THE SCOTS COLLEGE

YEAR 12

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

EXTENSION TWO MATHEMATICS

AUGUST 2002

TIME ALLOWED: 3 HOURS *[plus 5 minutes reading time]*

OUTCOMES ASSESSED:

- E3 - Uses the relationship between algebraic and geometric representations of complex numbers and of conic sections.
- E4 - Uses efficient techniques for algebraic manipulation required in dealing with conic sections and polynomials.
- E6 - Combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions.
- E7 - Uses techniques in slicing and volumes. Applies further techniques of integration to problems.
- E8 - Applies further techniques of integration.
- E5 - Uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces, resisted motion and circular motion.

QUESTION 1 [15 MARKS]

a. Given the complex number $z = 7 - 3i$. Find

(i) $|z|$

(ii) $\bar{z}$

(iii) $|z - \bar{z}|$

(iv) $\arg(z - \bar{z})$

b. Express $z = \dfrac{\sqrt{2}}{1-i}$ in modulus argument form and hence express z^5 in the form $x + iy$.

c. K, L, M, N are vertices of a square, in anti-clockwise order. Given that K and M represent the numbers $2+i$ and $2+3i$ respectively, find the coordinates of:

(i) L and N.

(ii) M, if the square is rotated clockwise through an angle of 90° about the origin.

d. In the Argand Plane, sketch the following:

(i) $|z - 12 + 3i| = 5$

(ii) $|z^2 - \bar{z}^2| \geq 4$

(iii) $\arg \dfrac{z - 1 + i}{z + 1 - i} = 0$

START A NEW BOOKLET

QUESTION 2 [15 MARKS]

a. The polynomial function $p(x) = x^4 - 4x^3 - 3x^2 + 50x - 52$ has a zero at $x = 3 - 2i$. Factorise $P(x)$ over the field of:

(i) rationals

(ii) reals

(iii) complex numbers

b. The equation $x^4+4x^3-3x^2-4x-2=0$ has roots $\alpha,\beta,\gamma,\delta$. Find the equation with roots:

$\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma},\frac{1}{\delta}$

c. The equation $2x^3-9x^2+7=0$ has roots α,β,γ. Find the equation with root $\alpha^3,\beta^3,\gamma^3$.

d. Solve $x^5+2x^4-2x^3-8x^2-7x-2=0$ if it has a root of multiplicity 4.

START A NEW BOOKLET

QUESTION 3 [15 MARKS]

a. Find the coordinates of the two foci on the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$

b. P and Q are variable points on the rectangular hyperbola $xy=c^2$.

(i) The tangent at Q passes through the foot of the ordinate of P. If P and Q have parameters p and q, show that $p=2q$.

(ii) Hence prove that the locus of the midpoint of PQ is a rectangular hyperbola and find its equation.

c. The hyperbola H has equation $xy=16$.

(i) Sketch this hyperbola and indicate on your diagram the positions and coordinates of all points at which the curve intersects the axes of symmetry.

(ii) $P\left(4p,\frac{4}{p}\right)$, where $p>0$, and $Q\left(4q,\frac{4}{q}\right)$ where $q>0$ are two distinct arbitrary points on H. Find the equation of chord PQ.

(iii) Prove that the equation of the tangent at P is $x+p^2y=8p$.

(iv) The tangents at P and Q intersect at T. Find the coordinates of T.

START A NEW BOOKLET

QUESTION 4 [15 MARKS]

a. Find:

(i) $\int \tan^{-1}x\,dx$

(ii) $\int \frac{dx}{\sin x\cos x}$

(iii) $\int \frac{\sec^2 x\,dx}{\tan^2 x-3\tan x+2}$

b. Leaving your answer in exact form, evaluate:

(i) $\int_0^1 \frac{dx}{x^2+8x+4}$

(ii) $\int_0^1 x^2e^{-x}dx$

START A NEW BOOKLET

QUESTION 5 [15 MARKS]

Sketch the following curves on separate axes, showing all relevant points.

a. (i) $y=\sin x$ and hence also sketch;

(ii) $y^2=\sin x$ for $-2\pi\le x\le 2\pi$.

b. (i) $y=x^3-4x$ and hence also sketch;

(ii) $y=|x^3|-4|x|$ for Domain $-3\le x\le 3$.

c. Sketch the graph of $y=\frac{x^2-6x+8}{x^2-x-6}$ clearly indicating all relevant points.

