 0$$

$$\frac{1}{\sqrt{y}}\left[\left(\frac{1}{\sqrt{y}}\right)^2 + l\right] = -m$$

$$\frac{1}{\sqrt{y}}\left(\frac{1}{y} + l\right) = -m$$

square both sides

$$\frac{1}{y}\left(\frac{1}{y^2} + 2\frac{l}{y} + l^2\right) = m^2$$

$$\frac{1}{y^2} + 2\frac{l}{y} + l^2 = m^2 y$$

$$1 + 2ly + l^2y^2 = m^2y^3$$

$$m^2y^3 - l^2y^2 - 2ly - 1 = 0$$

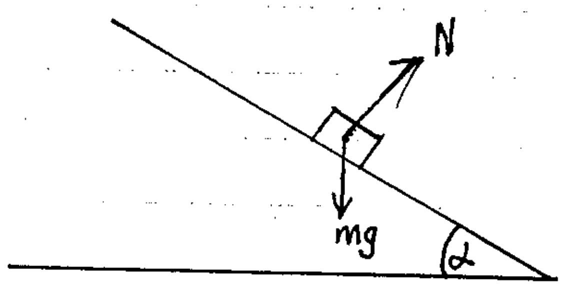
Note change of variable for convenience

ie required equation: $mx^3 - l^2x^2 - 2lx - 1 = 0$

$\therefore$ co-efficients are as shown

Question 6

(a) (i)



no lateral thrust

Resolving vertically : $N\cos\alpha - mg = 0$ ---- ①

Resolving Horizontally: $N\sin\alpha = \dfrac{mv^2}{R}$ ---- ②

Solving ① and ②

② ÷ ① $\dfrac{N\sin\alpha}{N\cos\alpha} = \dfrac{\frac{mv^2}{R}}{mg}$

$\tan\alpha = \dfrac{v^2}{Rg}$

(ii) $v = 72\,km/hr = \dfrac{72\,km}{1\,hr}$

$= \dfrac{72 \times 1000\,m}{1 \times 60 \times 60\,sec}$

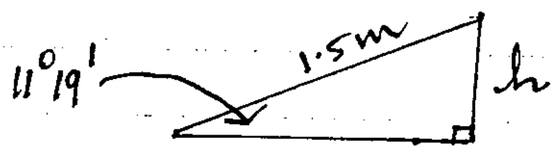
$= 20\,m/sec.$

$r = 200\,m$

$\tan\alpha = \dfrac{20 \times 20}{200 \times 10}$

$\tan\alpha = 0.2$

$\alpha = 11^\circ 19'$

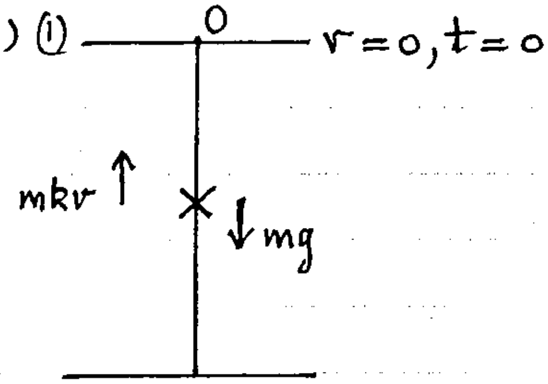


$\dfrac{h}{1.5} = \sin 11^\circ 19'$

$h = 1.5 \sin 11^\circ 19'$

$h = 0.2943\ldots\,m \doteqdot 29\,cm$

(b) (i)



$$F = ma$$

$$ma = mg - mkv$$

$$a = g - kv$$

$$\frac{dv}{dt} = g - kv$$

$$\frac{dt}{dv} = \frac{1}{g-kv}$$

$$\frac{dt}{dv} = -\frac{1}{k}\left(\frac{-k}{g-kv}\right)$$

$$t = -\frac{1}{k}\ln|g-kv| + c$$

Now $v=0$ when $t=0$

$$0 = -\frac{1}{k}\ln|g| + c$$

$$\therefore c = \frac{1}{k}\ln|g|$$

$$\therefore t = \frac{1}{k}\ln|g| - \frac{1}{k}\ln|g-kv|$$

$$t = \frac{1}{k}\ln\left|\frac{g}{g-kv}\right|$$

Question 6 (Continued)

