ST IGNATIUS COLLEGE RIVERVIEW

ASSESSMENT TASK 4

TRIAL HSC EXAMINATION

YEAR 12

2008

EXTENSION 2

Time allowed: 3 hours (+ 5 minutes reading time)

Instructions to Candidates

- Attempt all questions.
- There are eight questions. All questions are of equal value.
- All necessary working should be shown. Full marks may not be awarded if work is careless or badly arranged.
- The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.
- Approved calculators may be used. A table of standard integrals is provided.

- **Each question is to be started in a new booklet. Your number should be written clearly on the cover of each booklet.**

Question 1 [15 Marks] **Start a new answer booklet.**

(a) Find the following integrals :

(i) $\int \cos^{-1} x \, dx$ [2]

(ii) $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x \, dx$ [2]

(b) (i) Express $\dfrac{25}{(x+2)(2x-1)^2}$ in partial fractions [3]

(ii) Hence show that $\displaystyle\int_1^2 \frac{25}{(x+2)(2x-1)^2} = \frac{10}{3} - 2\ln\frac{3}{2}$ [2]

(c) (i) Using the substitution $x = a - t$, or otherwise,

prove that $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$ [2]

(ii) Hence, or otherwise, show that $\displaystyle\int_0^{\frac{\pi}{2}} x(\tfrac{\pi}{2} - x)\sin^2 x \, dx = \frac{\pi^3}{96}$ [4]

Question 2 [15 Marks] Start a new answer booklet.

(a) If $z = \dfrac{2-i}{1+i}$, where $z = x + iy$, find $\bar{z}$ in the form $(a + bi)$ [2]

(b) Find the square root of $(21 + 20i)$ in the form $(a + bi)$ [3]

(c) (i) Sketch the locus of $|z + 1 + i| \le 1$, where $z = x + iy$ [2]

(ii) Find the maximum and minimum values of $|z|$ in part (i) [2]

(d) (i) The complex number $z = x + iy$ is represented by the point P. [3]

If $\dfrac{z-1}{z-2i}$ is purely imaginary, show that the locus of P is a circle, excluding two points.

(ii) State the centre and the radius of this circle. [1]

(iii) Give the co-ordinates of the two excluded points and the reason for their exclusion. [2]

Question 3 [15 Marks] Start a new answer booklet.

(a) Sketch graphs (on separate number planes) of the following relations, without the use of calculus.

Each graph should be labelled clearly.

(i) $y=(x-1)(x+1)$ [1]

(ii) $y=|x-1|(x+1)$ [2]

(iii) $y=\dfrac{1}{(x-1)(x+1)}$ [2]

(iv) $y=\sqrt{(x-1)(x+1)}$ [2]

(v) $y=e^{(x-1)(x+1)}$ [2]

(vi) $y=\log_e(x-1)(x+1)$ [2]

(b) (i) Sketch on the same number plane $y=|x|-2$ and $y=4+3x-x^2$ [1]

(ii) Hence, or otherwise, solve $\dfrac{|x|-2}{4+3x-x^2}>0$ [3]

Question 4 **[15 Marks]** **Start a new answer booklet.**

(a) Write down the co-ordinates of the vertices and the foci for the hyperbola $xy = 2$ [3]

(b) P is a point $(a\sec\theta, b\tan\theta)$ which lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, with centre 0.

The tangent P meets the asymptote $y = \frac{b}{a}x$ at Q and the other asymptote at R.

The normal at P meets OQ at K

(i) Represent the above data with a suitable diagram [1]

(ii) Derive the equation of the tangent at P [2]

(iii) Prove that the co-ordinates of Q are [2]

$(a[\sec\theta + \tan\theta], b[\sec\theta + \tan\theta])$

(iv) If the co-ordinates of R are $(a[\sec\theta - \tan\theta], b[\tan\theta - \sec\theta])$ [1]
Prove that P is the midpoint of QR

(v) (α) If P is equidistant from Q, R and O, prove that the hyperbola is rectangular [2]

(β) Hence, prove that $Q\hat{K}P = P\hat{O}R$ [4]

Question 5 [15 Marks] Start a new answer booklet.

(a) Consider the polynomial $P(x) = x^4 - 2x^3 + 2x - 1$

(i) Show that $P(x) = 0$ has a multiple zero and state its value and multiplicity. [3]

(ii) Hence, fully factorise $P(x)$ [2]

(b) Consider the polynomial $f(x) = ax^3 + bx^2 + cx + d$ where *a*, *b*, *c* and *d* are real. [5]

Given that two of the roots of $f(x) = 0$ are $(1-2i)$ and -2, and that

$f(-1) = -8$, find *a*, *b*, *c*, and *d*.

(c) If α, β and γ are the roots of the equation $x^3 + 2x^2 - 3x + 4 = 0$ find [5]

A cubic equation whose roots are $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$.

Question 6 [15 Marks] **Start a new answer booklet.**

(a) A solid has its base the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$ [4]

If each section perpendicular to the major axis is an equilateral triangle,

show that the volume of the solid is $128\sqrt{3}$ cubic units.

(b) The region bounded by the curve $y=\log_e x$, the straight line $x=e$ and the x-axis is rotated about the straight line $x=e$. By taking slices parallel to the x-axis, find the exact volume generated. [5]

(c) Find the exact volume generated, by rotating the area bound by the curves [6]

$y=(x-1)^2$ and $y=x+1$, about the y-axis using the method of

cylindrical shells.

Question 7 **[15 Marks]** **Start a new answer booklet.**

(a)

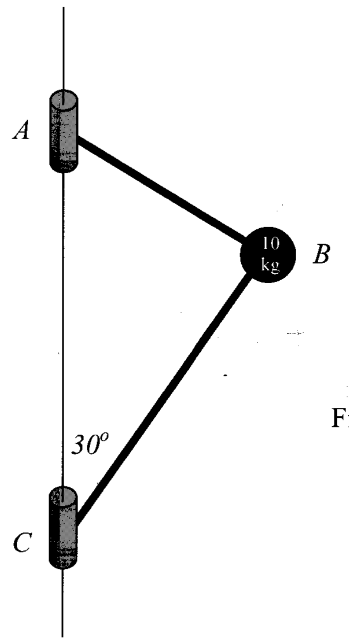


Figure not to scale

The above diagram shows a mass of 10 kilograms at B connected by light rods (at right angles) to sleeves A and C which revolve freely about the vertical axis AC but do not move vertically. The angle between the vertical axis AC and the light rod BC is 30^o. The acceleration due to gravity is g metres per second squared.

(i) Given AC is 2 metres, show that the radius of the circular path of rotation of B is $\dfrac{\sqrt{3}}{2}$ metres. [1]

(ii) Find the tensions in the rods AB and BC when the mass makes 90 revolutions per minute about the vertical axis. [5]

Question 7 Continued

(b) A particle P of mass m kg projected vertically upward with an initial velocity u metres per second is subjected to forces which create a constant vertical downward acceleration of magnitude g metres per second squared and an acceleration directed against the motion of magnitude kv when the speed is v metres per second squared. K is a constant

(i) Show, with the aid of a diagram, that the acceleration function Is given by $\ddot{x} = -g - kv$ [2]

(ii) Prove that the maximum height reached by the particle after time T is given by $T = \frac{1}{k}\log_e\left|\frac{g+ku}{g}\right|$ [3]

(ii) Prove that the maximum height is $\frac{1}{k}(u - gT)$ [4]

Question 8 [15 Marks] Start a new answer booklet.