START A NEW BOOKLET

QUESTION 6 [15 MARKS]

a. Determine the real values for k for which the equation

$$\frac{x^2}{19-k}+\frac{y2}{7-k}=1$$

defines respectively an ellipse and a hyperbola. Sketch the curve corresponding to the value $k=3$.

b. Evaluate the definite integral

$$\int_{-1}^{1}\frac{4+x^2}{4-x^2}dx$$

c. Sketch the region (R) which is completely bounded by the curves $y=\sin 2x$ and $y=\frac{1}{2}$ in the domain, $0\le x\le\frac{\pi}{2}$. Find the volume generated when R is rotated about the:

(i) x-axis

(ii) line $y=\frac{1}{2}$

START A NEW BOOKLET

QUESTION 7 [15 MARKS]

a. The base of a solid is a circular region of radius a units. Find the volume if every cross-section of a plane perpendicular to a certain diameter is a square with one side lying in the base.

b. Find, by the method of cylindrical shells, the volume of the solid generated when the region bounded by the curve $y=x^2+1$, the line $x=2$ and the coordinate axes is rotated about the line $x=3$.

START A NEW BOOKLET

QUESTION 8 [15 MARKS]

a. A parachutist of mass m falls to ground from an aircraft. Given that the air resistance per unit mass is proportional to the square of the speed V:

(i) Draw a diagram showing clearly the forces acting on the parachutist during his free fall.

(ii) Deduce that $\frac{d}{dx}(v^2)=2g-2kv^2$

(iii) Show that $v^2=\frac{g}{k}-Ae^{-2kx}$ satisfies the differential equation in part (ii) above and show that $A=\frac{g}{k}$.

(iv) Sketch the graph of v^2 against x and find an expression for the terminal speed of the parachutist during his free fall.

b. A particle of mass 10kg is found to experience a resistance, in Newtons, one tenth of the square of its velocity in m/sec, when it moves through the air. The particle is projected vertically upwards from 0 with a velocity of u m/sec, and the point A, vertically above 0, is the highest point reached by the particle before it starts to fall to the ground again. Assuming the value of g is 10 m/sec^2.

(i) Find the time taken for the particle to reach A from 0.

(ii) Show that the height OA is $50\ln(1+10^{-3}u^2)$.

(iii) Show that the particles velocity W m/sec when it reaches 0 again is given by

$$W^2=\frac{u^2}{1+10^{-3}u^2}$$

END OF PAPER

Standard Integrals

$$\int x^n dx = \frac{1}{n+1}x^{n+1}+C, \quad n\neq -1; x\neq 0, \text{if } n<0$$

$$\int \frac{1}{x}dx = \ln x + C, \quad x>0$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax}+C, \quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \quad a\neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C, \quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C, \quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}+C, \quad a>0, \ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right)+C, \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)+C, \quad a\neq 0$$

NOTE: $\ln x \equiv \log_e x, \quad x>0$

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Question 1

a) Given $z = 7 - 3i$

i) $|z| = \sqrt{49+9}$

$= \sqrt{58}$ $\quad \underline{1}$

ii) $\bar{z} = 7 + 3i$ $\quad \underline{1}$

iii) $|z - \bar{z}|$

$= |-6i|$

$= 6$ $\quad \underline{1}$

iv) $\arg(z - \bar{z})$

$= \arg(-6i)$

$= \frac{-\pi}{2}$ $\quad \underline{1}$

b) Express $z = \frac{\sqrt{2}}{1-i}$ in Mod arg form

$z = \frac{\sqrt{2}}{1-i} \times \frac{1+i}{1+i}$

$= \frac{\sqrt{2}(1+i)}{2}$

$= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ $\quad \underline{2}$

$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$.