(b) continued.

$$kt = \ln\left|\frac{g}{g-kv}\right|$$

$$\frac{g}{g-kv} = e^{kt}$$

$$\frac{g-kv}{g} = e^{-kt}$$

$$g-kv = ge^{-kt}.$$

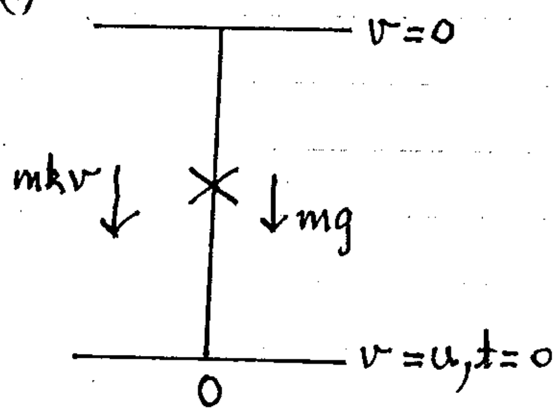
$$kv = g - ge^{-kt}$$

$$V = \frac{g}{k}\left(1-e^{-kt}\right) \qquad \text{-------} \; (1)$$

Note as $t \to \infty$, $1-e^{-kt} \to 1$ so $V \to \frac{g}{k}$

$\therefore$ Terminal velocity $w = \frac{g}{k}$. $\qquad$ --- (2)

(ii)



$$F = ma$$

$$ma = -mg - mkv$$

$$a = -(g+kv)$$

$$\frac{dv}{dt} = -(g+kv)$$

$$\frac{dt}{dv} = -\frac{1}{g+kv}$$

$$\frac{dt}{dv} = -\frac{1}{k}\cdot\frac{k}{g+kv}$$

$$t = -\frac{1}{k}\ln|g+kv| + c$$

<u>Now</u> when $t=0$, $v=u$.

$$0 = -\frac{1}{k}\ln|g+ku| + c$$

$$c = \frac{1}{k}\ln|g+ku|$$

$$\therefore\ t = \frac{1}{k}\ln|g+ku| - \frac{1}{k}\ln|g+kv|$$

$$t = \frac{1}{k}\ln\left|\frac{g+ku}{g+kv}\right|$$

When $v=0$ The time to maximum height is

$$t = \frac{1}{k}\ln\left|\frac{g+ku}{g}\right| \qquad \text{------} (3)$$

Sub (3) in (1)

$$V = \frac{g}{k}\left[1 - e^{-k\left(\frac{1}{k}\ln\left|\frac{g+ku}{g}\right|\right)}\right]$$

$$V = \frac{g}{k}\left(1 - e^{-\ln\left|1+\frac{k}{g}u\right|}\right)$$

$$V = W\left(1 - e^{-\ln\left|1+\frac{u}{W}\right|}\right) \qquad \text{using } (2)$$

$$v = w\left(1 - e^{-\ln\left|\frac{w+u}{w}\right|}\right)$$

$$v = w\left(1 - e^{\ln\left|\frac{w}{w+u}\right|}\right)$$

Note $\boxed{a^{\log_a x} = x}$

$$v = w\left(1 - \frac{w}{w+u}\right)$$

$$v = w\left(\frac{w+u-w}{w+u}\right)$$

$$v = \frac{wu}{w+u}$$

Question 7

(a) $(a+b)^{2m+1}$

Now for The expansion $(a+b)^n$ The General Term is given by $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