(a) (i) Prove by Mathematical Induction that if n is a positive integer, then $2^{(n+4)} > (n+4)^2$ [4]

(ii) By choosing a suitable substitution, or otherwise, show that if a is a positive integer, then $2^{3(a+2)} > 9(a+2)^2$ [2]

(b) (i) Write down the formula for $\tan(A+B)$ in terms of $\tan A$ and $\tan B$ [1]

(ii) Prove that $\tan(2\tan^{-1} x) = 2\tan(\tan^{-1} x + \tan^{-1} x^3)$ [3]

(c) Consider the curve C in the x-y plane defined by $\sqrt{|x|} + \sqrt{y} = 1$

(i) Write down the domain for C [1]

(ii) For $x > 0$, show that $\frac{dy}{dx} < 0$ [2]

(iii) Sketch a graph of C, paying close attention to the gradient of the curve at $x = 0$ [2]

STANDARD INTEGRALS

$\int x^n \, dx$	$= \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; n \neq 0, \text{ if } n < 0$
$\int \frac{1}{x} dx$	$= \ln x, \; x < 0$
$\int e^{ax} \, dx$	$= \frac{1}{a} e^{ax}, \; a \neq 0$
$\int \cos ax \, dx$	$= \frac{1}{a} \sin ax, \; a \neq 0$
$\int \sin ax \, dx$	$= -\frac{1}{a} \cos ax, \; a \neq 0$
$\int \sec^2 ax \, dx$	$= \frac{1}{a} \tan ax, \; a \neq 0$
$\int \sec ax \tan ax \, dx$	$= \frac{1}{a} \sec ax, \; a \neq 0$
$\int \frac{1}{a^2 + x^2} dx$	$= \frac{1}{a} \tan^{-1} ax \frac{x}{a}, \; a \neq 0$
$\int \frac{1}{\sqrt{a^2 - x^2}} dx$	$= \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$
$\int \frac{1}{\sqrt{x^2 - a^2}} dx$	$= \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$
$\int \frac{1}{\sqrt{x^2 + a^2}} dx$	$= \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE: $\ln x = \log_e x, \; x > 0$

Question 1

(a) (i) $I = \int \cos^{-1}x \, dx$

let $u = \cos^{-1}x \Rightarrow u' = -\dfrac{1}{\sqrt{1-x^2}}$

$v' = 1 \Rightarrow v = x$

$I = (\cos^{-1}x) \times x - \int -\dfrac{x}{\sqrt{1-x^2}} dx$ ← This integral can be done by substitution,

$\underline{\underline{I = x\cos^{-1}x - \sqrt{1-x^2} + c}}$

(ii) $\displaystyle\int_0^{\pi/2} \sin^3 x \cos^2 x \, dx$.

$I = \displaystyle\int_0^{\pi/2} \sin^2 x \sin x \cos^2 x \, dx$

$I = \displaystyle\int_0^{\pi/2} (1-\cos^2 x)\cos^2 x \sin x \, dx$

Note $\sin^2 x = 1 - \cos^2 x$

$I = -\displaystyle\int_1^0 (1-u^2)u^2 \, du$

let $u = \cos x$

$\therefore du = -\sin x \, dx$

when $x = \frac{\pi}{2} \Rightarrow u = 0$

$x = 0 \Rightarrow u = 1$

$I = \displaystyle\int_0^1 (u^2 - u^4) \, du$

$= \left[\dfrac{u^3}{3} - \dfrac{u^5}{5}\right]_0^1$

$= \left(\frac{1}{3} - \frac{1}{5}\right) - 0$

$= \underline{\underline{\dfrac{2}{15}}}$

ling

Question 1

(b) (i) Let $\dfrac{25}{(x+2)(2x-1)^2} \equiv \dfrac{a}{x+2} + \dfrac{10}{2x-1} + \dfrac{c}{(2x-1)^2}$

$\therefore\ 25 \equiv a(2x-1)^2 + b(x+2)(2x-1) + c(x+2)$

$25 \equiv x^2(4a+2b) + x(3b+c-4a) + (2c-2b-a)$

Equating co-efficients.

$4a+2b=0 \Rightarrow 2a+b=0$ --- ①

$3b+c-4a=0$ --- ②

$2c-2b+a=25$ --- ③

Solving from ① $b=-2a$

Sub. in ② and ③

$c-10a=0$ --- (2a)

$2c+5a\ 25$ --- (3a)

$20a+5a=25$

$a=1$

$\left.\begin{array}{l} a=1 \\ \therefore b=-2 \\ c=10 \end{array}\right\}$

(ii) $I = \displaystyle\int_1^2 \frac{25}{(x+2)(2x-1)^2}\,dx$

$= \displaystyle\int_1^2 \left[\frac{1}{x+2} - \frac{2}{2x-1} + \frac{10}{(2x-1)^2}\right]dx$

$= \Big[\ln|x+2| - \ln|2x-1| - 5(2x-1)^{-1}\Big]_1^2$

$= \left[\ln 4 - \ln 3 - \frac{5}{3} - (\ln 3 - \ln 1 - 5)\right]$

$= \ln 4 - \ln 3 - \frac{5}{3} - \ln 3 + 0 + 5$

$= \frac{10}{3} - 2\ln 2 - 2\ln 3 = \underline{\underline{\frac{10}{3} - 2\ln\left(\frac{3}{2}\right)}}$

Question 1

(c) (i)

$$I=\int_0^a f(x)\,dx$$

let $x=a-t$
$dx=-dt$
when $x=a,\ t=0$
$x=0\ \ t=a.$

$$=-\int_a^0 f(a-t)\,dt$$

$$=\int_0^a f(a-t)\,dt$$

$$=\int_0^a f(a-x)\,dx$$

let $x=t$
$\therefore dx=dt.$

(ii) $$I\int_0^{\frac{\pi}{2}} x\left(\tfrac{\pi}{2}-x\right)\sin^2 x\,dx \quad ----[A]$$

$$=\int_0^{\pi/2}\left(\tfrac{\pi}{2}-x\right)\left[\tfrac{\pi}{2}-\left(\tfrac{\pi}{2}-x\right)\right]\sin^2\left(\tfrac{\pi}{2}-x\right)dx$$

$$I=\int_0^{\pi/2}\left(\tfrac{\pi}{2}-x\right)x\cos^2 x\,dx \quad ----[B]$$

$$2I = A+B$$

$$=\int_0^{\pi/2} x\left(\tfrac{\pi}{2}-x\right)\sin^2 x\,dx+\int_0 x\left(\tfrac{\pi}{2}-x\right)\cos^2 x\,dx$$

$$=\int_0^{\pi/2} x\left(\tfrac{\pi}{2}-x\right)\left[\sin^2 x+\cos^2 x\right]dx$$

Note $\cos^2 x+\sin^2 x=1$

$$2I=\int_0^{\pi/2} x\left(\tfrac{\pi}{2}-x\right)dx=\int_0^{\pi/2}\left(\tfrac{\pi x}{2}-x^2\right)dx$$

$$2I=\left[\frac{\pi^2 x}{4}-\frac{x^3}{3}\right]_0^{\pi/2}=\frac{\pi^3}{16}-\frac{\pi^3}{24}=\frac{\pi^3}{48}$$

$$I=\frac{\pi^3}{96}$$

Question 2

(a) $Z = \frac{2-i}{1+i} \times \frac{1-i}{1-i} = \frac{2-3i+i^2}{1-i^2} = \frac{2-3i-1}{1+1} = \frac{1-3i}{2}$

$\overline{Z} = \frac{1}{2} + \frac{3}{2}i$

(b) let $\sqrt{21+20i} = a+bi$ where a and b are real. #

square both sides

$21+20i = a^2+2abi+b^2i^2$

$21+20i = a^2-b^2+2abi$

equate real and imaginary parts.

