SAINT IGNATIUS' COLLEGE

Trial Higher School Certificate

2010

EXTENSION 2 MATHEMATICS

Directions to Students

• Reading Time : 5 minutes	• Total Marks **120**
• Working Time : 3 hours	
• Write using blue or black pen. (sketches in pencil).	• Attempt Question 1 – 8
• Board approved calculators may be used	• All questions are of equal value
• A table of standard integrals is provided at the back of this paper.	
• All necessary working should be shown in every question.	
• **Answer each question in the booklets provided and clearly label your name and teacher's name.**	

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Question 1 (Start a new Booklet) Marks

(a) Find $\int \cos^2 x \sin x \, dx$ 2

(b) (i) Use partial fractions to show 2

$$\frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{2-x}{x^2+4}$$

(ii) Hence evaluate 3

$$\int_0^2 \frac{8dx}{(x+2)(x^2+4)}$$

(c) Use integration by parts to find $\int \cos^{-1} x \, dx$ 2

(d) Find $\int \frac{dx}{\sqrt{x^2+5}}$ 1

(e) (i) Prove that if $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$ then $I_n = \frac{n-1}{n} I_{n-2}$ 4

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^7 x \, dx$ 1

Question 2 (Start a new Booklet) Marks

(a) Given $z = 3 - 4i$, find $z\bar{z}$ 2

(b) (i) Express $z = -1 + i\sqrt{3}$ in modulus–argument form. 2

(ii) Hence or otherwise find z^5 in the form $a + ib$. 2

(c) Find all pairs of real numbers x and y that satisfy 4

$$(x + iy)^2 = 12 - 16i$$

(d) On an Argand diagram, illustrate the region that satisfies 3

$$0 \le \arg(z + 4) \le \frac{2\pi}{3} \text{ and } |z + 4| \le 4$$

(e) The diagram below represents a square *OABC*. The point *C* represents the complex number $2 + 3i$.

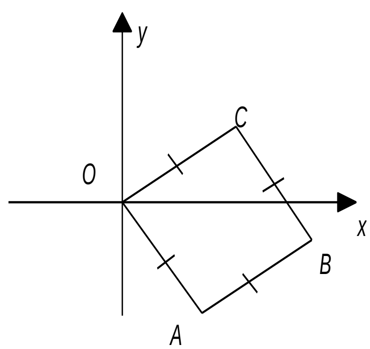


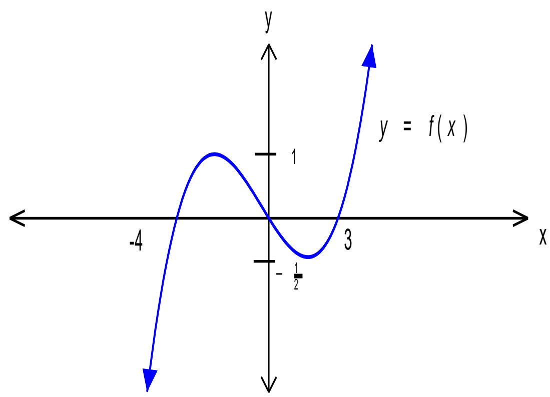
Diagram not to scale.

(i) Find the complex number that represents the point *A*. 1

(ii) Hence or otherwise, find the coordinates of the point *B*. 1

Question 3 (Start a new Booklet) Marks

(a) Consider the graph below.



On separate number planes, sketch the following:

(i) $y = \dfrac{1}{f(x)}$ 2

(ii) $y = f(|x|)$ 2

(iii) $y = [f(x)]^2$ 2

(iv) $y^2 = f(x)$ 3

(v) $y = \ln f(x)$ 3

(b) Sketch, showing all asymptotes, the graph of 3

$$y^2 = \frac{x^2}{x^2 + 2}$$

Question 4 (Start a new Booklet) Marks

(a) Consider the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$.

(i) Find its eccentricity. 1

(ii) State the co-ordinates of the foci. 1

(iii) State the equations of the directrices. 1

(iv) State the equations of the asymptotes. 1

(b) $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its centre at the origin O. A line through O, parallel to the tangent at P, meets the ellipse at Q and R.

(i) Show that the equation of the line QR is $ay\sin\theta + bx\cos\theta = 0$. 3

(ii) Show that the coordinates of Q and R are $(-a\sin\theta, b\cos\theta)$ and $(a\sin\theta, -b\cos\theta)$. 3

(iii) Find the perpendicular distance from P to the line QR. 2

(iv) Show that the area of ΔPQR is independent of θ . 3

Question 5 (Start a new Booklet) Marks

(a) If $P(x) = 2x^3 + 5x + 1$ has roots α, β, γ then find the value of $\alpha^3 + \beta^3 + \gamma^3$ 3

(b) Factorise $x^4 + 7x^2 - 8$ into the product of linear factors over the complex field. 2

(c) Consider the polynomial $P(x) = x^4 + Bx^3 + Cx^2 - 24x + 36$.
The equation $P(x) = 0$ has a double root at $x = 2$.