(1) $T_{m+1} = \binom{2m+1}{m} a^{2m+1-m} b^m$

$$T_{m+1} = \binom{2m+1}{m} a^{m+1} b^m$$

$$T_{m+2} = \binom{2m+1}{m+1} a^{2m+1-(m+1)} b^{m+1}$$

$$T_{m+2} = \binom{2m+1}{m+1} a^m b^{m+1}$$

$$T_{m+1} + T_{m+2} = \binom{2m+1}{m} a^{m+1} b^m + \binom{2m+1}{m+1} a^m b^{m+1}$$

$$= a^m b^m \left[a\binom{2m+1}{m} + b\binom{2m+1}{m+1} \right]$$

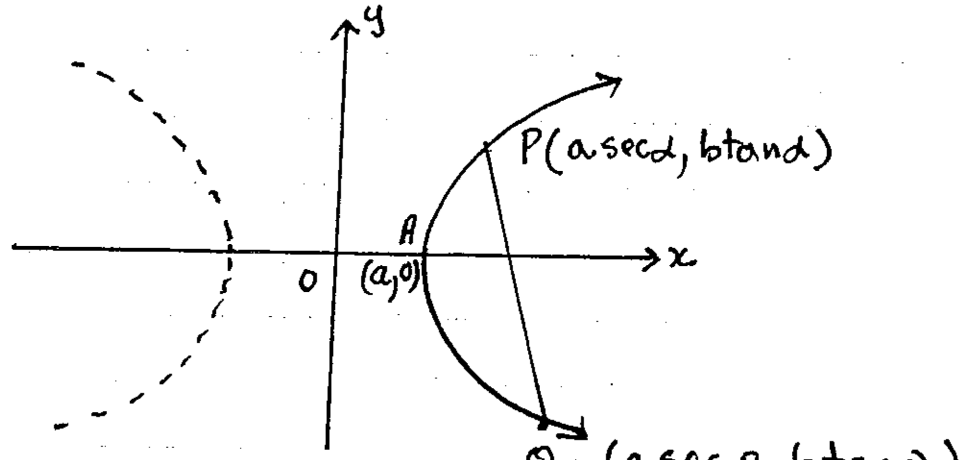
$$= a^m b^m \left[a\,\frac{(2m+1)!}{m!(m+1)!} + b\,\frac{(2m+1)!}{(m+1)!\,m!} \right]$$

$$= a^m b^m \frac{(2m+1)!}{m!(m+1)!} \left[a+b \right]$$

$$T_{m+1} + T_{m+2} = a^m b^m \binom{2m+1}{m}$$

Question 7 (Continued)

(b)



(i) (γ)

Gradient of PQ: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{b\tan\alpha - b\tan\beta}{a\sec\alpha - a\sec\beta}$$

$$m = \frac{b\left(\frac{\sin\alpha}{\cos\alpha} - \frac{\sin\beta}{\cos\beta}\right)}{a\left(\frac{1}{\cos\alpha} - \frac{1}{\cos\beta}\right)}$$

$$m = \frac{b}{a}\left(\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\beta - \cos\alpha}\right)$$

$$m = -\frac{b}{a}\left[\frac{\sin(\alpha-\beta)}{\cos\alpha - \cos\beta}\right]$$

$$m = -\frac{b}{a}\;\frac{{}^{\#}2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$m = \frac{b\cos\left(\frac{\alpha-\beta}{2}\right)}{a\sin\left(\frac{\alpha+\beta}{2}\right)}$$

Note from the Question data this change is just an application of $\sin 2A = 2\sin A\cos A \Rightarrow \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$ etc.

Question 7 (continued)

(θ) $y - y_1 = m(x - x_1)$

$$y - b\tan\alpha = \frac{b\cos\left(\frac{\alpha-\beta}{2}\right)}{a\sin\left(\frac{\alpha+\beta}{2}\right)}\left(x - a\sec\alpha\right)$$

$$ya\sin\left(\frac{\alpha+\beta}{2}\right) - ab\tan\alpha.\sin\left(\frac{\alpha+\beta}{2}\right) = x.b\cos\left(\frac{\alpha-\beta}{2}\right) - ab\sec\alpha\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$xb\cos\left(\frac{\alpha-\beta}{2}\right) - ya\sin\left(\frac{\alpha+\beta}{2}\right) = ab\sec\alpha\cos\left(\frac{\alpha-\beta}{2}\right) - ab\tan\alpha\sin\left(\frac{\alpha+\beta}{2}\right)$$

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{\cos\alpha}\left[\cos\left(\frac{\alpha-\beta}{2}\right) - \sin\alpha\sin\left(\frac{\alpha+\beta}{2}\right)\right]$$

$$\ast\updownarrow$$

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{\cos\alpha}\left[\cos\left(\alpha - \frac{\alpha+\beta}{2}\right) - \sin\alpha.\sin\left(\frac{\alpha+\beta}{2}\right)\right]$$

