SAINT IGNATIUS' COLLEGE

Trial Higher School Certificate

2007

MATHEMATICS EXTENSION 2

12:30pm – 3:35pm
Tuesday 28th August 2007
Directions to Students

• Reading Time : 5 minutes	• Total Marks **120**
• Working Time : 3 hours	
• Write using blue or black pen. (sketches in pencil).	• Attempt Question 1 – 8
• Board approved calculators may be used	• All questions are of equal value
• A table of standard integrals is provided at the back of this paper.	
• All necessary working should be shown in every question.	
• **Answer each question in the booklets provided and clearly label your name and teacher's name.**	

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Show that $\sin(A+B)+\sin(A-B)=2\sin A\cos B$ 1

(ii) Hence find the indefinite integral $\int \sin 5x \cos 3x \, dx$ 2

(b) Evaluate $\displaystyle\int_0^5 \frac{t\,dt}{\sqrt{t+4}}$ 2

(c) Evaluate $\displaystyle\int_{-1}^{1} 3^x \, dx$ correct to three significant figures. 2

(d) Evaluate $\displaystyle\int_0^{\frac{1}{3}} \frac{dx}{\sqrt{1-9x^2}}$ 2

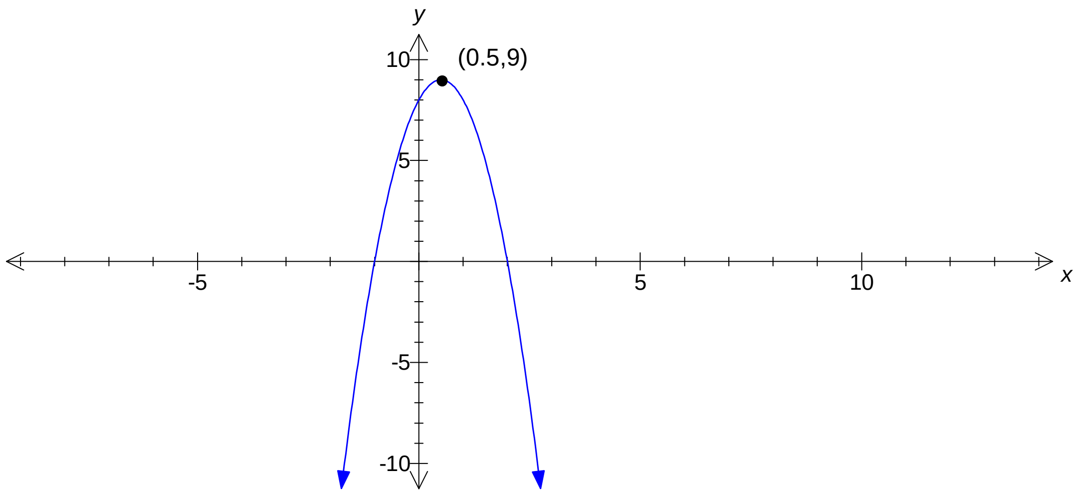
(e) Find $\displaystyle\int \frac{1}{1+\sin x} dx$, using the substitution $t=\tan\frac{x}{2}$ 3

(f) Use integration by parts to find $\displaystyle\int \frac{\cos^{-1} x}{\sqrt{1+x}} dx$ 3

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Prove that $f\left(x\right)=\dfrac{x^3}{\sin x}$ is an even function. 2

(b) A sketch of $f\left(x\right)=-4\left(x+1\right)\left(x-2\right)$ is shown below.



With the aid of the above diagram, and without the use of calculus, draw a separate half page sketch for each of the following.

(i) $y=\left|f\left(x\right)\right|$ 1

(ii) $y=f\left(2x\right)$ 1

(iii) $y=f\left(-x\right)$ 1

(iv) $y=\dfrac{1}{f\left(x\right)}$ 2

(v) $y=\sqrt{f\left(x\right)}$ 2

(vi) $y=\log_e f\left(x\right)$ 2

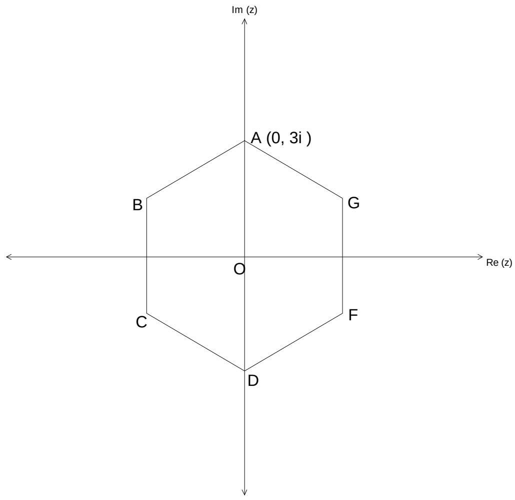
(c) Using calculus, show that $e^{-x}+x-1\geq 0$ for real x. 4

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Express the following in the form $(x+iy)$, where x and y are real: 2

$$\frac{i^2-1}{i}+\frac{1}{1+i}$$

(b)



The Argand diagram above, shows a regular hexagon with vertex A at the point $(0,3i)$. O is the centre of the hexagon.

(i) Copy the diagram into your writing booklet.

(ii) On your diagram show the region within the hexagon in which both the inequalities $|z|\le 2$ and $-\frac{\pi}{6}\le \arg z\le \frac{\pi}{6}$ are satisfied. 2

(iii) Find in the form $|z-z_1|=R$, the equation of the circle through the points O, B and F. 1

(iv) Find the complex numbers, in modulus argument form, represented by the points B and C. 2

(v) The hexagon is rotated anticlockwise about the origin through an angle of $\frac{\pi}{4}$. Express in the form $r(\cos\theta+i\sin\theta)$, where θ is the principal argument, the complex numbers represented by the new positions of B and C. 3

Question 3 continues on page 5

Question 3 (continued)

(c) If $1 - 2i$ is a root of the equation $z^2 - (3+i)z + k = 0$,

(i) explain why the conjugate $1 + 2i$ cannot be a root to the equation 1

(ii) show that the other root is $2 + 3i$ 1

(iii) find the value of k 1

(iv) hence, or otherwise, find the two square roots of $-24 + 10i$. 2

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) A polynomial $p(x) = x^n + ax^2 - 2$ has a factor of $(x-1)$ and leaves a remainder of -6 on division by $(x+2)$.

Find:

(i) the value of a 1

(ii) the value of n 1

(iii) the zeros of $p(x)$. 2

(b) Find the values of a and b so that $p(x) = 2x^3 - (2a+1)x^2 + (2+b)x - 1$ has a double root at $x = 1$. 4

(c) If l, m, n are the roots of the equation $x^3 - 2x + 5 = 0$,

(i) find the cubic equation whose roots are $2l$, $2m$, $2n$. 2

(ii) find the value of $l^3 + m^3 + n^3$. 2

(d) Find all the values of k for which the polynomial equation $3x^4 - 4x^3 + k = 0$ has no real roots. 3

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) An ellipse has the equation $\frac{x^2}{25}+\frac{y^2}{16}=1$. O is the centre of the ellipse and S and S' are the foci.

(i) Find

(α) the eccentricity 1

(β) the co-ordinates of the foci 1

(γ) the equations of the directrices. 1

(ii) Make a third of a page sketch of the ellipse showing the features found in part (i) 2

(iii) If $P(x_{0,}y_0)$ is a point on the ellipse show that $(PS+PS')$ is constant. You may mark point P in quadrant one of the above mentioned diagram. 2

(iv) Show that the equation of the tangent at $P(x_0,y_0)$ is 2

$$\frac{xx_0}{25}+\frac{yy_0}{16}=1.$$

(v) The tangent at P meets the nearer focus at R. If S is the nearer focus to P,

(α) write down the co-ordinates of R. 1

(β) find expressions for the gradients of PR and SR in terms of x_0 and y_0. 2

(γ) show that the angle PSR is a right angle. 1

(b) A hyperbola has its centre at the origin and asymptotes $y=\pm\frac{2}{3}x$. Find its equation. 2

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The region bounded by the curve $y = x(2-x)$ and the x–axis is rotated about the y-axis. Find the volume of the solid of revolution by taking slices perpendicular to the y-axis. 4

(b) The region bounded by the curve $y = \ln x$, the line $y = 1$ and the co-ordinate axes is rotated about the x-axis.