$a^2-b^2=21$ ---- ①

$ab = 10$ ---- ②

Sub $b=\frac{10}{a}$ in ①

$a^2 - \frac{100}{a^2} = 21$

$a^4-21a^2-100=0$

$(a^2-25)(a^2+4)=0$

$a^2-25=0$ only #

$a=\pm 5$

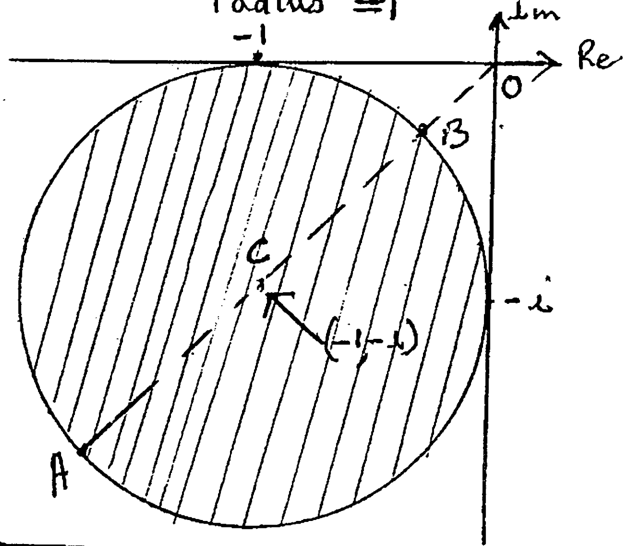
when $a=5$, $b=2$

$a=-5$, $b=-2$

Square roots are $(5+2i)$ and $(-5-2i)$

(c)(i) $|Z-(-1-i)|\le 1$

Circle Centre $(-1,-1)$
radius $=1$



(ii) Maximum value of $|Z| = OA$

$OA = OB+AB$

$(\sqrt{2}-1)+2$

$=\sqrt{2}+1$

Minimum value of $|Z| = OB$

$OB = OC - BC$

$=\sqrt{2}-1$

Question 2

(d)(i) If $z = x + iy$

$$\frac{z-1}{z-2i} = \frac{x+iy-1}{x+iy-2i} = \frac{(x-1)+iy}{x+(y-2)i} \times \frac{x-(y-2)i}{x-(y-2)i}$$

$$= \frac{x(x-1)-y(y-2)i^2 + (x-1)(y-2)i + xyi}{x^2-(y-2)^2 i^2}$$

$$= \frac{x(x-1)+y(y-2)+(xy-xy+2x+y-2)i}{x^2+(y-2)^2}$$

But $\frac{z-1}{z-2i}$ is purely imaginary.

Hence $\frac{x(x-1)+y(y-2)}{x^2+(y-2)^2} = 0$ ie the real part is zero.

$$x^2 - x + y^2 - 2y = 0.$$

$$x^2 - x + \tfrac{1}{4} + y^2 - 2y + 1 = 1 + \tfrac{1}{4}$$

$$(x-\tfrac{1}{2})^2 + (y-1)^2 = \tfrac{5}{4} = \left(\tfrac{\sqrt{5}}{2}\right)^2.$$

(ii) This is a circle with centre $(\frac{1}{2}, -1)$ and radius $\frac{\sqrt{5}}{2}$ units

(iii) Now $\left(\frac{z-1}{z-2i}\right)$ is undefined if z is $\underline{\underline{(0,2)}}$ ie $(0+2i)$

and for $\underline{\underline{(1,0)}}$, $\left(\frac{z-1}{z-2i}\right)$ would be zero & $\in$ Reals.
which does not meet the stated condition.

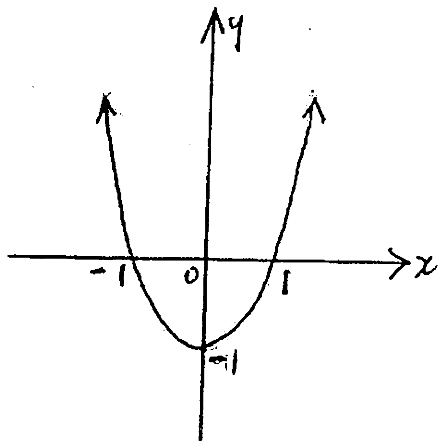
Note An alternative argument can be used by considering

$$\arg\left(\frac{z-1}{z-2i}\right) = \arg(z-1) - \arg(z-2i) = \pm\frac{\pi}{2}$$

The locus is a semi circle above and below.
The join of $P(1,0)$ and $Q(0,2)$ excluding P and Q.

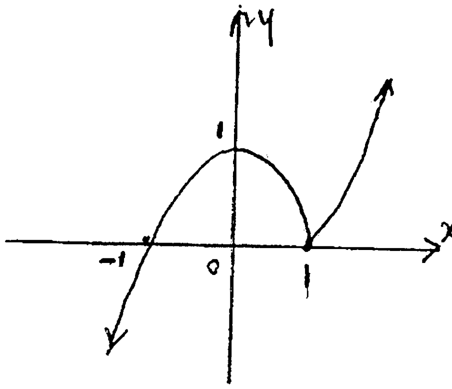
Question 3

(a) (i)



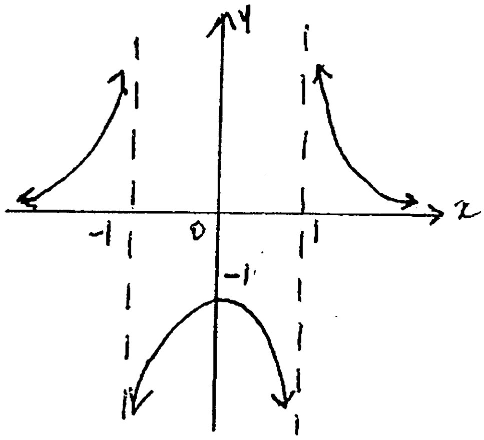
$y = (x-1)(x+1)$

(ii)