Now $\cos\left(\alpha - \frac{\alpha+\beta}{2}\right) = \cos\alpha\cos\left(\frac{\alpha+\beta}{2}\right) + \sin\alpha\sin\left(\frac{\alpha+\beta}{2}\right)$

$$\therefore \frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{\cos\alpha}\times\cos\alpha\cos\left(\frac{\alpha+\beta}{2}\right)$$

ie $\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$

* A change of convenience.

$$\frac{\alpha-\beta}{2} = \alpha - \frac{\alpha+\beta}{2}.$$

A bit tricky ?

Question 7 continued

(ii) Gradient of AP : $m_1 = \dfrac{b\tan\alpha}{a\sec\alpha - a}$

Gradient of AQ : $m_2 = \dfrac{b\tan\beta}{a\sec\beta - a}$

Now $m_1 m_2 = -1$ because $\widehat{PAQ} = 90°$

$$\frac{b\tan\alpha}{a(\sec\alpha - 1)} \times \frac{b\tan\beta}{a(\sec\beta - 1)} = -1$$

$$\frac{\frac{\sin\alpha}{\cos\alpha} \times \frac{\sin\beta}{\cos\beta}}{\left(\frac{1}{\cos\alpha} - 1\right)\left(\frac{1}{\cos\beta} - 1\right)} = -\frac{a^2}{b^2}$$

$$\frac{\sin\alpha\sin\beta}{(1-\cos\alpha)(1-\cos\beta)} = -\frac{a^2}{b^2}$$

$$\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \times 2\sin\frac{\beta}{2}\cos\frac{\beta}{2}}{2\sin^2\frac{\alpha}{2} \times 2\sin^2\frac{\beta}{2}} = -\frac{a^2}{b^2}$$

$$\cot\frac{\alpha}{2}\cot\frac{\beta}{2} = -\frac{a^2}{b^2}$$

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = -\frac{b^2}{a^2}$$

Now $b^2 = a^2(e^2 - 1)$

$$\frac{b^2}{a^2} = e^2 - 1$$

$$-\frac{b^2}{a^2} = 1 - e^2$$

$$\therefore \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = 1 - e^2$$

Question 8

(a) $y = \frac{\pi}{2} - x$

$\frac{dy}{dx} = -x$

$-dy = dx$

when $x = \frac{\pi}{2}$

$y = \frac{\pi}{2} - \frac{\pi}{2} = 0$

when $x = 0$

$y = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x}\,dx$$

$$I = \int_{\frac{\pi}{2}}^{0} \frac{1}{1+\tan\left(\frac{\pi}{2}-y\right)} \times -1 \times dy$$

$$= -\int_{\frac{\pi}{2}}^{0} \frac{1}{1+\cot y}\,dy$$

Note $\int_a^b f(x)dx = -\int_b^a f(x)dx$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot y}\,dy$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot x}\,dx$$

Note: change of variable for convenience.

$$\text{Now}\quad 2I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x}\,dx + \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot x}\,dx$$

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{1}{1+\tan x} + \frac{1}{1+\cot x}\right) dx$$

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{1}{1+\tan x} + \frac{1}{1+\frac{1}{\tan x}}\right) dx$$

Question 8 (continued)

(b) $z\bar{z} + (1-2i)z + (1+2i)\bar{z} \le 3$

Let $z = x + iy$

$\therefore$ inequality becomes :