(i) By dividing the resulting solid into cylindrical shells, show that each shell has an approximate volume : $\delta v = 2\pi y e^{y}\,\delta y$ where δy is the thickness of the shell. 2

(ii) Hence calculate the volume of the solid. 2

(c) The base of a particular solid is the circle $x^2 + y^2 = 8$. Find the volume of the solid if every cross section to the x-axis is an isosceles – right angled triangle with the hypotenuse in the base of the solid. 4

(d) Show that the straight line $lx + my + n = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2l^2 - b^2m^2 = n^2$. 3

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Mr Kirkpatrick's mathematics class brought him a ride in a Gondola at Queenstown in New Zealand, during a recent trip. When Mr Kirkpatrick's hands were H metres above the Earth's surface, he dropped overboard his packet of beer nuts of mass m kg. The packet of beer nuts encounters air resistance proportional to its velocity v (which is in metres per second), that is the resistive force is equal to mkv.
Taking Mr Kirkpatrick's hands as the origin and downwards displacement as positive:

(i) Write down an equation of motion representing the passage of the packet of beer nuts. 2

(ii) Find the terminal velocity , w, of the packet of beer nuts. 1

(iii) Show that the equation of motion in part (i) can be written as $\ddot{x} = k(w - v)$. 1

(iv) Show that the displacement, x metres, of the packet of beer nuts from Mr Kirkpatrick's hands is given by: $x = -\frac{v}{k} - \frac{w}{k}\ln\left(\frac{w-v}{w}\right)$. 4

(v) If the packet reaches the Earth's surface with a velocity of u metres per second, show that $\ln\left(1 - \frac{u}{w}\right) + \frac{u}{w} + \frac{kH}{w} = 0$. 1

(vi) Consider the moment when the packet of beer nuts has reached 75% of its terminal velocity.
Find:

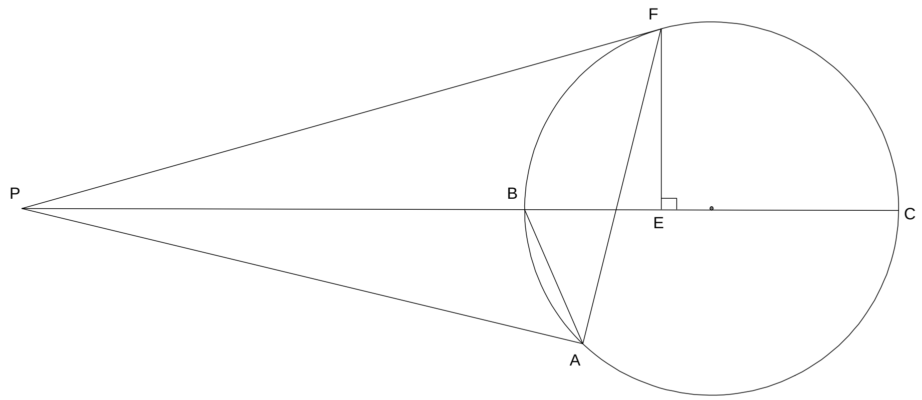
(α) the time, t seconds, for this moment to be reached. 3

(β) the distance fallen at this moment. 3

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Draw a neat half page sketch of the graph for $y^2 = x^2\left(4 - x^2\right)$. 3

(b)



In the above diagram *AB* and *BC* are chords of a circle, and *F* is on the arc *ABC* such that arc *AF* is equal to arc *FC*. *E* is the foot of the perpendicular from *F* to the chord *BC*. *CB* is extended to *P* so that *PE* = *EC*. (Note that *B* is inside the triangle *APF*)

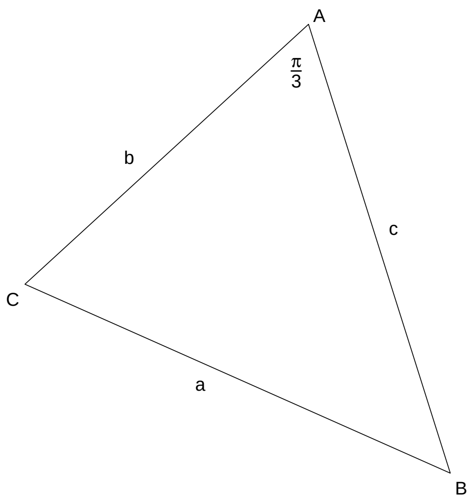
(i) Show that the triangle *APF* is isosceles. 3

(ii) Show that *AB* + *BE* = *EC*. 4

Question 8 continues on page 11

Question 8 (continued)

(c)



A triangle ABC has sides of varying length a, b and c with a fixed interior angle of $BAC =$ as shown in the above diagram.

Use the cosine rule to show that:

(i) $a^2 \geq bc$, and hence, 3

(ii) the area of triangle $ABC \leq \dfrac{a^2\sqrt{3}}{4}$ $ABC \leq \dfrac{a^2\sqrt{3}}{4}$ 2

End of examination

Mathematics
Extension 2 Solutions 2007

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Solutions

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Trial Higher School Certificate

2007

MATHEMATICS EXTENSION 2

- This is a study resource for HSC preparation
- This document is designed to help students understand the questions
- The solutions should be treated as aids only
- There may be better solutions to some questions.

Question 1

(a)(i) $LHS = \sin(A+B) + \sin(A-B)$

$$= \sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B$$

$$= 2\sin A\cos B = RHS.$$

(ii) $$\int \sin 5x\cos 3x\,dx = \frac{1}{2}\int(\sin 8x + \sin 2x)\,dx$$

$$= \frac{1}{2}\left(-\frac{1}{8}\cos 8x - \frac{1}{2}\cos 2x\right) + c$$

$$= -\frac{1}{16}\cos 8x - \frac{1}{4}\cos 2x + c$$

(b) $$I = \int_0^5 \frac{t\,dt}{\sqrt{t+4}}$$

Let $u = t+4$

$du = dt$

when $t=5,\ u=9$

$t=0,\ u=4$

$$I = \int_4^9 \frac{u-4}{u^{1/2}}\,du$$

$$I = \int_4^9 \left(u^{\frac{1}{2}} - 4u^{-\frac{1}{2}}\right) du$$

$$I = \left[\frac{2}{3}u^{\frac{3}{2}} - 2\times 4u^{\frac{1}{2}}\right]_4^9$$

$$I = \left[\frac{2}{3}u^{\frac{3}{2}} - 8u^{\frac{1}{2}}\right]_4^9$$

$$I = \left[\left(\frac{2}{3}\times 27 - 24\right) - \left(\frac{2}{3}\times 8 - 8\times 2\right)\right]$$

$$I = 18 - 24 - \frac{16}{3} + 16$$

$$I = 34 - 29\tfrac{1}{3}$$

$$I = 4\tfrac{2}{3}$$

(c) $\displaystyle\int_{-1}^{1} 3^x\,dx = \frac{1}{\ln 3}\Big[3^x\Big]_{-1}^{1}$

$= \dfrac{1}{\ln 3}\left(3 - \frac{1}{3}\right)$

$= \dfrac{8}{3\ln 3}$

$= 2.43$ c.3.s.f Calculater Face 2.427304604

(d) $\displaystyle\int_0^{\frac{1}{3}} \frac{dx}{\sqrt{1-9x^2}}$

<u>Note</u> $1 - 9x^2 = 9\left(\frac{1}{9} - x^2\right)$

$\sqrt{1-9x^2} = 3\sqrt{\frac{1}{9} - x^2}$

$= \dfrac{1}{3}\displaystyle\int_0^{\frac{1}{3}} \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}}\,dx$