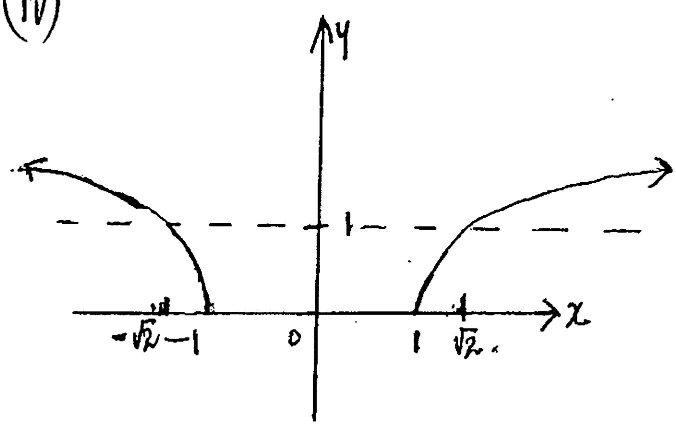
$y = |x-1|(x+1)$

(iii)



$y = \dfrac{1}{(x-1)(x+1)}$

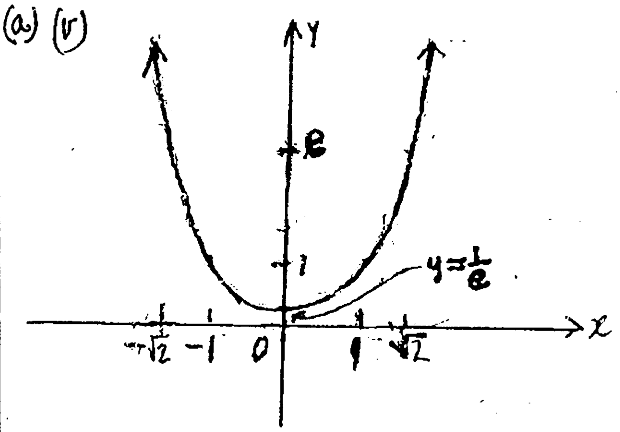
(iv)



$y = \sqrt{(x-1)(x+1)}$

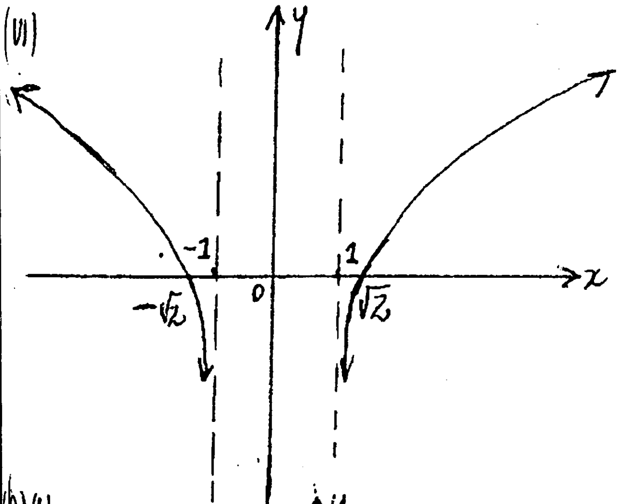
Question 3

(a) (v)



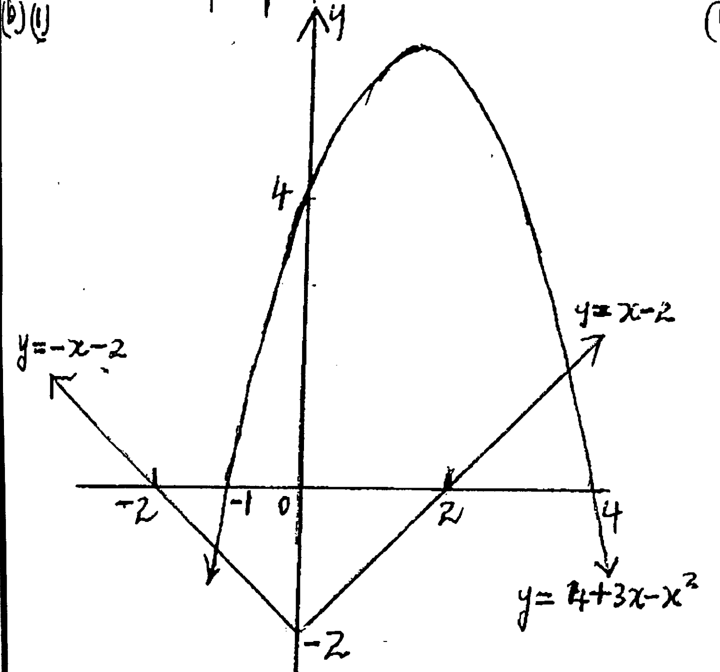
$$y = e^{(x-1)(x+1)}$$

(vi)



$$y = \log_e (x-1)(x+1)$$

(b) (i)



$y = 4+3x-x^2$

$y = -(x^2-3x-4)$

$y = -(x-4)(x+1)$

(ii) Now for $\frac{|x|-2}{4+3x-x^2} > 0$

$y = |x|-2$ and $y = 4+3x-x^2$ must have the same sign

Both are positive for $\underline{\underline{2<x<4}}$

Both are negative for $\underline{\underline{-2<x<-1}}$

Hence the solution for the inequality will contain both of these sets.

Question 4

(a) $xy=2$

Eccentricity $e=\sqrt{2}$

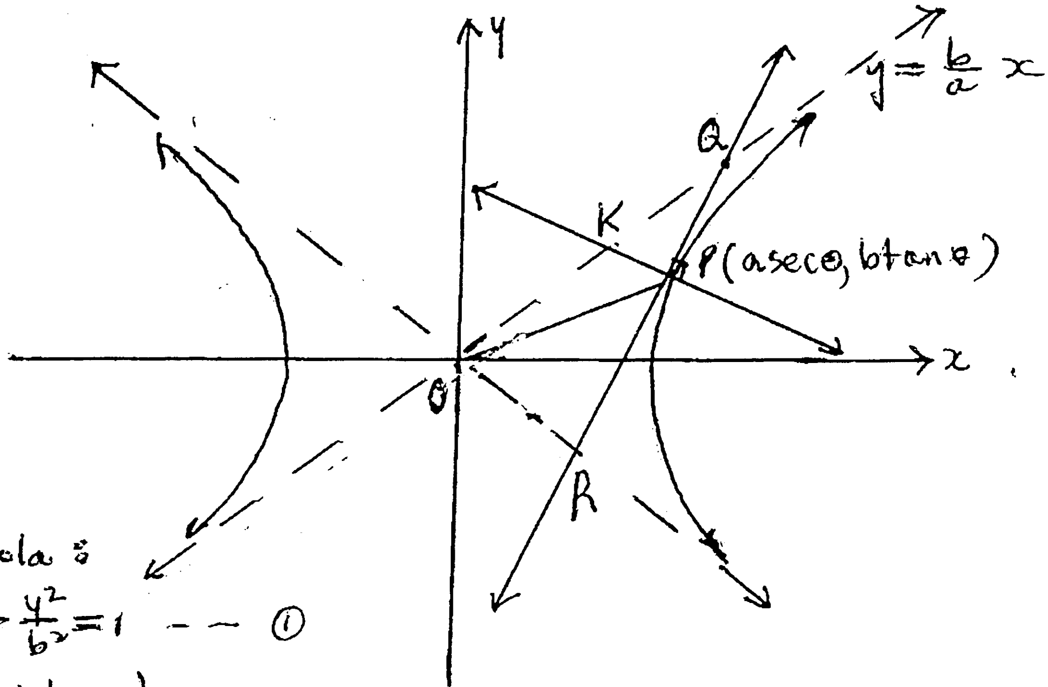
Foci are $(c\sqrt{2}, c\sqrt{2})$ and $(-c\sqrt{2}, -c\sqrt{2})$; Now $c=\sqrt{2}$

