Total marks – 120
Attempt Questions 1–8
All questions are of equal value

SCEGGS Darlinghurst 2005 Ext.2 trial

Answer each question in a SEPARATE writing booklet.

Marks

Question 1 (15 marks)

(a) Integrate:

(i) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$ **1**

(ii) $\displaystyle\int \frac{x^2}{x^2-9}\,dx$ **3**

(iii) $\int \sin^{-1} x\,dx$ using Integration by Parts. **3**

(b) Prove that $\displaystyle\int_0^{\frac{\pi}{3}} \frac{d\theta}{1+\sin\theta} = \sqrt{3}-1$ using the substitution $t=\tan\dfrac{\theta}{2}$. **4**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sin^3\theta\cos^3\theta\,d\theta$. **4**

Marks

Question 2 (15 marks) BEGIN A NEW BOOKLET

(a) Consider the complex number $u = 1-\sqrt{3}i$.

(i) Express u in mod-arg form. **1**

(ii) Evaluate u^{6n} if n is an integer. **2**

(iii) On an Argand Diagram, sketch the curve $|z-u|=2$ showing important features. **2**

(iv) Find the maximum value of $|z|$ on the curve. **1**

(b)

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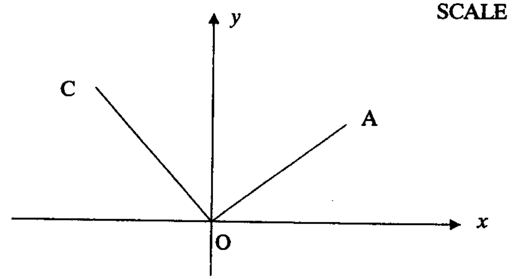

OABC is a square on an Argand Diagram where O is the origin.
The points A and C represent the complex numbers z and iz respectively.

(i) Find the complex number represented by B. **1**

(ii) The square is now rotated about O through $\frac{\pi}{4}$ in an anti-clockwise direction. Prove that the new position of B is given by the complex number $\sqrt{2}\,i\,z$. **2**

Question 2 continues on page 4

 Marks

(c) In an Argand Diagram, an equilateral triangle has its vertices on the circle centre the origin, radius 2 units. One of the vertices is represented by the point whose argument is π.

(i) Find the 3 vertices in Cartesian form. **2**

(ii) Prove that the complex numbers represented by these vertices form the roots of the equation $z^3 + 8 = 0$. **3**

(iii) Find the area of the triangle. **1**

End of Question 2

Question 3 (15 marks) BEGIN A NEW BOOKLET **Marks**

(a) The curve $y = f(x) = \dfrac{\log_e x}{x}$ is shown below.

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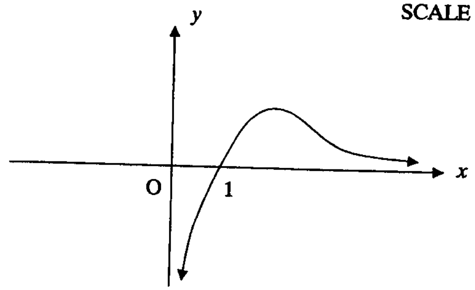


Given the maximum stationary point is $\left(e, \dfrac{1}{e}\right)$, sketch the following curves showing essential features, taking at least $\dfrac{1}{3}$ page for each.

(i) $y = |f(x)|$ **1**

(ii) $y = f(|x|)$ **1**

(iii) $y = f(x+1)$ **2**

(iv) $y = \dfrac{1}{f(x)}$ **3**

Question 3 continues on page 6

Marks

Question 3 (continued)

(b) Consider the polynomial **3**

$$P(x) = x^4 + 2x^3 + x^2 - 1$$

It is given that one zero is $\frac{-1}{2} + \frac{\sqrt{3}}{2}i$.

Find the other three zeros.

(c) Consider $f(x) = x^3 - 3cx$ (c is a constant).

(i) Prove that $f(x) = 0$ has only one real root if $c < 0$. **2**

(ii) Prove that $x^3 - 3cx = k$ has 3 real different roots if: **3**

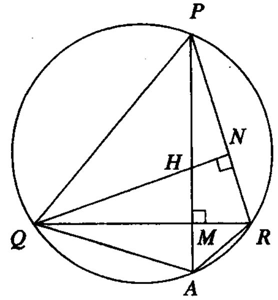
$$|k| < 2c\sqrt{c}$$

End of Question 3

Marks

Question 4 (15 marks) BEGIN A NEW BOOKLET

(a) **4**



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P, Q, R and A lie on the circumference of a circle.
$PA \perp QR$ meeting QR at M.
$QN \perp PR$ meeting PA at H.
Let $\angle MQA = x°$.

Prove QR bisects HA.

(b) A solid has as its base the ellipse $\frac{x^2}{4} + y^2 = 1$ in the $x - y$ plane. Find the volume of the solid such that every cross section by a plane parallel to the y axis is a semi circle with its diameter in the $x - y$ plane. **4**

A diagram and a clear explanation should accompany your solution.

(c) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$ where n is a non-negative integer.

(i) Use Integration by Parts to prove that **3**

$$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx \quad \text{for } n \geq 2.$$

(ii) Deduce that $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geq 2$. **2**

(iii) Evaluate I_4. **2**

Question 5 (15 marks) BEGIN A NEW BOOKLET **Marks**

(a) The equation $x^3 - x^2 - 3x + 5 = 0$ has roots α, β and γ.