$= \dfrac{1}{3}\left[\sin \dfrac{x}{\frac{1}{3}}\right]_0^{\frac{1}{3}}$

$= \dfrac{1}{3}\Big[\sin 3x\Big]_0^{\frac{1}{3}}$

$= \dfrac{1}{3}\Big[\sin 1 - \sin 0\Big]$

$= \dfrac{1}{3} \times \dfrac{\pi}{2}$

$= \dfrac{\pi}{6}$

(e) $\int \frac{1}{1+\sin x}\,dx$

when $t = \tan\frac{x}{2}$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}.$$

$$2\,dt = \left(1+\tan^2\frac{x}{2}\right)dx$$

$$\frac{2\,dt}{1+t^2} = dx$$

$$I = \int \frac{1}{1+\frac{2t}{1+t^2}}\,\frac{2\,dt}{1+t^2}$$

$$I = 2\int \frac{1+t^2}{(1+t^2+2t)(1+t^2)}\,dt$$

$$I = 2\int \frac{1}{(1+t)^2}\,dt$$

$$I = 2\int (1+t)^{-2}\,dt$$

$$I = 2\frac{(1+t)^{-1}}{1\times -1} + c$$

$$I = -\frac{2}{1+t} + c.$$

$$I = -\frac{2}{1+\tan\frac{x}{2}} + c$$

(f) $\displaystyle\int \frac{\cos^{-1}x}{\sqrt{1+x}}\,dx$

Let $u = \cos^{-1}x \Rightarrow u' = -\dfrac{1}{\sqrt{1-x^2}}$

$v' = (1+x)^{-\frac{1}{2}} \Rightarrow v = \dfrac{(x+1)^{\frac{1}{2}}}{\frac{1}{2}\times 1} = 2\sqrt{x+1}$

$$I = 2\sqrt{x+1}\cos^{-1}x - \int -\frac{1}{\sqrt{1-x^2}} \times 2\sqrt{x+1}\,dx$$

$$I = 2\sqrt{x+1}\cos^{-1}x + 2\int \frac{\sqrt{x+1}}{\sqrt{1-x^2}}\,dx$$

$$I = 2\sqrt{x+1}\cos^{-1}x + 2\int \frac{1}{\sqrt{1-x}}\,dx$$

$$I = 2\sqrt{x+1}\cos^{-1}x + 2\int (1-x)^{-\frac{1}{2}}\,dx$$

$$I = 2\sqrt{x+1}\cos^{-1}x + 2\times \frac{(1-x)^{\frac{1}{2}}}{\frac{1}{2}} \times -1 + c$$

$$I = 2\sqrt{x+1}\cos^{-1}x - 4\sqrt{1-x} + c$$

Question 2

(a) $f(x) = \dfrac{x^3}{\sin x}$

$$f(-x) = \frac{(-x)^3}{\sin(-x)}$$

$$= \frac{-x^3}{-\sin x}$$

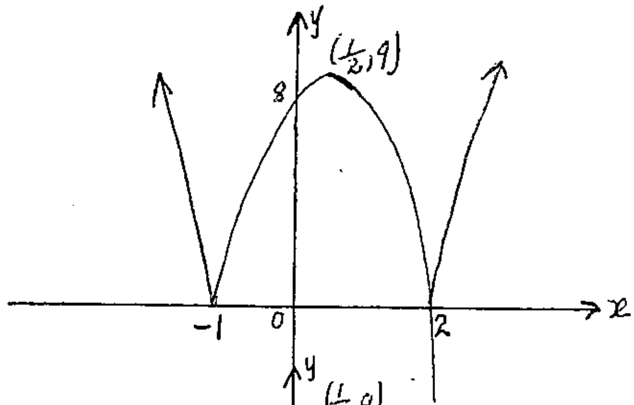
$$= \frac{x^3}{\sin x}$$

$$= f(x)$$

$\therefore$ $f(x)$ is an even function

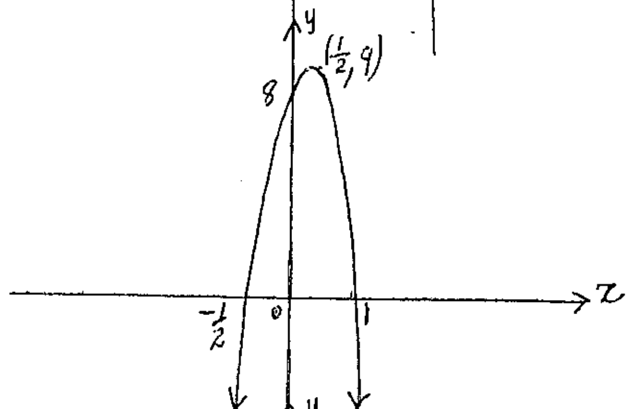
(b) $f(x) = -4(x+1)(x-2)$

(i)



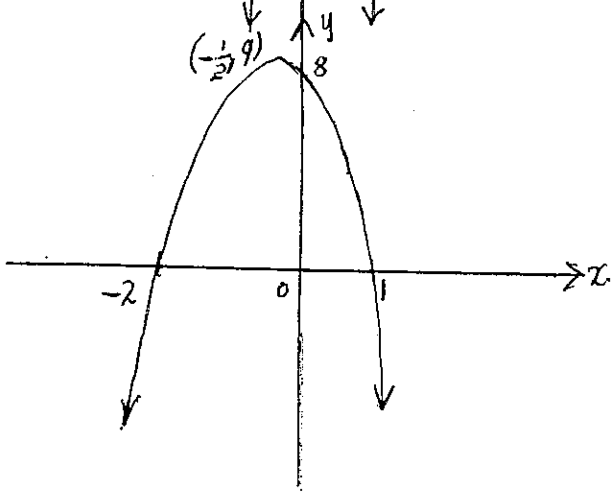
$y = |f(x)|$

(ii)



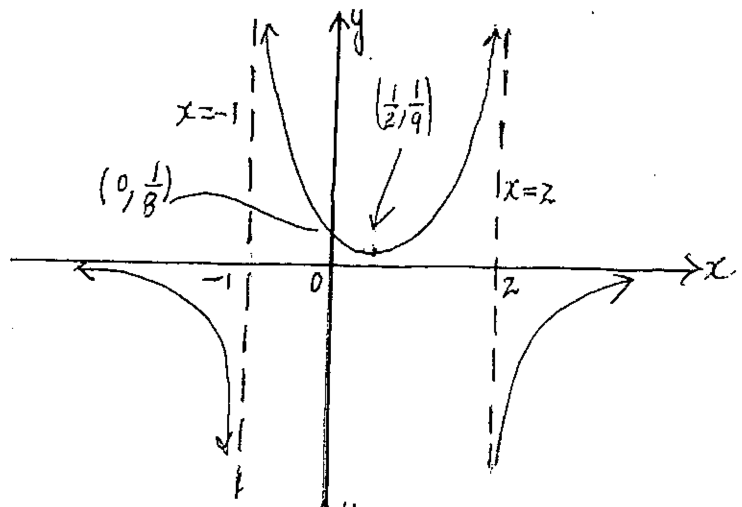
$y = f(2x)$

(iii)



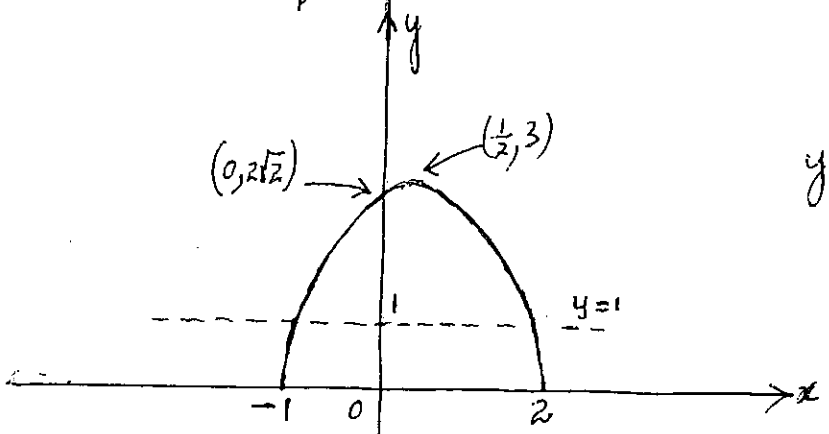
$y = f(-x)$

(iv)



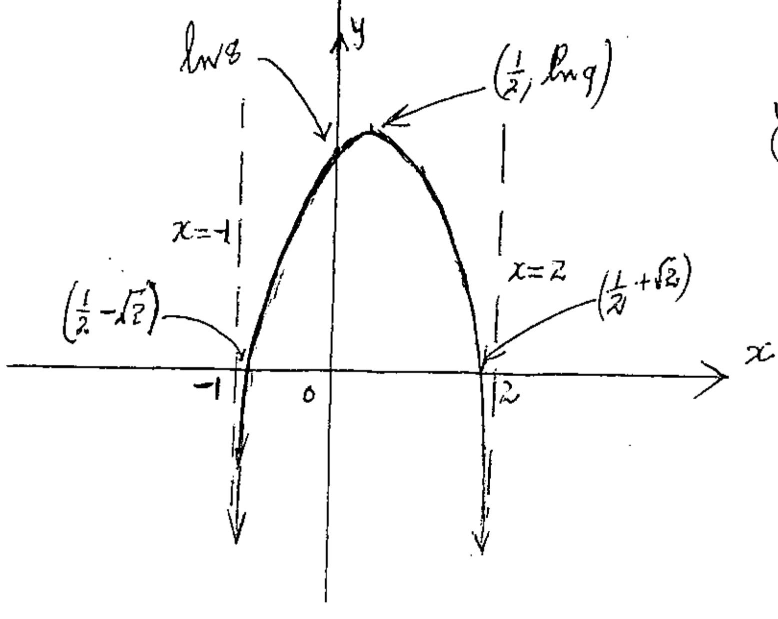
$y = \frac{1}{f(x)}$

(v)



$y = \sqrt{f(x)}$

(vi)



$y = \ln f(x)$

(c) $e^{-x}+x-1 \geqslant 0$

Let $y = e^{-x}+x-1$

Now $\frac{dy}{dx} = -e^{-x}+1$

$\frac{d^2y}{dx^2} = e^{-x} = \frac{1}{e^x} > 0$ for all x in the domain of the reals.

