ST IGNATIUS COLLEGE RIVERVIEW

ASSESSMENT TASK 4

TRIAL HSC EXAMINATION

YEAR 12

2009

EXTENSION 2

Time allowed: 3 hours (+ 5 minutes reading time)

Instructions to Candidates

- Attempt **all** questions
- There are eight questions. All questions are of equal value.
- Show all necessary working. Full marks may not be awarded for careless or poorly arranged work.
- The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.
- Board approved calculators and templates may be used.
- Each question must be returned in a ***separate*** writing booklet marked Q1, Q2 etc
- **Each booklet must have your name**.

Question 1 [15 marks] Start a new answer booklet.

(a) If $w=1+2i$ and $z=2-3i$, express in the form $a+bi$;

(i) $w\overline{z}$ 2

(ii) $\dfrac{w}{z}$. 2

(b) Solve for z where $z \in C$: $z^2+2iz+2=0$. 2

(c) Form a monic quadratic equation whose roots are $4i$ and $(3+i)$. 2

(d) Graph the region in the argand diagram which simultaneously satisfies $1 \le |z-i| \le 2$ and $\operatorname{Im}(z) \ge 0$. Mark all intercepts. 3

(e) Suppose that $z=1+\sqrt{3}i$ and $w=(\cos\theta+i\sin\theta)z$, where $-\pi \le \theta \le \pi$.

(i) Find the argument of z. 1

(ii) Given that w is purely imaginary and $\operatorname{Im}(w) > 0$, find the exact value of:

(α) θ 1

(β) $\arg(z+w)$ 2

Question 2 **[15 marks]** **Start a new answer booklet.**

(a) (i) Show that $(2+i)$ is a root of the equation $2z^3-5z^2-2z+15=0$. 2

(ii) Find the other roots. 2

(b) If α, β and γ are the roots of the equation $x^3+mx+n=0$, find in terms of m and n a cubic equation in 'x' with roots $\alpha+\beta-\gamma,\ \beta+\gamma-\alpha,\ \gamma+\alpha-\beta$. 3

(c) If α, β and γ are the roots of the equation $2y^3-y+4=0$, evaluate:

(i) $\alpha^3+\beta^3+\gamma^3$ 2

(ii) $\alpha^4+\beta^4+\gamma^4$ 3

(d) Find the value of t so that the equation $5x^5-3x^3+t=0$ has two equal positive roots. 3

Question 3 [15 marks] **Start a new answer booklet.**

(a) Find the indefinite integrals:

(i) $\int \frac{\sec^2(\ln x)}{x} dx$ [1]

(ii) $\int \frac{1}{\sqrt{x^2 - 6x}} dx$ [2]

(iii) $\int \frac{1}{\sqrt{6x - x^2}} dx$ [2]

(b) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{5 - 4\cos x}$ [4]

(c) (i) If $U_m = \int_0^1 \left(1 - x^2\right)^{\frac{m}{2}} dx$, where m is a positive integer, show that [4]

$U_m = \frac{m}{m+1} U_{m-2}$.

(ii) Hence evaluate U_5. [2]

Question 4 **[15 marks]** **Start a new answer booklet.**

(a) Let $f(x)=x^3-3x^2-x+3$. On separate diagrams, and without using calculus, sketch the following curves. Scale should be clearly indicated.

(i) $y=f(x)$ [1]

(ii) $y=|f(x)|$ [1]

(iii) $y=f(|x|)$ [2]

(iv) $y=\sqrt{f(x)}$ [2]

(v) $\sqrt{y}=f(x)$ [2]

(vi) $y=\tan^{-1} f(x)$ [2]

(vii) $y=e^{f(x)}$ [2]

(b) For $f(x)=\dfrac{1}{4x-5-x^2}$ prove that $-1\le f(x)<0$. [3]

Question 5 [15 marks] Start a new answer booklet.