$\therefore$ foci are $(2,2)$ and $(-2,-2)$

Vertices (c,c) and $(-c,-c)$

When $c=2$ $(\sqrt{2},\sqrt{2})$ and $(-\sqrt{2},\sqrt{2})$

(b) (i)



Hyperbola :

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ --- ①

$P(a\sec\theta, b\tan\theta)$

Asymptote OKQ : $y=\frac{b}{a}x$ --- ②

(ii) tangent at P : $\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$

$$\frac{dy}{dx}=\frac{b^2x^2}{a^2y^2}$$

Gradient of tangent $m_1=\frac{b^2 a\sec\theta}{a^2\times b\tan\theta}=\frac{b\sec\theta}{a\tan\theta}$

Tangent : $y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)$

$ay\tan\theta-ab\tan^2\theta=bx\sec\theta-ab\sec^2\theta$

$bx\sec\theta-ay\tan\theta=ab\sec^2\theta-ab\tan^2\theta$

Question 4(b)

(ii) $bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta)$

$\div ab$: $\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$ ---③ Note $\sec^2\theta - \tan^2\theta = 1$

(iii) Solving ② and ③

sub ② in ③

$$\frac{x\sec\theta}{a} - \frac{b}{a}\left(\frac{\tan\theta}{b}\right)x = 1$$

$$\frac{x\sec\theta}{a} - \frac{x\tan\theta}{a} = 1$$

$$\frac{x}{a}(\sec\theta - \tan\theta) = 1$$

$$x = \frac{a}{\sec\theta - \tan\theta} \times \left(\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\right)$$

$$x = a(\sec\theta + \tan\theta)$$

Sub in ② $y = \dfrac{b}{a} \times a(\sec\theta + \tan\theta) = b(\sec\theta + \tan\theta)$,

$\therefore$ Q is $(a[\sec\theta + \tan\theta], b[\sec\theta + \tan\theta])$

(iv) For the co-ordinates of the midpoint of QR.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$M = \left(\frac{a[\sec\theta + \tan\theta] + a(\sec\theta - \tan\theta)}{2}, \frac{b(\sec\theta + \tan\theta) + b(\tan\theta - \sec\theta)}{2}\right)$$

$$M = (a\sec\theta, b\tan\theta)$$

Now this is P.

Question 4

(b) (v) Consider $OP = PQ$

$$\text{ie } OP^2 = PQ^2$$

using $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$$(a\sec\theta)^2 + (b\tan\theta)^2 = (a\sec\theta - a(\sec\theta + \tan\theta))^2 + (b\tan\theta - b(\sec\theta + \tan\theta))^2$$

$$a^2\sec^2\theta + b^2\tan^2\theta = a^2\tan^2\theta + b^2\sec^2\theta.$$

$$a^2\sec^2\theta - a^2\tan^2\theta = b^2\sec^2\theta - b^2\tan^2\theta$$

$$a^2(\sec^2\theta - \tan^2\theta) = b^2(\sec^2\theta - \tan^2\theta)$$

$$\underline{\underline{a^2 = b^2}}$$

which is the condition for a rectangular hyperbola.

(iv) $\widehat{KOR} = 90°$ (rectangular hyperbola ⇒ asymptotes ⊥).
$\widehat{KPR} = 90°$ (tangent ⊥ Normal)

∴ ORPK is a cyclic Quadrilateral.

$\widehat{QKP} = \widehat{PRO}$ (External ∠ of a cyclic Quadrilateral equals interior opposite angle)

But $\widehat{PRO} = \widehat{POR}$ (Isosceles Δ PRO, $PR = OR$)

∴ $\underline{\underline{\widehat{QKP} = \widehat{POR}}}$

Question 5

(a) (i) $P(x) = x^4 - 2x^3 + 2x - 1$

$P'(x) = 4x^3 - 6x^2 + 2$.

$P''(x) = 12x^2 - 12x$

$= 12x(x-1)$

When $P''(x) = 0$

$x = 0$ or 1

Consider $x = 1$

$P'(1) = 4 - 6 + 2 = 0$

$P(1) = 1 - 2 + 2 - 1 = 0$

Now $P(1) = P'(1) = P''(1)$

Hence $x = 1$ is a root of multiplicity 3 for $P(x) = 0$

(ii) Now $(x-1)^3$ is a factor of $P(x)$

Now consider $x = -1$ in $P(x)$

$P(-1) = (-1)^4 - 2(-1)^3 + 2(-1) - 1$

$= 1 + 2 - 2 - 1 = 0$

$\therefore (x+1)$ is a factor.

Hence $P(x) = (x-1)^3(x+1)$.

Question 5

(b) $f(x) = ax^3 + bx^2 + cx + d$.

Considering $f(x) = 0$

Now one root is $(1-2i)$ and its conjugate $(1+2i)$ is a root because the co-efficients of $f(x)$ are real.

So the 3 roots of $f(x) = 0$ are $(1 \pm 2i)$ and 2.

If $f(-1) = -8$ Then

$$-a + b - c + d = -8$$

$$\text{ie } a - b + c - d = 8 \quad \text{--- --- ---} \quad (1)$$

Sum of roots : $\alpha + \beta + \gamma = -\frac{b}{a}$.

$$1 + 2i + 1 - 2i - 2 = -\frac{b}{a}$$

$$0 = -\frac{b}{a}.$$

$$\boxed{b = 0} \quad \text{--- --- ---} \quad (2)$$

Product of roots : $\alpha\beta\gamma = -\frac{d}{a}$

$$(1-2i)(1+2i)(-2) = -\frac{d}{a}.$$

$$-2(1-2i^2) = \frac{d}{a}$$

$$d = 10a \quad \text{--- --- --- ---} \quad (3)$$

Sum of roots 2 at a time : $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$.

$$(1+2i)(1-2i) + (1+2i)\times -2 + (1-2i)\times -2 = \frac{c}{a}$$

$$5 + -2 - 4i - 2 + 4i = \frac{c}{a}$$

$$1 = \frac{c}{a}$$

$$c = a \quad \text{--- --- ---} \quad (4)$$

sub (2), (3), (4) in (1) $\quad a + 0 + a - 10a = 8$

$$a = -1, \quad c = -1, \quad d = -10.$$

Question 5

(c) $x^3+2x^2-3x+4=0$ ----- [A]

roots α, β, γ

$\alpha+\beta+\gamma=-2$ --- ①

$\alpha\beta+\alpha\gamma+\beta\gamma=-3$ --- ②

$\alpha\beta\gamma=-4$ --- ③

Now $\alpha\beta+\beta\gamma+\gamma\alpha=$ sum for required equation.

$\alpha\beta\times\beta\gamma+\alpha\beta\times\gamma\alpha+\beta\gamma\times\gamma\alpha=\alpha\gamma\beta^2+\beta\gamma\alpha^2+\alpha\beta\gamma^2$

$=\alpha\beta\gamma(\alpha+\beta+\gamma)=-4\times-2=8$

This is the sum of roots taken 2 at a time for the required equation.