Note: The meaning of a negative index $a^{-m} = \frac{1}{a^m}$

Now since $\frac{d^2y}{dx^2} > 0$ we can say that the curve.

$y = e^{-x}+x-1$ is always concave up.

When $\frac{dy}{dx} = 0$

$-e^{-x}+1 = 0$

$e^{x} = 1$

$e^{x} = e^{0}$

$\therefore x = 0$

and when $x = 0$, $y = e^{0}+0-1$

$y = 0$

$\therefore$ minimum turning point at $(0,0)$

Hence $y \geqslant 0$ for all x hence

$e^{-x}+x-1 \geqslant 0$ for real x.

Question 3

(a) $\dfrac{i^2-1}{i} + \dfrac{1}{1+i}$

$= \dfrac{-1-1}{i} + \dfrac{1}{1+i} = \dfrac{-2}{i} + \dfrac{1}{1+i}$

$= \dfrac{-2}{i}\times\dfrac{i}{i} + \dfrac{1(1-i)}{(1+i)(1-i)}$

$= 2i + \dfrac{1-i}{1+1}$

$= \dfrac{1}{2} + 2i - \dfrac{1}{2}i$

$= \dfrac{1}{2} + \dfrac{3}{2}i \Leftarrow$

$= \dfrac{-2(1+i)+i}{i(1+i)}$

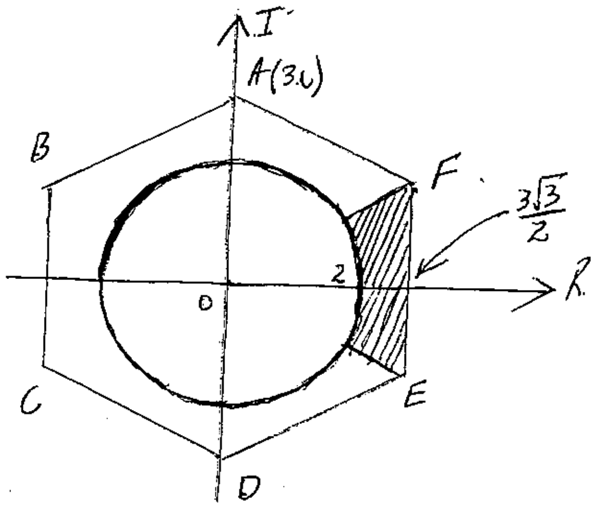
$= \dfrac{-2-2i+i}{i+i^2}$

$= \dfrac{-2-i}{i-1}\times\dfrac{i+1}{i+1}$

$= \dfrac{-2i-2-i^2-i}{i^2-1}$

$= \dfrac{-1-3i}{-2}$

(b) (i) and (ii)



(iii) Circle has centre $A(0, 3i)$ and radius 3.

Equation : $|z-3i| = 3$

(iv) B is given by : $3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

C is given by : $3\left(\cos -\frac{5\pi}{6} + i\sin -\frac{5\pi}{6}\right)$

(v) New position of B : $3\left(\cos\left(\frac{5\pi}{6}+\frac{\pi}{4}\right) + i\sin\left(\frac{5\pi}{6}+\frac{\pi}{4}\right)\right)$

$= 3\left(\cos -\frac{11\pi}{12} + i\sin -\frac{11\pi}{12}\right)$

New position of C : $3\left(\cos\left(-\frac{5\pi}{6}+\frac{\pi}{4}\right) + i\sin\left(-\frac{5\pi}{6}+\frac{\pi}{4}\right)\right)$

$= 3\left(\cos -\frac{7\pi}{12} + i\sin -\frac{7\pi}{12}\right)$

(c) $(1-2i)$ a root of $z^2-(3+i)z+k=0$

(i) Theorem: If polynomial equation co-efficients are real then complex roots exist in conjugate pairs. Hence $(1+2i)$ is not a root.

(ii) Sum of roots : $\alpha+\beta = -\frac{[-(3+i)]}{1}$

$$\alpha+\beta = 3+i$$

If $\alpha = 1-2i$

Then $1-2i+\beta = 3+i$

$$\therefore \beta = 2+3i$$

(iii) Product of roots : $\alpha\beta = \frac{k}{1}$

$$\alpha\beta = k.$$

substitution : $(1-2i)(2+3i) = k.$

$$k = 2+3i-4i-6i^2$$
$$k = 2+6-i$$
$$k = 8-i$$

(iv) Now considering $z^2-(3+i)z+(8-i)=0$

$$z = \frac{(3+i) \pm \sqrt{(3+i)^2-4(1)(8-i)}}{2(1)}$$

$$z = \frac{(3+i) \pm \sqrt{9+6i+i^2-32+4i}}{2}$$

$$z = \frac{(3+i) \pm \sqrt{-24+10i}}{2}.$$

Note here that there is an alternative method for finding square roots

Let one of the square roots be $(a+bi)$

$$\therefore 1-2i = \frac{(3+i)+a+bi}{2}.$$

$$2-4i = 3+i+a+bi.$$
$$-1-5i = a+bi$$

Equating real and imaginary parts.

$$a=-1, \quad b=-5$$

Similarly it can also be shown that $a=1, b=5$.

$\therefore$ The two square roots are $\pm(1+5i)$

Question 4

(a) $p(x) = x^n + ax^2 - 2$

(i) when $x = 1$, $p(x) = 0$

$\therefore 0 = (1)^n + a(1)^2 - 2$

$\therefore a = 1$

(ii) When $p(x) = -6$, $x = -2$

$\therefore -6 = (-2)^n + 1\times(-2)^2 - 2$

$-6 = (-2)^n + 4 - 2$

$-6 = (-2)^n + 2$

$(-2)^n = -8$

$(-2)^n = (-2)^3$

$n = 3$

(iii) $\therefore p(x) = x^3 + x^2 - 2$

Now $p(x) \div (x-1)$

$$
\begin{array}{r|l}
 & x^2 + 2x + 2 \\
x-1 & x^3 + x^2 + 0x - 2 \\
 & x^3 - x^2 \\
\hline
 & 2x^2 + 0x \\
 & 2x^2 - 2x \\
\hline
 & 2x - 2 \\
 & 2x - 2 \\
\hline
 & 0 + 0
\end{array}
$$

$\therefore p(x) = (x-1)(x^2 + 2x + 2)$

When $x^2 + 2x + 2 = 0$

$x = \frac{-2 \pm \sqrt{4-8}}{2}$

$x = \frac{-2 \pm \sqrt{4i^2}}{2}$

$x = \frac{-2 \pm 2i}{2} = -1 \pm i$

Hence solutions for $p(x) = 0$ are $x = 1$ or $x = -1 \pm i$

(b) $p(x) = 2x^3 - (2a+1)x^2 + (2+b)x - 1$

$p'(x) = 6x^2 - 2(2a+1)x + (2+b)$

Now There is a double root at $x = 1$

$\therefore p(1) = p'(1) = 0$

So $2 - (2a+1)\times 1 + (2+b)\times 1 - 1 = 0$.

$2 - 2a - 1 + 2 + b - 1 = 0$.

$-2a + b + 2 = 0$

$2a - b = 2$ --- ①

And $6 - 2(2a+1)\times 1 + 2 + b = 0$.

$6 - 4a - 2 + 2 + b = 0$.

$-4a + b + 6$

$4a - b = 6$ --- ②

Solving: ① − ② $-2a = -4$

$a = 2$

Sub in ① $4 - b = 2$

$b = 2$

Answer $a = 2$, $b = 2$

(b) l, m, n roots of $x^3 - 2x + 5 = 0$

(i) Let $y = 2x \Rightarrow x = \frac{y}{2}$

Substitute:

$\left(\frac{y}{2}\right)^3 - 2\left(\frac{y}{2}\right) + 5 = 0$

$\frac{y^3}{8} - y + 5 = 0$.