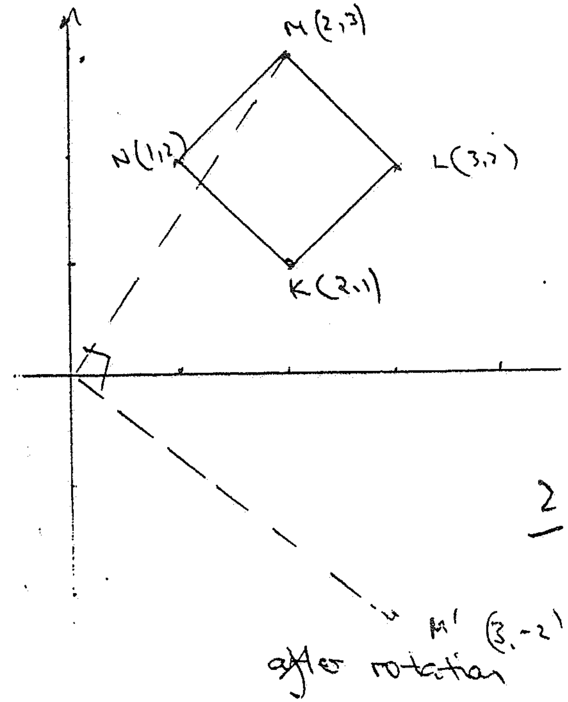
$\Rightarrow z^5 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^5$

$= \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

$= -\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}$. $\quad \underline{2}$

$= -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$.

c)



$\underline{2}$

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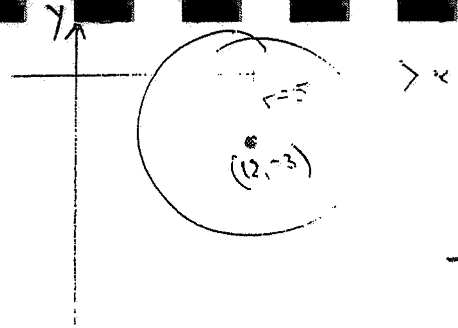
1) $|z - 12 + 3i| = 5$

Circle centre $(12, -3)$

radius 5



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ii) $|z^2 - \bar{z}^2| \geq 4$

$|(z + \bar{z})(z - \bar{z})| \geqslant 4$

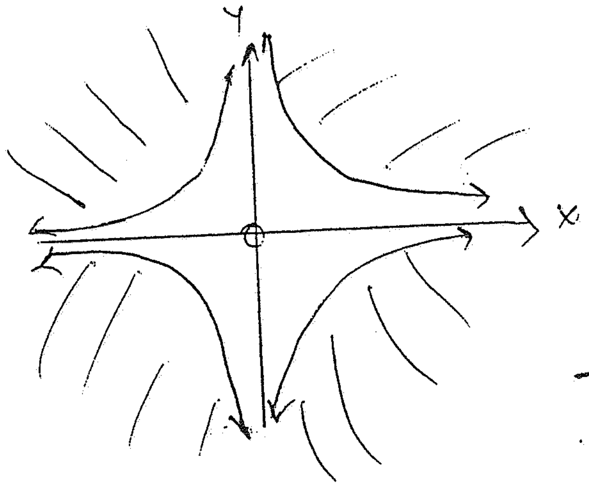
$|(2x)(2iy)| \geqslant 4$

$|xyi| \geqslant 1$

$|xy| \geqslant 1$

Consider $xy = 1$

and $xy = -1$



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iii) $\arg \dfrac{z - 1 + i}{z + 1 - i} = 0$