$(x+iy)(x-iy) + (1-2i)(x+iy) + (1+2i)(x-iy) \le 3$

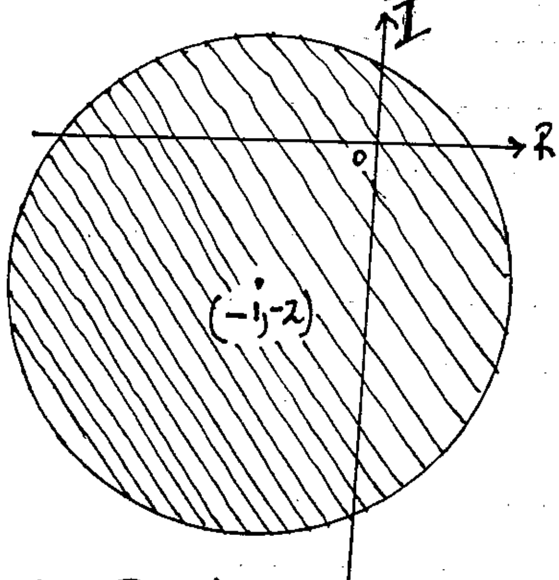
$x^2 + y^2 + x + iy - 2xi + 2y + x - iy + 2xi + 2y \le 3$

$x^2 + y^2 + 2x + 4y \le 3$

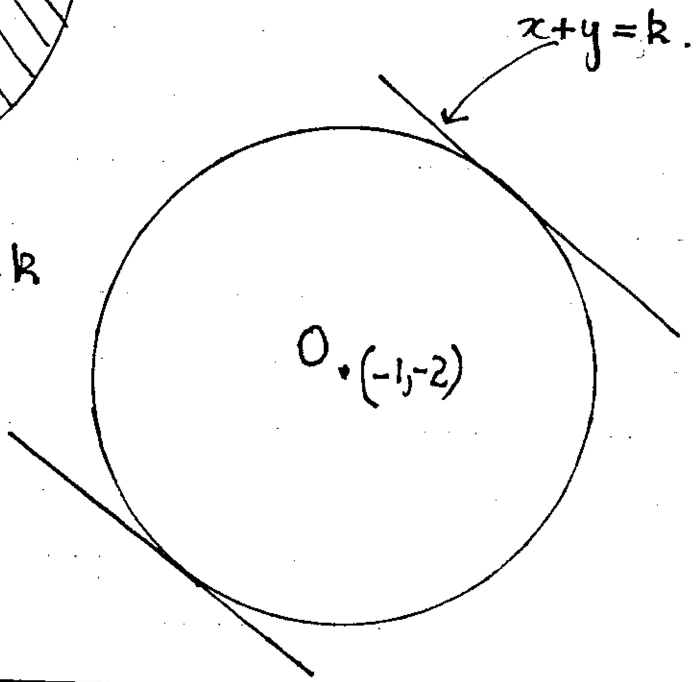
$x^2 + 2x + (+1)^2 + y^2 + 4y + (+2)^2 \le 3 + (+1)^2 + (+2)^2$

$(x+1)^2 + (y+2)^2 \le 8$

The locus is the region enclosed (and inclusive of the boundary) of the circle centre $(-1,-2)$; radius $2\sqrt{2}$



(ii) Let The line $x+y=k$ touch The circle $(x+1)^2 + (y+2)^2 = 8$



$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{1}{1+\tan x} + \frac{\tan x}{1+\tan x} \right) dx$$

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{1+\tan x}{1+\tan x} \right) dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1 \, dx$$

$$2I = \left[x \right]_0^{\frac{\pi}{2}}$$

$$2\pi = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}.$$

Note $0.785398163.....$
$\fallingdotseq 0.785$ c. 3. d. p.

Solving $(x+1)^2+(y+2)^2=8$ --- ①

$x+y=k$ --- ②

simultaneously.

sub ② in ① ie $y=k-x$ --- (2a)

$(x+1)^2+(k-x+2)^2=8$

$x^2+2x+1+k^2-2kx+x^2+4(k-x)+4=8$

$2x^2+2x-2kx-4x+1+k^2+4k-8+4=0$

$2x^2-2(k+1)x+k^2+4k-3=0$

For tangency $\Delta=0$

$[2(k+1)]^2-4(2)(k^2+4k-3)=0$

$(k+1)^2-2(k^2+4k-3)=0$

$k^2+2k+1-2k^2-8k+6=0$

$-k^2-6k+7=0$

$k^2+6k-7=0$

$(k+7)(k-1)=0$

$k=1$ or $k=-7$

Thus for any (x,y) satisfying

$(x+1)^2+(y+2)^2\leq 8$

$-7\leq x+y\leq 1$

Thus the maximum value of $(x+y)$ is 1 and the minimum value of $(x+y)$ is -7.