$y^3 - 8y + 40 = 0$.

reverting to The variable x:

The required equation is $x^3 - 8x + 40 = 0$

C (ii) If l, m, n are the roots of the equation $x^3 = 2x - 5$ Then:

$$l^3 = 2l - 5 \quad --- \ (1)$$
$$m^3 = 2m - 5 \quad --- \ (2)$$
$$n^3 = 2n - 5 \quad --- \ (3)$$

Add these equations:

$$l^3 + m^3 + n^3 = 2(l + m + n) - (5 \times 3)$$

Now the sum of the roots of the given equation is zero. ie $(l + m + n) = 0$

$$\therefore l^3 + m^3 + n^3 = -15.$$

(d) Considering $3x^4 - 4x^3 + k = 0$.

Let $p(x) = 3x^4 - 4x^3 + k$.

$$p'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$$
$$p''(x) = 36x^2 - 24x = 12x(3x-2)$$

For stationary points $p'(x) = 0 \therefore x = 0$ or 1

When $x = 0$ $p(x) = k$

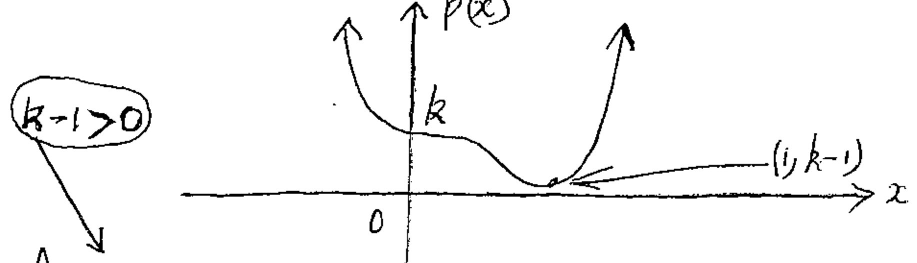
and $p''(x) = 0$

It appears That $(0, k)$ is a horizontal point of inflection

When $x = 1$ $p(x) = k - 1$

and $p''(x) = 12 > 0$

$\therefore (1, k-1)$ is a minimum turning point.



If $k > 1$ The point $(1, k-1)$ is above The x axis and hence $(3x^4 - 4x^3 + k) = 0$ has no real roots.

Question 5

(a) $\frac{x^2}{25} - \frac{y^2}{16} = 1$

$a = 5, b = 4$

(i) $b^2 = a^2(e^2 - 1)$

$16 = 25(e^2 - 1)$

$\frac{16}{25} = e^2 - 1$

$e^2 = \frac{25}{25} + \frac{16}{25}$

$e^2 = \frac{41}{25}$

$e = \frac{\sqrt{41}}{5}, \ e > 0$

(ii) S is $(ae, 0)$; $S' = (-ae, 0)$

$ae = \frac{5 \times \sqrt{41}}{5} = \sqrt{41}$

$\therefore S = (\sqrt{41}, 0)$ and $S' = (-\sqrt{41}, 0)$

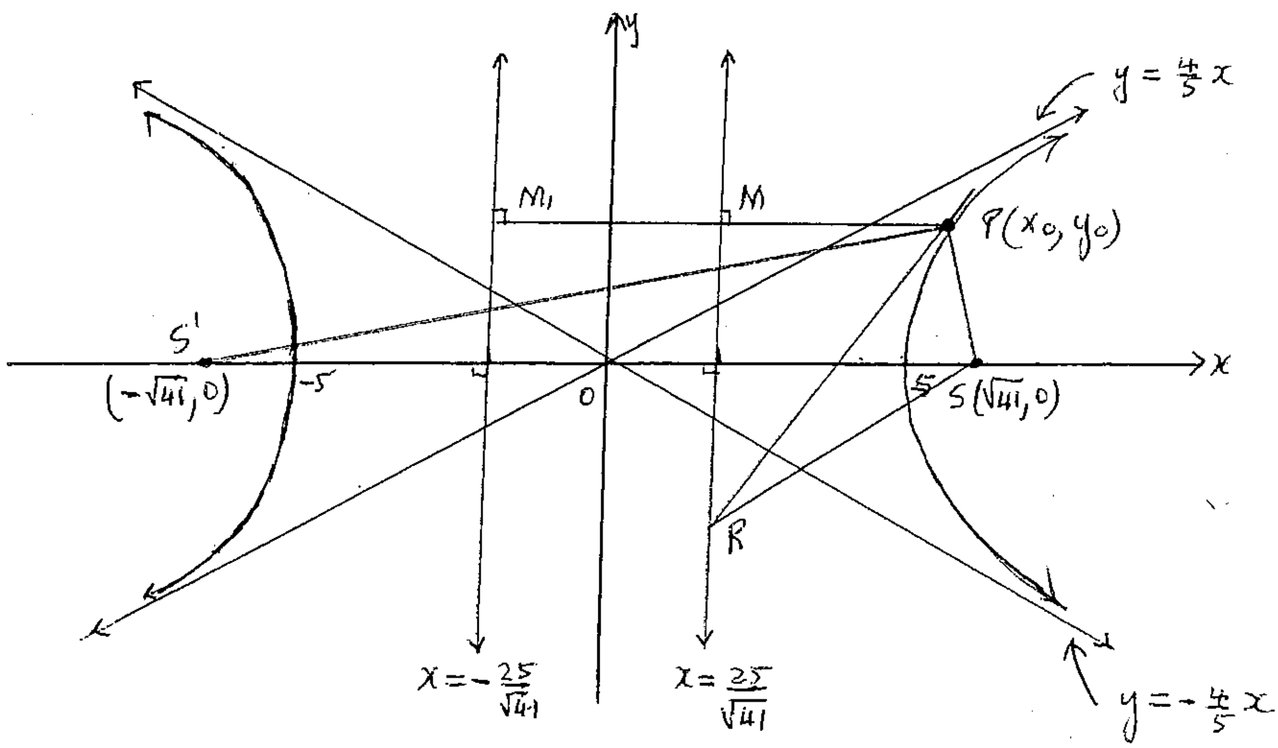
(iii) $y = \pm \frac{b}{a} x$

$y = \pm \frac{4}{5} x$

(iv) $x = \pm \frac{a}{e}$

$x = \pm \frac{25}{\sqrt{41}}$

(b)



(c) $\dfrac{PS}{PM} = e$ and $\dfrac{PS'}{PM'} = e$ This is from the locus definition of a Hyperbola.

$$PS - PS' = ePM - ePM'$$
$$= e(PM - PM')$$
$$|PS - PS'| = \left|\frac{\sqrt{41}}{5}\left[x_0 - \frac{25}{\sqrt{41}} - \left(x_0 + \frac{25}{\sqrt{41}}\right)\right]\right|$$
$$= \left|\frac{\sqrt{41}}{5}\left(x_0 - \frac{25}{\sqrt{41}} - x_0 - \frac{25}{\sqrt{41}}\right)\right|$$
$$= \left|\frac{\sqrt{41}}{5}\left(-\frac{50}{\sqrt{41}}\right)\right|$$
$$= 10$$ ie a constant (This is the length of the major axis).

(d) Equation of tangent:

$$\frac{2x}{25} - \frac{y}{8}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{16x}{25y}$$

Gradient of tangent: $m = \dfrac{16x_0}{25y_0}$

tangent: $y - y_1 = m(x - x_1)$
$$y - y_0 = \frac{16x_0}{25y_0}(x - x_0)$$
$$25y_0y - 25y_0^2 = 16x_0x - 16x_0^2$$
$$16x_0x - 25y_0y = 16x_0^2 - 25y_0^2$$
$$\frac{x_0x}{25} - \frac{y_0y}{16} = \frac{x_0^2}{25} - \frac{y_0^2}{16} \quad \#$$

ie $$\frac{x_0x}{25} - \frac{y_0y}{16} = 1 \quad - - - - \quad (1)$$

\# Note. (x_0, y_0) satisfies the equation of the hyperbola

Hence $\dfrac{x_0^2}{25} - \dfrac{y_0^2}{16} = 1$

(e) (i) Solving ① with $x_R = \frac{25}{\sqrt{41}}$ ---- ② for the co-ordinates of R.