$\arg(z - 1 + i) - \arg(z + 1 - i) = 0$

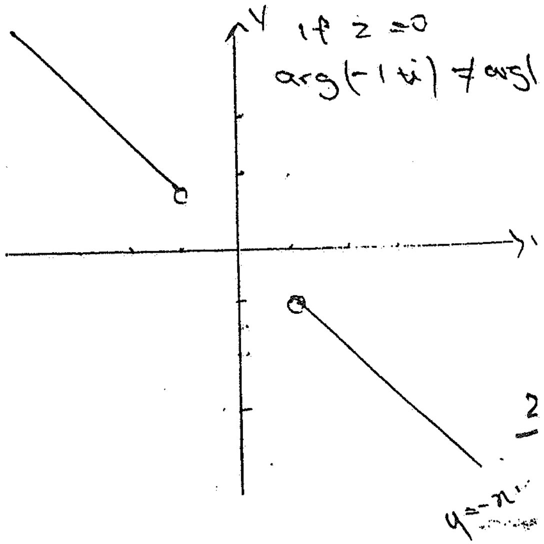
$\tan^{-1} \dfrac{y+1}{x-1} = \tan^{-1} \dfrac{y-1}{x+1}$

$\therefore \dfrac{y+1}{x-1} = \dfrac{y-1}{x+1}$

$2x + 2y = 0$

$x + y = 0$

$y = -x$.



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ii) For the particle on the return trip, let the origin and the direction be the same as above

$$m\ddot{x} = -mg + \frac{v^2}{10}$$

$$a = v\frac{dv}{dx} = -10 + \frac{v^2}{100}$$

$$\frac{v\,dv}{-10 + \frac{v^2}{100}} = \frac{100\,v\,dv}{v^2 - 1000} = dx$$

$$\int_0^{-w} \frac{100\,v\,dv}{v^2 - 1000} = \int_H^0 dx$$

note velocity $= -w$ when it reaches the ground.

$$-H = 50\left[\ln(v^2 - 1000)\right]_0^{-w}$$

$$= 50 \ln \frac{w^2 - 1000}{-1000}$$

$$\therefore H = -50\ln\left(1 - \frac{w^2}{1000}\right)$$

$$\therefore 50\ln\left(1 + \frac{u^2}{1000}\right) = -50\ln\left(1 - \frac{w^2}{1000}\right)$$

$$\Rightarrow \ln\left(1 + \frac{u^2}{1000}\right) = -\ln\left(1 - \frac{w^2}{1000}\right)$$

END SOLUTIONS

Question Two

a) $P(x) = x^4 - 4x^3 - 3x^2 + 50x - 52$

has a zero value at $x = 3 - 2i$

If $x = 3-2i$ is a factor then $x = 3+2i$ is also a factor.

ie $(x - 3 + 2i)(x - 3 - 2i)$ is a factor.

$\therefore$ $x^2 - 6x + 13$ is a factor.

$\therefore$ i) $P(x) = (x^2 - 6x + 13)(x^2 + 2x - 4)$ over a rational field 1+1

ii) Consider from above $x^2 + 2x - 4$

$$x = \frac{-2 \pm \sqrt{4 - 4\times1\times-4}}{2}$$

$$= \frac{-2 \pm \sqrt{20}}{2}$$

$$= \frac{-2 \pm 2\sqrt{5}}{2}$$ 1+1

$x = -1 \pm \sqrt{5}$.

$P(x) = (x^2 - 6x + 13)(x + 1 - \sqrt{5})(x + 1 + \sqrt{5})$ over real field

iii) $(x - 3 + 2i)(x - 3 - 2i)(x + 1 - \sqrt{5})(x + 1 + \sqrt{5})$ over complex field. 1+1

$\alpha, \beta, \gamma, \delta$

New equation has roots

$x = \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$

$\therefore \alpha = \frac{1}{x}$

Therefore new eqn is

$\Rightarrow \left(\frac{1}{x}\right)^4 + 4\left(\frac{1}{x}\right)^3 - 3\left(\frac{1}{x}\right)^2 - 4\left(\frac{1}{x}\right) - 2 = 0$