$\alpha\beta\times\beta\gamma\times\gamma\alpha=\alpha^2\beta^2\gamma^2=(\alpha\beta\gamma)^2=(-4)^2=16$.

This is the product of the roots for the required equation.

Required equation:

$x^3-\begin{pmatrix}\text{Sum of}\\ \text{roots}\end{pmatrix}x^2+\begin{pmatrix}\text{sum of}\\ \text{roots 2 at}\\ \text{a time}\end{pmatrix}x-\begin{pmatrix}\text{product of}\\ \text{roots}\end{pmatrix}=0$

ie $x^3+3x^2+8x-16=0$

Note A more elegant approach, try: $\alpha\beta\gamma\left(\frac{1}{\gamma}\right); \alpha\beta\gamma\left(\frac{1}{\beta}\right); \alpha\beta\gamma\left(\frac{1}{\alpha}\right)$

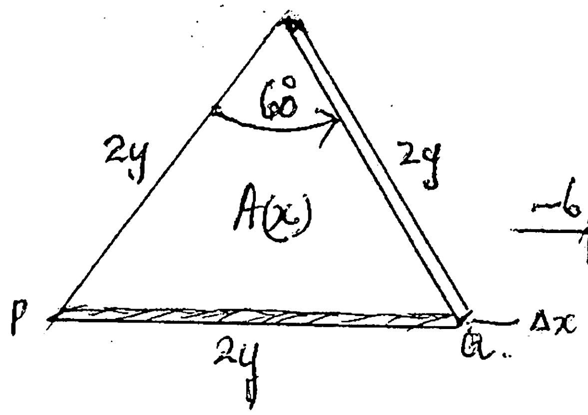
$\Rightarrow -\frac{4}{\gamma}; -\frac{4}{\beta}, -\frac{4}{\alpha}$

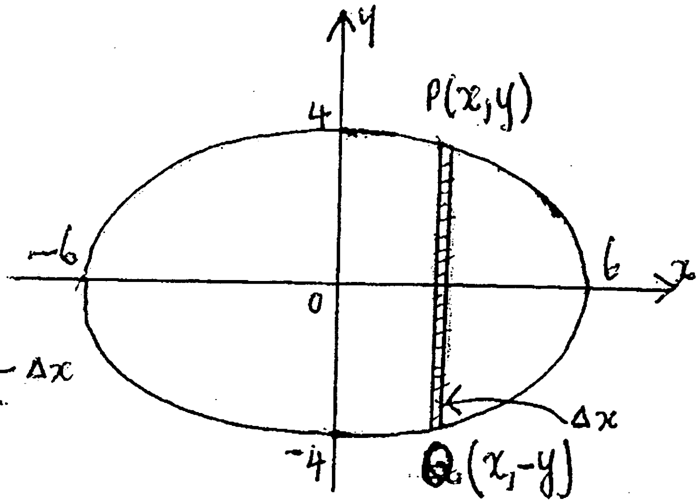
and use the transformation: $y=-\frac{4}{x}$

Hence sub $x=-\frac{4}{y}$ in [A] above etc.

Question 6

(a) $\dfrac{x^2}{36} + \dfrac{y^2}{16} = 1$





$A(x) = \frac{1}{2} \times 2y \times 2y \times \sin 60°$

$= 2y^2 \times \frac{\sqrt{3}}{2}$

$= y^2\sqrt{3}$

$A(x) = 16\left(1 - \frac{x^2}{36}\right)\sqrt{3}$

$\Delta V \doteqdot 16\sqrt{3}\left(1 - \frac{x^2}{36}\right)\Delta x$

$$V = \lim_{\Delta x \to 0} \sum_{x=-6}^{x=6} 16\sqrt{3}\left(1 - \frac{x^2}{36}\right)\Delta x$$

$$V = 16\sqrt{3}\int_{-6}^{6}\left(1 - \frac{x^2}{36}\right)dx$$

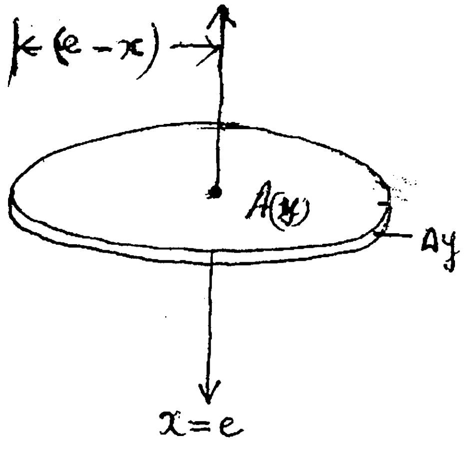
$$V = 32\sqrt{3}\int_{0}^{6}\left(1 - \frac{x^2}{36}\right)dx$$

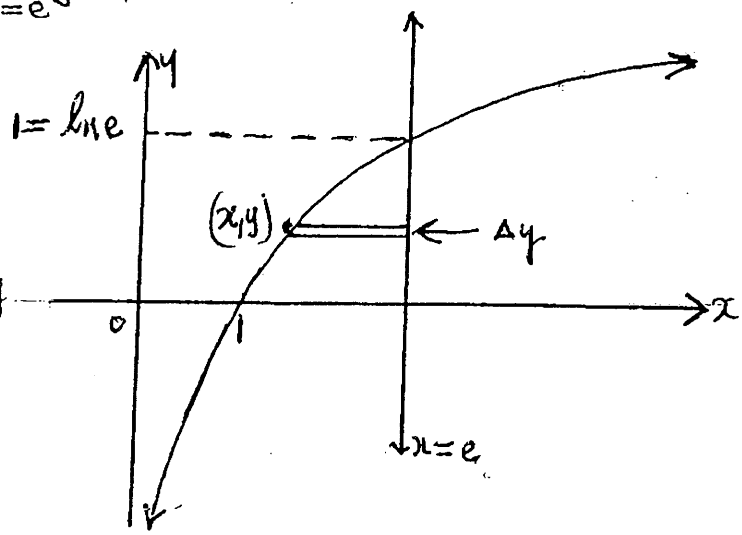
$$= 32\sqrt{3}\left[x - \frac{x^3}{108}\right]_0^6 = 32\sqrt{3}\left[6 - \frac{216}{108}\right] = 32\sqrt{3} \times 4$$

$= \underline{\underline{128\sqrt{3}\text{ units}^3}}$

Question 6

(b) $y = \ln x \Rightarrow x = e^y$





$$A(y) = \pi(e-x)^2$$

$$\Delta V \doteqdot \pi(e-x)^2 \Delta y$$
$$\Delta V \doteqdot \pi(e-e^y)^2 \Delta y$$

$$V = \lim_{\Delta y \to 0} \sum_{y=0}^{y=1} \pi(e^2 - 2eye + e^{2y})\Delta y$$

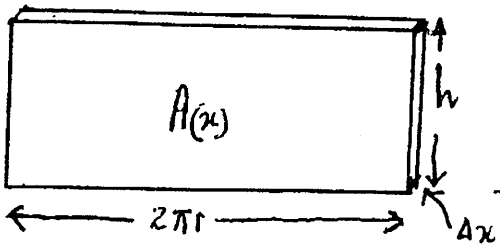
$$V = \pi \int_0^1 (e^2 - (2e)y^e + e^{2y})\,dy$$