Sub ② in ① $\frac{x_0}{25} \times \frac{25}{\sqrt{41}} - \frac{y_0 y}{16} = 1$

$$\frac{x_0}{\sqrt{41}} - 1 = \frac{y_0 y}{16}$$

$$y_R = \frac{16}{y_0}\left(\frac{x_0}{\sqrt{41}} - 1\right)$$

The co-ordinates of R are given by (x_R, y_R)

(ii) Gradients : $m = \frac{y_2 - y_1}{x_2 - x_1}$

$$m_{PS} = \frac{y_0 - 0}{x_0 - \sqrt{41}} = \frac{y_0}{x_0 - \sqrt{41}}$$

$$m_{SR} = \frac{\frac{16}{y_0}\left(\frac{x_0}{\sqrt{41}} - 1\right) - 0}{\frac{25}{\sqrt{41}} - \sqrt{41}} = \frac{\frac{16}{y_0\sqrt{41}}\left(x_0 - \sqrt{41}\right)}{-\frac{16}{\sqrt{41}}}$$

(iii) For perpendicular lines $m_1 m_2 = -1$

$$m_{PS} \times m_{SR} = \frac{y_0}{x_0 - \sqrt{41}} \times \frac{\sqrt{41}}{-16} \times \frac{16}{y_0\sqrt{41}}\left(x_0 - \sqrt{41}\right)$$

$$= -1$$

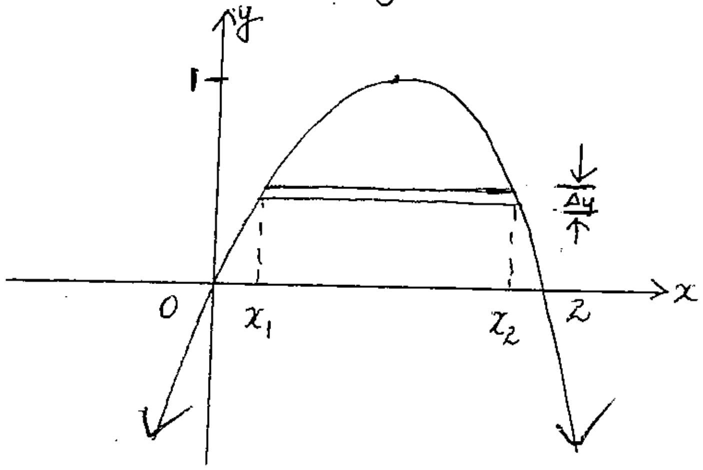
Therefore $PS \perp SR$

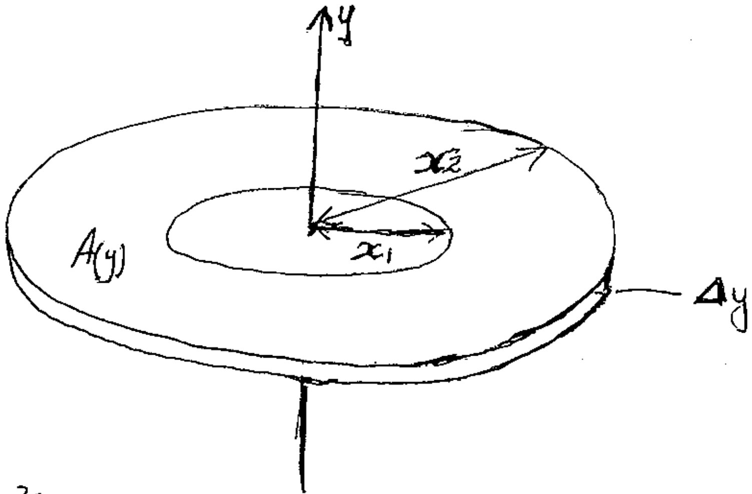
Hence $\widehat{PSR} = 90^\circ$ ie a right angle.

Question 6

(a)

$y = 2x - x^2 = x(2-x)$





$A(y) = \pi x_2^2 - \pi x_1^2$

$\Delta V \doteqdot \pi(x_2^2 - x_1^2)\Delta y.$

$$V = \lim_{\Delta y \to 0} \sum_{y=0}^{y=1} \pi \times 4\sqrt{1-y}\,\Delta y$$

$$V = 4\pi \int_0^1 (1-y)^{\frac{1}{2}}\,dy$$

$$V = 4\pi \left[\frac{(1-y)^{\frac{3}{2}}}{\frac{3}{2}\times -1}\right]_0^1$$

$$V = 4\pi \times -\tfrac{2}{3}\left[0-1\right]$$

$V = \dfrac{8\pi}{3}$ units ^{3}

Note To find x_1 and x_2 solve the quadratic equation in x where 'y' is a fixed value

$$x^2 - 2x + y = 0$$

$$x = \frac{2 \pm \sqrt{4-4y}}{2}$$

$$x = 1 \pm \sqrt{1-y}$$

$x_2 = 1 + \sqrt{1-y}$

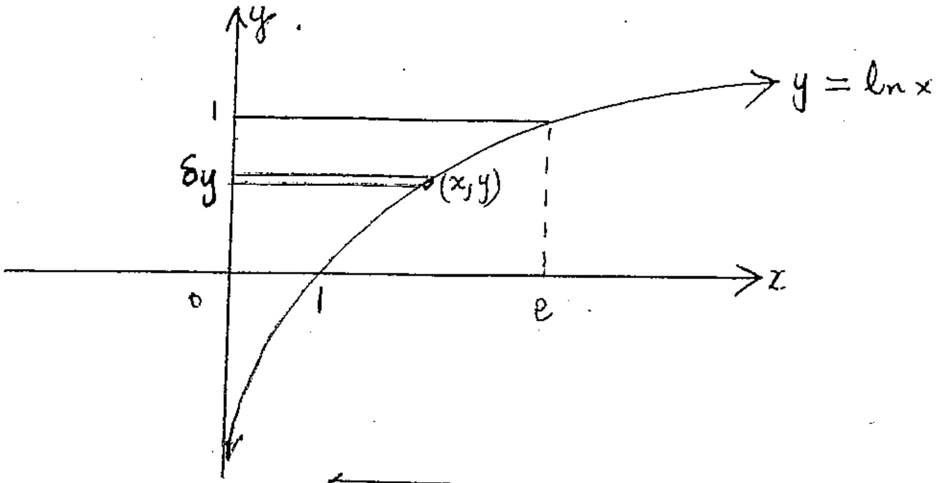
$x_1 = 1 - \sqrt{1-y}.$

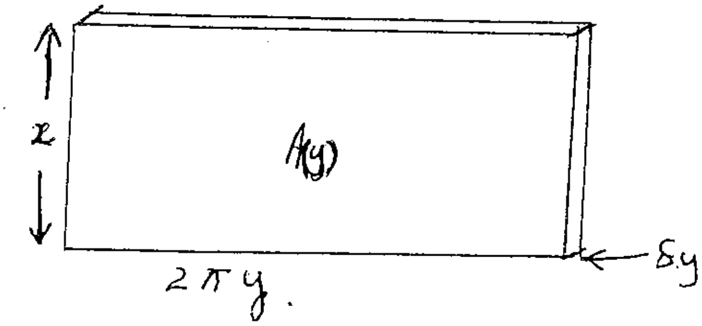
And $x_2^2 - x_1^2 = (x_2 + x_1)(x_2 - x_1)$

$= 2\left(2\sqrt{1-y}\right)$

$= 4\sqrt{1-y}$

(b)





$A(y) = 2\pi y \times x = 2\pi x y$

$V \doteqdot 2\pi x y \,\Delta y.$

$\delta V \doteqdot 2\pi \times e^y \times y\, \delta y$

ie $\delta V \doteqdot 2\pi y e^y \,\delta y$

$$V = \lim_{\delta x \to 0} \sum_{y=0}^{y=1} 2\pi y e^y \,\delta y$$

$$V = 2\pi \int_0^1 y e^y \, dy$$

Now integration by parts with $u = y \Rightarrow u' = 1$
$v' = e^y \Rightarrow v = e^y$

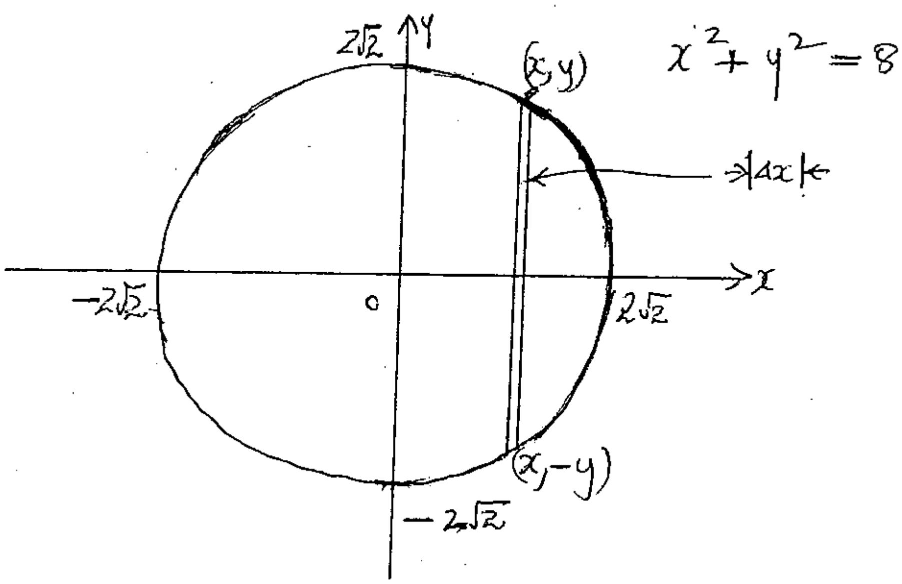
$$V = 2\pi \left\{ \left[y e^y \right]_0^1 - \int_0^1 e^y \, dy \right.$$