$\Rightarrow \frac{1}{x^4} + \frac{4}{x^3} - \frac{3}{x^2} - \frac{4}{x} - 2 = 0$ 3

$\Rightarrow 1 + 4x - 3x^2 - 4x^3 - 2x^4 = 0$

c) $2x^3 - 9x^2 + 7 = 0$ has roots α, β, γ

Find equation with roots $\alpha^3, \beta^3, \gamma^3$

New equation has roots

$x = \alpha^3, \beta^3, \gamma^3$

$\therefore x^{1/3} = \alpha$

$\frac{10 + v^2}{100} = \ldots$

$100 \int_u^0 \frac{dv}{1000 + v^2} = \int_0^t -dt$

$\therefore t = \sqrt{10} \tan^{-1} \frac{u\sqrt{10}}{100}$

ii) $a = v\frac{dv}{dx} = -10 - \frac{v^2}{100}$

$\frac{v\,dv}{10 + \frac{v^2}{100}} = \frac{100v\,dv}{1000 + v^2} = -dx$

$100 \int_u^0 \frac{v\,dv}{1000 + v^2} = \int_0^H -dx$

$-H = 50\left[\ln(1000 + v^2)\right]_u^0$

$= 50 \ln \frac{1000}{1000 + u^2}$

$\therefore H = 50 \ln \frac{1000 + u^2}{1000}$

$= 50 \ln\left(1 + \frac{u^2}{1000}\right)$

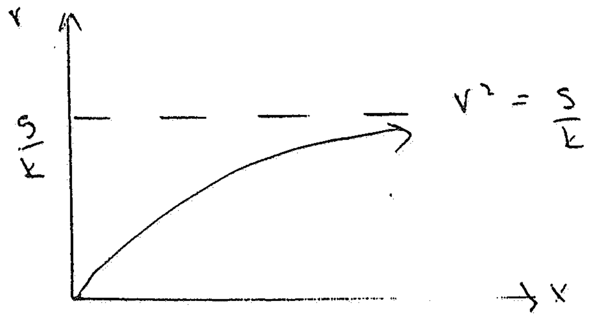
from $v^2 = \frac{g}{k} - Ae^{\ldots}$ $\quad A = \frac{g}{k}$

$$v^2 = \frac{g}{k} - \frac{g}{k}e^{-2kx}.$$

$$v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$$

$$\lim_{x \to \infty} v^2 = \lim_{x \to \infty} \frac{g}{k}\left(1 - e^{-2kx}\right)$$

$v^2 = \frac{g}{k}$ $\Rightarrow$ terminal speed $= \sqrt{\frac{g}{k}}$



8b) 1) Let The upward direction be +ve and O be origin

$$ma = -mg - \frac{v^2}{10}$$

$$a = \frac{dv}{dt} = -g - \frac{v^2}{10m} = -10 - \frac{v^2}{100} \quad (m = 10\text{kg})$$

$$\frac{dv}{10 + \frac{v^2}{100}} = -dt$$

∴ New equation.