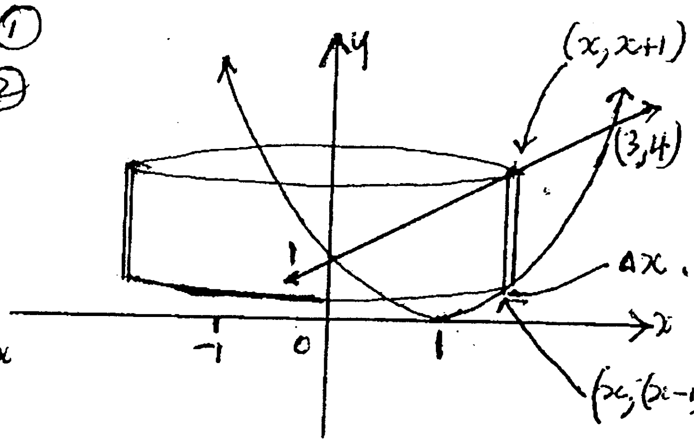
$$V = \pi \left[e^2 y - (2e)e^y + \frac{1}{2}e^{2y} \right]_0^1$$

$$V = \pi\left[\left(e^2 - 2e^2 + \frac{e^2}{2}\right) - \left(0 - 2e + \frac{1}{2}\right)\right]$$
$$V = \pi\left(-\frac{e^2}{2} + 2e - \frac{1}{2}\right)$$
$$V = \frac{\pi}{2}(4e - e^2 - 1) \text{ units}^3.$$

Question 6

(c) $y = x+1$ — ①
$y = (x-1)^2$ — ②





$r = x$

$h = (x+1)-(x-1)^2$

$= 3x - x^2$

$A(x) = 2\pi r h$

$= 2\pi x(3x-x^2)$

$\Delta(V) \doteqdot 2\pi(3x^2-x^3)\Delta x$

Note: Intersection of st line and parabola can be obtained by solving simultaneously.

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{x=3} 2\pi(3x^2-x^3)\Delta x$$

$$V = 2\pi \int_0^3 (3x^2-x^3)\,dx$$

$$V = 2\pi \left[x^3 - \frac{x^4}{4}\right]_0^3$$

$$V = 2\pi\left(27 - \frac{81}{4}\right)$$

$$V = 2\pi\left(\frac{108-81}{4}\right)$$

$$V = \frac{27\pi}{4} \text{ units}^3$$

Question 7

(a)(i) In ΔABC

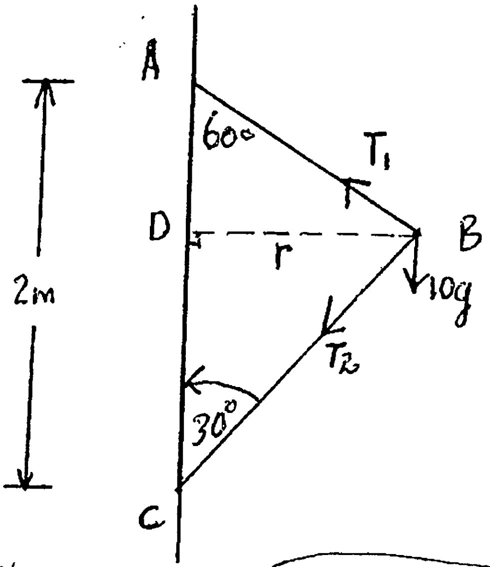
$$\frac{AB}{2} = \sin 30^\circ$$

$$AB = 2 \times \tfrac{1}{2} = 1 \text{ unit.}$$

In ΔABD

$$\frac{r}{1} = \sin 60^\circ$$

$$r = \frac{\sqrt{3}}{2} \quad (\text{ie } BD)$$



(ii) now 2π rad $=$ 1 revolution

$\omega = 90$ rpm

$$1 \text{ rev/min} = \frac{2\pi}{60} \text{ rad/sec}$$

$$90 \text{ rpm} = \frac{2\pi}{60} \times 90 \text{ rad/sec}$$

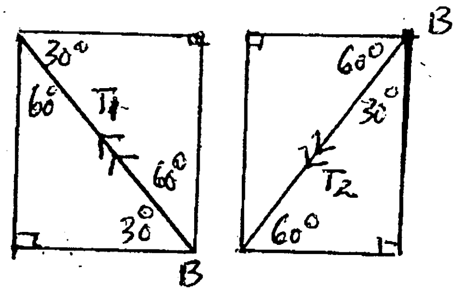
$$= 3\pi \text{ rad/sec}$$

Note for (ii) below, if using $g = 10$ then

$T_2 \doteqdot 290.03\text{ N}$

$T_1 \doteqdot 716.20\text{ N}$ } c. 2dp

Resolving Forces at B.



Vertically

$$T_1 \cos 60^\circ - T_2 \cos 30^\circ = 10g$$

$$T_1 \times \tfrac{1}{2} - T_2 \times \tfrac{\sqrt{3}}{2} = 10g$$

$$T_1 - T_2\sqrt{3} = 20g \quad \text{---} (1)$$

Horizontally

$$T_2 \cos 60^\circ + T_1 \cos 30^\circ = mr\omega^2$$

$$T_2 \times \tfrac{1}{2} + T_1 \times \tfrac{\sqrt{3}}{2} = 10 \times \tfrac{\sqrt{3}}{2} \times (3\pi)^2$$

$$T_2 + T_1\sqrt{3} = 90\sqrt{3}\pi^2 \quad \text{---} (2)$$

Solving (1) and (2)

from (1) $T_1 = 20g + T_2\sqrt{3}$ sub in (2)

$$T_2 + \sqrt{3}(20g + T_2\sqrt{3}) = 90\sqrt{3}\pi^2$$

$$T_2 + 20\sqrt{3}g + 3T_2 = 90\sqrt{3}\pi^2$$

$$4T_2 = 90\sqrt{3}\pi^2 - 20\sqrt{3}g$$

$$T_2 = \frac{90\sqrt{3}\pi^2 - 20\sqrt{3}g}{4}$$

$$T_2 = \frac{5\sqrt{3}\,(9\pi^2 - 2g)}{2}$$

$$T_1 = \frac{5(27\pi^2 + 2g)}{2}$$

Tension in BC

Tension in AB

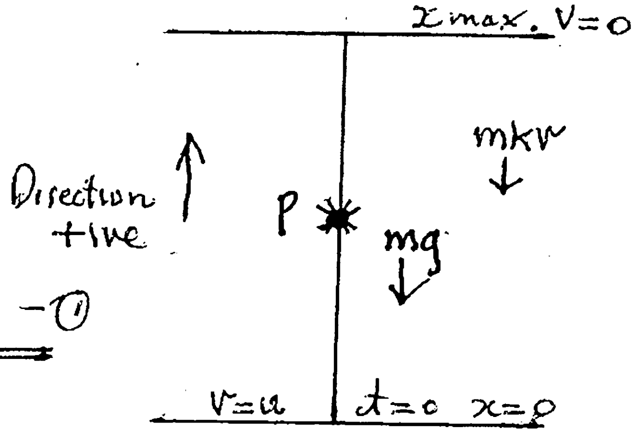
Question 7

(b) (i)