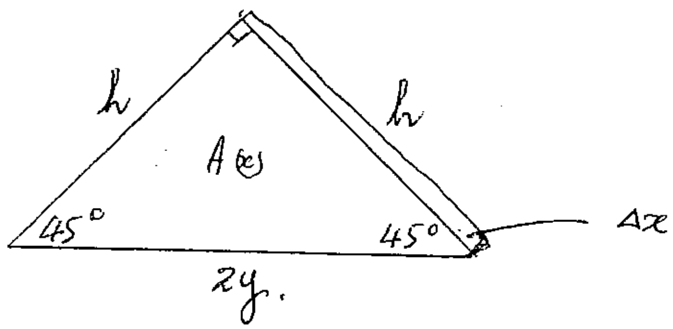
$$V = 2\pi \left\{ (e - 0) - \left[e^y \right]_0^1 \right\}$$

$V = 2\pi(e - e + 1)$

$V = 2\pi$ units^3

(c)





Now $\dfrac{h}{2y} = \cos 45° = \dfrac{1}{\sqrt{2}} \Rightarrow h = \dfrac{2y}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = y\sqrt{2}$.

$$A(x) = \frac{b \times h}{2} = \frac{y\sqrt{2} \times y\sqrt{2}}{2} = y^2$$

$$\Delta V \doteqdot y^2 \Delta x$$

$$\Delta V \doteqdot (8-x^2)\Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=-2\sqrt{2}}^{x=2\sqrt{2}} (8-x^2)\Delta x$$

$$V = 2\int_0^{2\sqrt{2}} (8-x^2)\,dx$$ <u>note</u> even function property.

$$V = 2\left[8x - \frac{x^3}{3}\right]_0^{2\sqrt{2}}$$

$$V = 2\left[16\sqrt{2} - \frac{16\sqrt{2}}{3} - 0\right]$$

$V = \dfrac{64\sqrt{2}}{3}$ units^3

(d) $lx+my+n=0$ ------ ①

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ---- ②

from ① $y=-\frac{lx+n}{m}$

Sub in ②

$$\frac{x^2}{a^2}+\frac{(lx+n)^2}{b^2m^2}=1$$

$$b^2m^2x^2+a^2(l^2x^2+2lnx+n^2)=a^2b^2m^2$$

$$b^2m^2x^2+a^2l^2x^2+2a^2lnx+a^2n^2-a^2b^2m^2=0$$

$$(b^2m^2+a^2l^2)x^2+2a^2lnx+a^2(n^2-b^2m^2)=0$$

In this quadratic equation $\Delta=0$ for tangency

$$(2a^2ln)^2-4(b^2m^2+a^2l^2)\times a^2(n^2-b^2m^2)=0$$

$\div 4a^2$: $a^2l^2n^2-(b^2m^2+a^2l^2)(n^2-b^2m^2)=0$.

$$a^2l^2n^2-(b^2m^2n^2-b^4m^4+a^2l^2n^2-a^2l^2b^2m^2)=0.$$

$$a^2l^2n^2-b^2m^2n^2+b^4m^4-a^2l^2n^2+a^2l^2b^2m^2=0.$$

$$-b^2m^2n^2+b^4m^4+a^2l^2b^2m^2=0$$

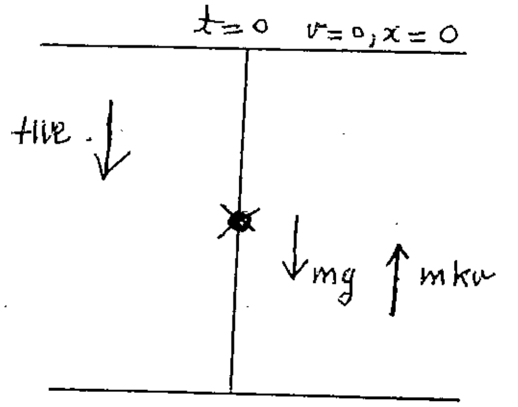
$$-m^2n^2+b^2m^4+a^2l^2m^2=0.$$

$$-n^2+b^2m^2+a^2l^2=0$$

ie $n^2=a^2l^2+b^2m^2$

Question 7

(a)



$F = ma$

$m\ddot{x} = mg - mkv$

$\ddot{x} = g - kv$ - - - ①

(b) For terminal velocity w let $\ddot{x} = 0$

$\therefore g - kw = 0$

$w = \frac{g}{k}$

(c) Now from ①

$\ddot{x} = k\left(\frac{g}{k} - v\right)$

$\ddot{x} = k(w - v)$ - - - ②

(d) In ② let $\ddot{x} = v\frac{dv}{dx}$

$v\frac{dv}{dx} = k(w-v)$

$\frac{dv}{dx} = k\left(\frac{w-v}{v}\right)$

$\frac{dx}{dv} = \frac{1}{k} \times \left(\frac{v}{w-v}\right)$

$\frac{dx}{dv} = \frac{1}{k}\left[-1 - \frac{w}{v-w}\right]$

Note

$\frac{v}{w-v} = -\left(\frac{v}{v-w}\right)$

$= -\left(\frac{v-w+w}{v-w}\right)$

$= -\left(1 + \frac{w}{v-w}\right)$

$= -1 - \frac{w}{v-w}$

(f) Now replacing $\ddot{x} = \frac{dv}{dt}$ in ②

(i) $$\frac{dv}{dt} = k(w - v)$$

$$\frac{dt}{dv} = \frac{1}{k} \times \left(\frac{1}{w-v}\right)$$

Note Several students used substitution here : Let $u = w - v$ etc

$$t = \frac{1}{k} \times -\ln|w-v| + c$$

$$t = -\frac{1}{k}\ln|w-v| + c.$$

Now when $t = 0,\ v = 0$

$$0 = -\frac{1}{k}\ln|w| + c$$

$$c = \frac{1}{k}\ln|w|$$

$$\therefore t = \frac{1}{k}\ln|w| - \frac{1}{k}\ln|w-v|$$

$$t = \frac{1}{k}\ln\left|\frac{w}{w-v}\right| \quad ---\ ④$$

Now 75% of terminal velocity means $v = 0{\cdot}75w$.

$$t = \frac{1}{k}\ln\left|\frac{w}{w - 0{\cdot}75w}\right|$$

Note $0{\cdot}25 = \frac{1}{4}$

$$t = \frac{1}{k}\ln\frac{1}{0{\cdot}25}$$

$$t = \frac{1}{k}\ln\frac{1}{\frac{1}{4}}$$

$$t = \frac{1}{k}\ln 4 \text{ seconds.}$$

ie the time to reach 75% of terminal velocity

$$x = \frac{1}{k}\left[-v - w\ln|v-w|\right] + c = -\frac{v}{k} - \frac{w}{k}\ln|v-w| + c$$

Now $x = 0$ when $v = 0$

$$0 = -\frac{w}{k}\ln|-w| + c.$$

$$c = \frac{w}{k}\ln|-w|$$

$$\therefore x = -\frac{v}{k} - \frac{w}{k}\ln|v-w| + \frac{w}{k}\ln|-w|$$

$$x = -\frac{v}{k} - \frac{w}{k}\left[\ln|v-w| - \ln|-w|\right]$$

$$x = -\frac{v}{k} - \frac{w}{k}\ln\left|\frac{v-w}{-w}\right|$$

$$x = -\frac{v}{k} - \frac{w}{k}\ln\left|\frac{w-v}{w}\right| \quad \text{- - - -} \ 3$$