(i) Find $\alpha + \beta + \gamma$. **1**

(ii) Find the equation whose roots are $2\alpha + \beta + \gamma,\ \alpha + 2\beta + \gamma,\ \alpha + \beta + 2\gamma$. **3**

(b) The points $P\left(2t, \frac{2}{t}\right)$ and $Q\left(2s, \frac{2}{s}\right)$ lie on the hyperbola $xy = 4$.

$$\left(t \neq 0, s \neq 0, t^2 \neq s^2\right)$$

(i) Prove that the equation of the tangent to the hyperbola at the point P is **2**

$$x + t^2 y = 4t.$$

(ii) Prove that the tangents at P and Q intersect at **2**

$$M\left(\frac{4st}{s+t}, \frac{4}{s+t}\right)$$

Suppose that $s = \frac{-1}{t}$

(iii) Prove that the locus of M is a straight line and state any conditions that may apply. **2**

(c) A tennis match between two players consists of a number of sets. The match continues until one of the players has won 3 sets.

Whenever Pat and John play, on average, for each set they play, there is a probability of $\frac{2}{3}$ that Pat wins and a probability of $\frac{1}{3}$ that John wins.

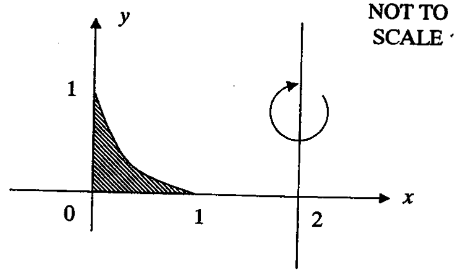
Find the probability that:

(i) Pat wins 2 of the first 3 sets. **1**

(ii) Pat wins the match. **4**

Question 6 (15 marks) BEGIN A NEW BOOKLET **Marks**

(a)



The region contained by the curve $y = (x-1)^2$ and the axes is rotated about the line $x = 2$.

(i) Taking slices perpendicular to the line of rotation prove that the volume obtained is **5**

$$\lim_{\delta y \to 0} \pi \sum_{y=0}^{1} \left(3 + 2\sqrt{y} - y\right)\delta y$$

(ii) Hence find this volume. **2**

(b) (i) Find the centre and radius of the circle $x^2 + y^2 + 4x = 0$. **2**

(ii) Prove that the line $y = mx + b$ will be a tangent to the circle if: **3**

$$4(mb + 1) = b^2$$

(iii) P is the point whose co-ordinates are $(k, 0)$. **3**
If P lies on the line $y = mx + b$ and is exterior to the circle, find possible values for k if the two tangents from P to the circle are perpendicular.

Marks

Question 7 (15 marks) BEGIN A NEW BOOKLET

(a) (i) Prove $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ using the substitution $y = a - x$. **1**

(ii) Hence evaluate $\int_0^1 x^2\sqrt{1-x}\,dx$ **3**

(b) (i) Prove $\cos(A+B)+\cos(A-B) = 2\cos A\cos B$. **1**

(ii) Hence solve: **3**

$$\cos 5x + \cos 3x - \cos x = 0 \text{ for } 0 \le x \le \frac{\pi}{2}$$

(c) An object of mass m kg is thrown vertically upwards. Air resistance is given by $R = 0.05m\,v^2$ where R is in newtons and $v\ ms^{-1}$ is the speed of the object. Take $g = 9.8\ ms^{-2}$.

(i) Explain why the equation of motion is **2**

$$\ddot{x} = -\frac{196+v^2}{20}$$

where x is the height of the object in metres above the point from which it is thrown.

If the initial velocity was $50ms^{-1}$, find:

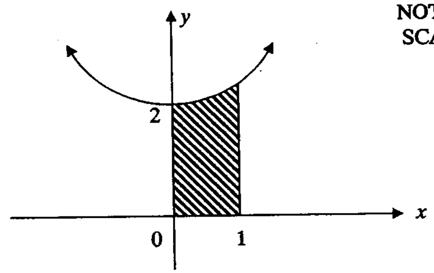
(ii) the maximum height attained. **3**

(iii) the time taken to reach this maximum height. **2**

Marks

Question 8 (15 marks) BEGIN A NEW BOOKLET

(a) **4**



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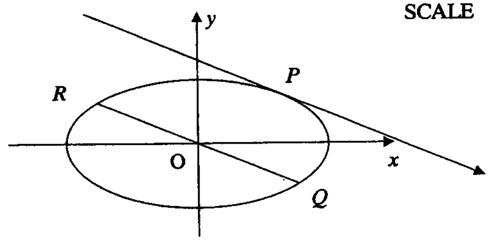
The curve $y = e^x + e^{-x}$ is shown.
Use the method of cylindrical skills to find the volume formed when the shaded region is rotated about the y axis.

(b)



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P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
RQ is the diameter of the ellipse parallel to the tangent at P.

(i) Prove that the equation of RQ is **1**

$$y = -\frac{bx\cos\theta}{a\sin\theta}$$

(ii) Hence find the co-ordinates of R and Q. **3**

Question 8 continues on the next page

Question 8 (continued) **Marks**

(b) (iii) Prove that the length of RQ is: **1**

$$2\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$$

(iv) Explain the relationship between this result and the length of the diameter of a circle centre the origin radius a units. **2**

(c) (i) Find the sum of: **1**

$$x + x^2 + x^3 + \cdots + x^n$$

(ii) Hence prove: **3**

$$x + 2x^2 + 3x^3 + 4x^4 + \cdots + nx^n = \frac{x}{(x-1)^2}\left[nx^{n+1} - (n+1)x^n + 1\right]$$

End of Paper

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