(i) Find the values of B and C. 3

(ii) Hence find all the solutions of $P(x) = 0$. 2

(d) (i) If $x = \cos\theta + i\sin\theta$, use De Moivre's Theorem to prove that $2\cos n\theta = x^n + \dfrac{1}{x^n}$ 1

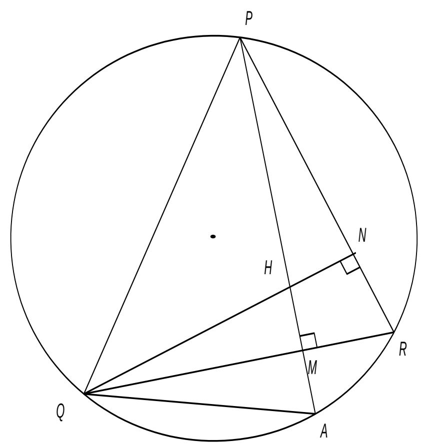
(ii) Hence or otherwise solve the equation 4

$$3x^4 - 5x^3 + 8x^2 - 5x + 3 = 0$$

Question 6 (Start a new Booklet) Marks

(a) Solve $\frac{x+1}{(x-1)(x+2)} \geq 0$ 2

(b)



P, Q, R and A lie on the circumference of a circle 4

$PA \perp QR$ meeting QR at M.

$QN \perp PR$ meeting PA at H.

Let $\angle MQA = x$.

(i) Copy the diagram into your writing booklet.

(ii) Prove QR bisects HA.

(c) The area enclosed by the curve $y=(x-2)^2$ and the line $y=4$ is rotated about the y-axis. Use the method of cylindrical shells to find the exact volume of the solid formed. 4

(d) A concrete beam of length 20 m has plane sides and cross sections parallel to ends which are rectangular. The beam measures 10 m by 12 m at one end and 5 m by 6 m at the other end.

(i) Find an expression in terms of h for the area of a cross-section of the beam that is h m from the smaller end. 3

(ii) Find the volume of the beam. 2

Question 7 (Start a new Booklet) Marks

(a) Find the general solution of the inequality $\cos\theta \geq \frac{\sqrt{3}}{2}$ 2

(b) A light inextensible string *OP* is fixed at the end *O* and is attached at the other end *P* to a particle of mass *m* which is moving uniformly in a horizontal circle whose centre is vertically below and distant *x* from *O*.

(i) Prove that the period of this motion is $2\pi\sqrt{\frac{x}{g}}$ seconds, where *g* is the acceleration due to gravity. 3

(ii) If the number of revolutions per second is increased from 2 to 3, find the change in *x*. (Take $g = 10\ m/s^2$) 3
Give your answer correct to the nearest millimetre.

(c) A particle of mass, *m*, falls vertically from rest under gravity in a medium for which the resistance to the motion is proportional to the square of the velocity (i.e. $R = mkv^2$).

(i) Write an equation for the acceleration ($\ddot{x}$) of the particle. 1

(ii) Show that the terminal velocity (*V*) is given by $V = \sqrt{\frac{g}{k}}$. 2

(iii) Show that the position, *x*, of the particle in terms of its velocity, *v* is given by $x = \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$ 4

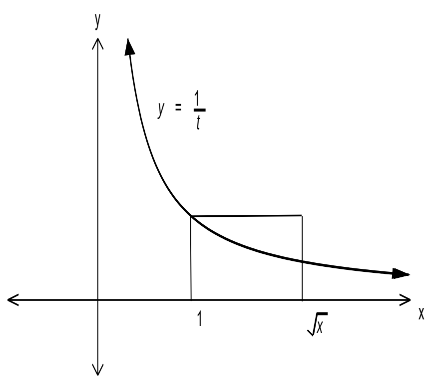
Question 8 (Start a new Booklet) Marks

(a) (i) Show that $\cos(A-B)-\cos(A+B)= 2\sin A\sin B$ 2

(ii) Hence evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin 5x \sin x \, dx$ 2

(b) A sequence T_n is such that $T_1 = 4$ and $T_2 = 8$ and $T_{n+2} = 6T_{n+1} - 5T_n$ 4
Prove by mathematical induction that $T_n = 5^{n-1} + 3$ for integers $n \geq 1$.

(c) The diagram represents the curve $y = \dfrac{1}{t}$ for $t > 0$



(i) If $x > 1$. show that $\displaystyle\int_1^{\sqrt{x}} \frac{1}{t}\, dt = \frac{1}{2}\ln x$ 2

(ii) Show that for $x > 1,\ 0 < \dfrac{1}{2}\ln x < \sqrt{x}$ 2

(iii) Use the inequality in (ii) to show that $\displaystyle\lim_{x \to \infty} \frac{\ln x}{x} = 0$ 3

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$