(e) Now when $x = H$, $v = u$ sub in

$$H = -\frac{u}{k} - \frac{w}{k}\ln\left|\frac{w-u}{w}\right|$$

$$kH = -u - w\ln\left|1 - \frac{u}{w}\right|$$

$$w\ln\left|1 - \frac{u}{w}\right| + u + kH = 0$$

$$\ln\left|1 - \frac{u}{w}\right| + \frac{u}{w} + \frac{kH}{w} = 0$$

(ii) Back to equation ④

$$t = \frac{1}{k}\ln\left|\frac{W}{W-v}\right|$$

$$kt = -\ln\left|\frac{W-v}{W}\right|$$

$$\ln\left|\frac{W-v}{W}\right| = -kt$$

$$\frac{W-v}{W} = e^{-kt}$$

$$W-v = We^{-kt}$$

$$v = W - We^{-kt}.$$

$$\frac{dx}{dt} = W - We^{-kt}$$

$$x = Wt + \frac{W}{k}e^{-kt} + c$$

Now when $x = 0$, $t = 0$

$$0 = 0 + \frac{W}{k}e^{0} + c$$

$$c = -\frac{W}{k}.$$

$$x = Wt + \frac{W}{k}\left(e^{-kt} - 1\right).$$

when $t = \frac{1}{k}\ln 4$

$$x = \frac{W}{k}\ln 4 + \frac{W}{k}\left(e^{-k \times \frac{1}{k}\ln 4} - 1\right)$$

$$x = \frac{W}{k}\left[\ln 4 + e^{-\ln 4} - 1\right]$$

$$x = \frac{W}{k}\left(\ln 4 + e^{\ln 4^{-1}} - 1\right)$$

$$x = \frac{W}{k}\left(\ln 4 + 4^{-1} - 1\right)$$

$$x = \frac{W}{k}\left(\ln 4 - \frac{3}{4}\right) \text{ metres}.$$

This is the distance fallen at 75% of the terminal velocity

(ii) Alternative approach:
Substitute $v = 0.75W$ in equation ③ part (iv)

$$x = -\frac{0.75W}{k} - \frac{W}{k}\ln\left(\frac{W - 0.75W}{W}\right)$$

$$x = -\frac{W}{k}\left[0.75 + \ln\left(\frac{W(1-0.75)}{W}\right)\right]$$

$$x = -\frac{W}{k}\left[\frac{3}{4} + \ln 0.25\right]$$

$$x = -\frac{W}{k}\left(\frac{3}{4} + \ln\frac{1}{4}\right)$$

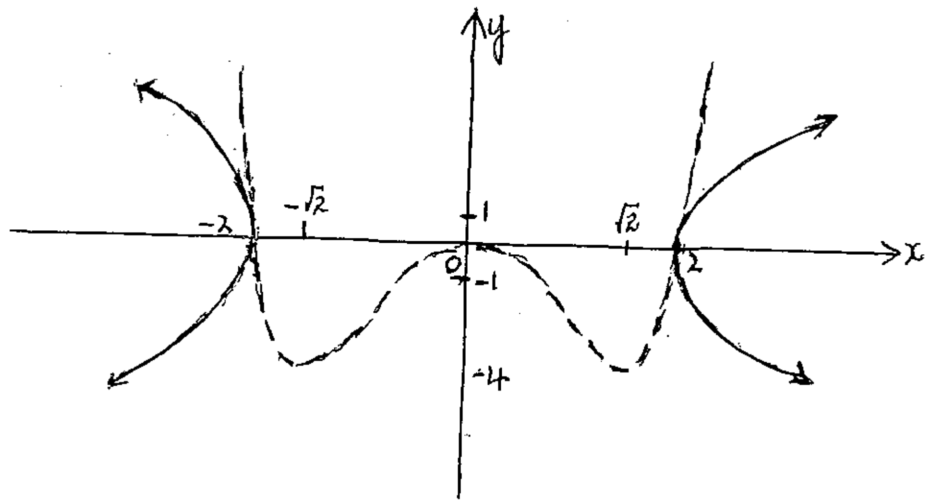
$$x = -\frac{W}{k}\left(\frac{3}{4} + \ln 4^{-1}\right)$$

$$x = -\frac{W}{k}\left(\frac{3}{4} - \ln 4\right)$$

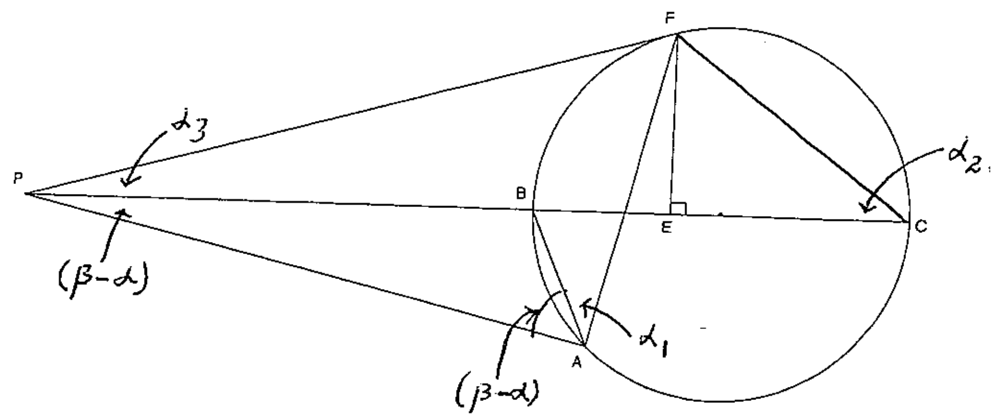
$$x = \frac{W}{k}\left(\ln 4 - \frac{3}{4}\right) \text{ metres.}$$

Question 8

(a) $y^2 = x^2(x^2-4)$ Dotted Graph ; $g(x) = x^2(x^2-4)$



(b)



(i) arc AF = arc FC {given} $\therefore$ $AF = FC$ --- ①

But in ΔPFC, EF perpendicularly bisects base PC.

$\therefore$ ΔPFC is isosceles {converse of The Theorem for an isosceles Δ which states That The altitude is The perpendicular bisector of The base}

$\therefore$ $PF = FC$ ------- ②

from ① and ② $AF = PF$ $\therefore$ ΔAPF is isosceles.

(ii) Now $\alpha_1 = \alpha_2$ {Angles in The same segment Theorem – BF subtends equal angles at A and C}

And $\alpha_2 = \alpha_3$ {base angles of isosceles ΔPFC}

Now since ΔAPF is isosceles

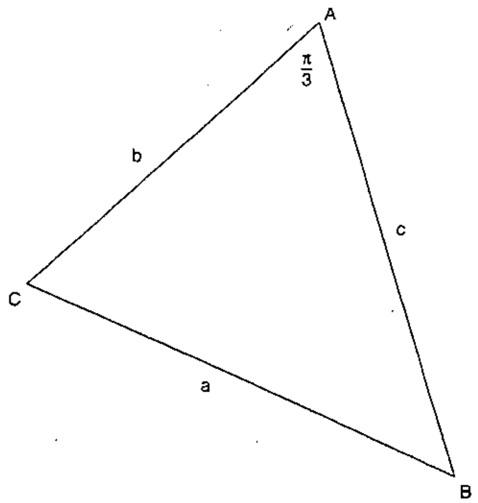
$\underline{\angle FPA} = \underline{\angle FAP} = \beta$, say

$\therefore$ $\underline{\angle BPA} = \underline{\angle BAP} = \beta - \alpha$

$\therefore$ ΔPBA is isosceles and $\therefore$ $AB = BP$ --- ③

Hence, $EC = EP = EB + BP = EB + AB$ (from ③) $= AB + BE$.

(c)



(i)

using the cosine rule :

$$a^2 = b^2 + c^2 - 2bc\cos\frac{\pi}{3}$$

$$\therefore a^2 = b^2 + c^2 - 2bc\left(\tfrac{1}{2}\right)$$

$$\therefore a^2 = b^2 + c^2 - bc$$

Now $a^2 - bc = b^2 + c^2 - 2bc$ (subtracting bc from both sides)

$$\therefore a^2 - bc = (b-c)^2$$

Now $(b-c)^2$ is a perfect square

$\therefore a^2 - bc \geq 0$ (note: equality if $b = c$)

$\therefore a^2 \geq bc$ --- ①

(ii) The area of $\Delta ABC = \frac{1}{2}bc\sin\frac{\pi}{3}$

$$\text{Area} = \frac{1}{2}bc \times \frac{\sqrt{3}}{2}$$

$$\text{Area} = \frac{\sqrt{3}}{4}bc$$

$\text{Area} \leq \frac{\sqrt{3}}{4}a^2$ (Using part (i) ①)