$$2x\left(\tfrac{1}{3}\right)^3 - 9\left(x^{1/3}\right)^2 + 7 = 0$$

$$2x^{3/3} - 9x^{2/3} + 7.$$

$$9x^{2/3} = 2x + 7$$

$$\left(9x^{2/3}\right)^3 = (2x+7)^3$$

$$729x^2 = 8x^3 + 84x^2 + 294x + 343$$ 3

$$\Rightarrow 8x^3 - 645x^2 + 294x + 343 = 0$$

d) Solve $x^5 + 2x^4 - 2x^3 - 8x^2 - 7x - 2 = 0$ of multiplicity 4

$$P(x) = x^5 + 2x^4 - 2x^3 - 8x^2 - 7x - 2$$

$$P'(x) = 5x^4 + 8x^3 - 6x^2 - 16x - 7$$

$$P''(x) = 20x^3 + 24x^2 - 12x - 16$$

$$P'''(x) = 60x^2 + 48x - 12 = 0$$

$$12\left[5x^2 + 4x - 1\right] = 0$$ 3

$$12\left[(5x-1)(x+1)\right] = 0$$

$$x = \frac{1}{5} \qquad x = -1$$

a) $\frac{x^2}{25} + \frac{y^2}{16} = 1$

$a^2 = 25 \quad b^2 = 16$

$a = 5 \quad b = 4$

$b^2 = a^2(1-e^2)$

$16 = 25(1-e^2)$

$16 = 25 - 25e^2$

$25e^2 = 25 - 16$

$e^2 = \frac{9}{25}$

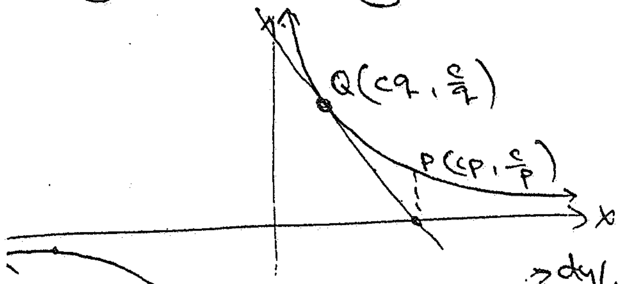
$e = \frac{3}{5}$

generally $S(ae, 0) \quad S'(-ae, 0)$

$\therefore S(3,0) \quad S'(-3,0)$

2

b) P and Q are variable points on rectangula hyperbola. $xy = c^2$



$xy = c^2$

$y = \frac{c^2}{x}$

$y = c^2 x^{-1}$

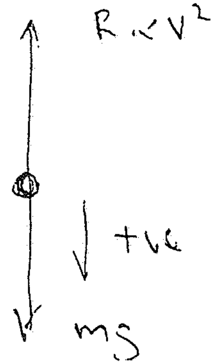
$dy/dx = -c^2 x^{-2}$

$dy/dx = -\frac{c^2}{x^2}$

at Q gradient $= \frac{-c^2}{c^2 q^2}$

$m = -\frac{1}{q^2}$

a) i)



$R = mkv^2$ let mk be const

ii) $m\ddot{x} = mg - mkv^2$

$\ddot{x} = g - kv^2$

using $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g - kv^2$

$\frac{d}{dx}(v^2) = 2g - 2kv^2$

ii) Show that $V^2 = \frac{g}{k} - Ae^{-2kx}$ satisfies (ii)

$v^2 = \frac{g}{k} - Ae^{-2kx}$

$\frac{d}{dx}(v^2) = \frac{d}{dx}\left(\frac{g}{k} - Ae^{-2kx}\right)$ Differentiate R.H.S.

$\frac{d}{dx}(v)^2 = 2Ake^{-2kx}$

$= 2kAe^{-2kx}$

$= 2k\left(\frac{g}{k} - v^2\right)$

$= 2g - 2kv^2$

$v^2 = \frac{g}{k} - Ae^{-2kx}$

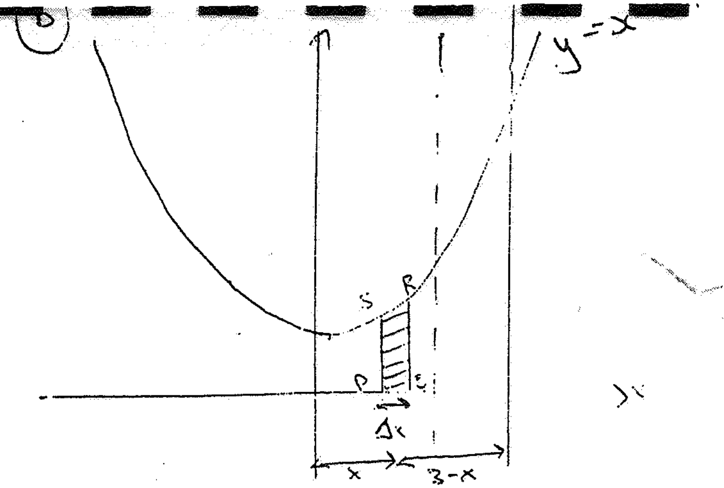
$\therefore Ae^{-2kx} = \frac{g}{k} - v^2$