(a) For what values of k does the equation $\frac{x^2}{6-k}+\frac{y^2}{k-2}=1$ represent:

(i) a circle? [1]

(ii) an ellipse? [1]

(iii) a hyperbola? [1]

(b) For the hyperbola $16x^2-9y^2=144$, find:

(i) its eccentricity; [1]

(ii) the co-ordinates of the foci; [1]

(iii) the equations of the asymptotes. [1]

(c) (i) Prove that the equation of the chord joining the points $P\left(ct_1,\frac{c}{t_1}\right)$ and $Q\left(ct_2,\frac{c}{t_2}\right)$ on the curve $xy=c^2$ is $x+t_1t_2y=c(t_1+t_2)$. [2]

(ii) If the chord PQ in part (i) is a normal at P, prove that $1+t_1^3t_2=0$. [2]

(d) Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a>b$ with a point $P(a\cos\theta,b\sin\theta)$ on it.

(i) Derive the equation of the tangent at P. [2]

(ii) If this tangent, the directrix and the major axis are concurrent on the positive side of the x axis, prove that $\theta=\cos^{-1}\left(\frac{\sqrt{a^2-b^2}}{a}\right)$. [3]

Question 6 [15 marks] **Start a new answer booklet.**

The solution for each of the parts (a) to (c) in this question should contain a neatly labelled diagram(s).

(a) Using the method of cylindrical shells find the volume of the solid formed when the region bounded by the curve $y = 2x^2 + 1$ and the x-axis between $x = 0$ and $x = 2$ is rotated about the y-axis. [5]

(b) The base of a solid is a region enclosed by the circle $x^2 + y^2 = 4$. Any cross section of the solid formed by a plane perpendicular to the x-axis is an equilateral triangle. Find the exact volume of the solid. [5]

(c) The curve $y = \sin x$ is rotated about the line $y = 1$. Use a slicing technique to find the volume of the solid of revolution formed by the portion from $x = 0$ to $x = \dfrac{\pi}{2}$. [5]

Question 7 **[15 marks]** **Start a new answer booklet.**

(a) A particle falls from rest, and there is an air resistance of $\frac{v^2}{10}$ per unit of mass , when its velocity is v metres per sec. Taking acceleration due to gravity as 10 metres per $\sec^2$;

(i) show why the acceleration is given by $\ddot{x}=10-\frac{v^2}{10}$. [2]

(ii) Find an expression for time in terms of velocity. [3]

(iii) Find an expression for velocity in terms of position x metres. [2]

(iv) What is the terminal or maximum velocity of the particle? [1]

(v) Find in exact form, the ratio of the times it takes the particle to attain $\frac{1}{2}$ and $\frac{1}{5}$ of its terminal velocity. [3]

(b) A 4kg mass, attached by a light inelastic string of length 60cm long to a fixed point, describes a horizontal circle at uniform angular velocity. [4]

Calculate the maximum number of revolutions per second that the pendulum will be able to complete if, for safety reasons, the greatest tension allowable in the string is 400 Newtons.

Question 8 **[15 marks]** **Start a new answer booklet.**

(a) (i) Show that if $\theta = \tan^{-1} x + \tan^{-1} y$, then $\tan\theta = \dfrac{x+y}{1-xy}$ [2]

(ii) If $\phi = \tan^{-1} x + \tan^{-1} y + \tan^{-1} z$, find an expression for $\tan\phi$ in terms of x, y and z. [3]

Deduce that if $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \dfrac{\pi}{2}$ then $xy + yz + zx = 1$.

(b) (i) Using the binomial theorem expand $\left(1+\dfrac{1}{n}\right)^k$. [1]

(ii) Given that $e = \lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$, prove that e can also be written as [2]

$1+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!}+\ldots\ldots\ldots\ldots\ldots$

(c) Show that $\dfrac{1}{n!} < \dfrac{1}{2^{n-1}}$ for all positive integral values $n \geq 3$, without the use of proof by Induction. [3]

(d) Given that $x^m \times y^n = (x+y)^{m+n}$, prove that $\dfrac{dy}{dx} = \dfrac{y}{x}$. [4]

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x > 0$