$F = ma$

$ma = -mg - mkv$

$\underline{\underline{a = -g - kv}} \quad ---\ (1)$



(ii) Let $a = \frac{dv}{dt}$

$$\frac{dv}{dt} = -g - kv$$

$$\frac{dv}{dt} = -(g + kv)$$

$$\frac{dt}{dv} = -\frac{1}{g + kv}$$

$$t = -\frac{1}{k}\ln|g + kv| + c$$

Now when $t = 0$, $v = u$.

$$0 = -\frac{1}{k}\ln|g + ku| + C$$

$$C = \frac{1}{k}\ln|g + ku|$$

$$t = \frac{1}{k}\ln|g + ku| - \frac{1}{k}\ln|g + kv|$$

$$t = \frac{1}{k}\ln\left|\frac{g + ku}{g + kv}\right|$$

Now for max height $v = 0$, $t = T$

$$\underline{\underline{T = \frac{1}{k}\ln\left|\frac{g + ku}{g}\right|}} \quad ----\ (2)$$

Question 7

(b)(ii) back to (b)(i) let $a = v\frac{dv}{dx}$

$$v\frac{dv}{dx} = -(g+kv)$$

$$\frac{dv}{dx} = -\frac{g+kv}{v}$$

$$\frac{dx}{dv} = -\frac{v}{g+kv}$$

$$\frac{dx}{dv} = \frac{g}{k}\left(\frac{1}{g+kv}\right) - \frac{1}{k}$$

Note $\frac{v}{g+kv} = \frac{\frac{1}{k}(g+kv) - \frac{g}{k}}{g+kv}$

or

$$\begin{array}{r|l} & \frac{1}{k} \\ kv+g & v + 0 \\ & v + \frac{g}{k} \\ \hline & -\frac{g}{k} \end{array}$$

$$x = \frac{g}{k}\times\frac{1}{k}\ln|g+kv| - \frac{v}{k} + c$$

$$x = \frac{g}{k^2}\ln|g+kv| - \frac{v}{k} + c$$

when $x = 0$, $v = u$.

$$0 = \frac{g}{k^2}\ln|g+ku| - \frac{u}{k} + c$$

$$c = \frac{u}{k} - \frac{g}{k^2}\ln|g+ku|$$

$$\therefore x = \frac{g}{k^2}\ln\left|\frac{g+kv}{g+ku}\right| + \left(\frac{u}{k} - \frac{v}{k}\right)$$

When $v = 0$ for max height

$$x = \frac{g}{k^2}\ln\left|\frac{g}{g+ku}\right| + \frac{u}{k}.$$

$$x = -\frac{g}{k^2}\ln\left|\frac{g+ku}{g}\right| + \frac{u}{k}.$$

for max height

$$x = -\frac{g}{k}T + \frac{u}{k} = \frac{1}{k}(u - gT)$$

using ② from part (i)

Question 8

(a)(i) $2^{(n+4)} > (n+4)^2$

For $n=1$

$2^5 > 5^2$

$32 > 25$ which is true

Consider the result to be true for $n=k$.

$\therefore \quad 2^{(k+4)} > (k+4)^2$

Consider $n=k+1$

$$
\begin{aligned}
2^{(k+1+4)} &= 2\left(2^{k+4}\right) \\
&> 2(k+4)^2 \\
&= 2(k^2+8k+16) \\
&= 2k^2+16k+32 \\
&= k^2+10k+25+k^2+6k+7 \\
&= (k+5)^2+k^2+6k+7
\end{aligned}
$$

Now since k is a positive integer $k^2+6k+7>0$

Hence $2^{(k+5)} > (k+5)^2$

ie $2^{[(k+1)+4]} > [(k+1)+4]^2$

This is the required inequality ie $(k+1)$ in place of n:
So if the result is true for $n=k$, Then it is true for $n=k+1$.

Now the result is true for $n=1$, hence true for $n=2$ and $n=3$ and so on for all positive integers n

Question 8

(a) (ii) Now let $3(a+2)=n+4$

$\therefore 3a+2=n$

Replace 'n' by $3a+2$ in the part (i) result.

$$2^{3a+2+4} > (3a+2+4)^2$$
$$= (3a+6)^2$$
$$= [3(a+2)]^2$$
$$= 9(a+2)^2$$
$$\therefore 3^{3(a+2)} > 9(a+2)^2$$

(b) (i) $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A \tan B}$

(ii) $\tan(2\tan^{-1}x) = 2\tan\left(\tan^{-1}x + \tan^{-1}x^3\right)$

Consider RHS $= 2\tan(A+B)$

Where $A = \tan^{-1}x \Rightarrow \tan A = x$

$B = \tan^{-1}x^3 \Rightarrow \tan B = x^3$

$$\text{RHS} = 2\left(\frac{\tan A + \tan B}{1-\tan A\tan B}\right)$$
$$= 2\left(\frac{x+x^3}{1-x^4}\right)$$
$$= \frac{2x(1+x^2)}{(1-x^2)(1+x^2)}$$
$$= \frac{2x}{1-x^2}$$
$$= \frac{2\tan A}{1-\tan^2 A}$$
$$= \tan 2A$$
$$= \tan(2\tan^{-1}x)$$
$$= \text{LHS}$$

Question 8

(c) $\sqrt{|x|} + \sqrt{y} = 1 \Rightarrow \sqrt{y} = 1 - \sqrt{|x|}$ ($\sqrt{y}$ cannot be less than zero)

(i) $0 \le |x| \le 1$

$|x| \le 1$

$-1 \le x \le 1$

(ii) If $x > 0$ then

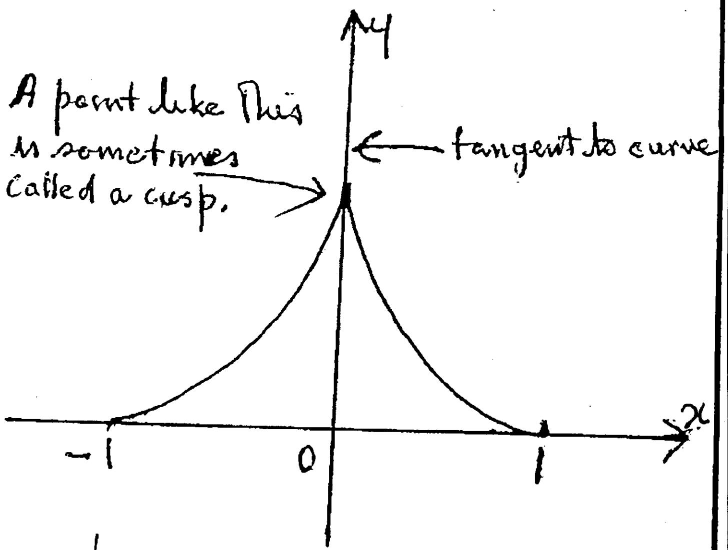
$\sqrt{x} + \sqrt{y} = 1$

$x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1$

$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} < 0$

as both $\sqrt{y}$ and $\sqrt{x}$ are both > 0



Note when $x=0$, $\frac{dy}{dx}$ is undefined $\Rightarrow$ vertical tangent.

(iii) If (x, y) is on curve then so is $(-x, y)$ hence the y axis is an axis of symmetry

Note when $x = \pm 1$, $y = 0$

Note $0 \le y \le 1$