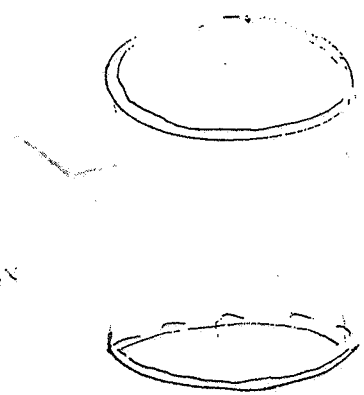
when $x = 0 \quad v = 0$

$\therefore 0 = \frac{g}{k} - Ae^0$

$\therefore A = \frac{g}{k}$

(b)





∴

$V = \pi\left[(3-x)^2 - [3-(x+\Delta x)]^2\right] y$

$= \pi\left[6\Delta x - 2x\Delta x - \Delta x^2\right] y$

$(\Delta x)^2$ negl

$= \pi(6-2x)\,y\,\Delta x$

$= \pi(6-2x)(x^2+1)\Delta$

Strip PQRS rotated about $x = 3$

$C = 2\pi r$

inner radius $= 3-(x+\Delta$

outer radius $= 3-x$.

hence

$V = \pi\int_0^2 (6x^2 - 2x^3 + 6 - 2x)\,dx$

$= \pi\left[2x^3 - \frac{x^4}{2} + 6x - x^2\right]_0^2$

$= 16\pi$ units^3

∴ tangent at Q is:

$y - \frac{c}{q} = -\frac{1}{q^2}(x - cq)$

$x + q^2 y = 2cq$

Cut x-axis at $y = 0$

at This point $x = cp$ ie $P(cp, \frac{c}{p})$

$\therefore cp = 2cq$ 3

$\boxed{p = 2q}$

ii) locus.

Midpoint of PQ

$x = \frac{cp + cq}{2}$ $\quad y = \frac{\frac{c}{p} + \frac{c}{q}}{2}$

$x = \frac{c(p+q)}{2}$ $\quad y = \frac{c}{2}\left(\frac{1}{p} + \frac{1}{q}\right)$

Note for $p = 2q$

$x = \frac{3cq}{2}$ $\quad y = \frac{c}{2}\left(\frac{1}{2q} + \frac{1}{q}\right)$

$2x = 3cq$ $\quad y = \frac{c}{2}\left(\frac{1+2}{2q}\right)$

$q = \frac{2x}{3c}$ $\quad y = \frac{c}{2}\left(\frac{3}{2q}\right)$

$y = \frac{3c}{4q}$ $\rightarrow 4qy = 3c$

$q = \frac{3c}{4y}$

Question 7.

$\int_{-1} -1 + \frac{8}{4-x^2}\,dx$

a) i) $\int \tan^{-1} x \, dx$

$\int 1 \times \tan^{-1} x \, dx$

$u = \tan^{-1} x \qquad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{1+x^2} \qquad v = x$

$\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$

$\Rightarrow x \tan^{-1} x - \int x \cdot \frac{1}{1+x^2}\,dx$

$\Rightarrow x \tan^{-1} x - \int \frac{x}{1+x^2}\,dx$

$\Rightarrow x \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c$

3

ii) $\int \frac{dx}{\sin x \cos x}$

$\Rightarrow \int \frac{2}{2\sin x \cos x}\,dx$

$\Rightarrow 2\int \frac{dx}{2\sin x \cos x}$

$\Rightarrow 2\int \frac{dx}{\sin 2x}$

$\Rightarrow 2\int \frac{1}{\sin 2x}\,dx$

$\Rightarrow 2\int \operatorname{cosec} 2x \, dx$

$\Rightarrow \frac{2\operatorname{cosec} 2x(\operatorname{cosec} 2x + \cot 2x)}{\operatorname{cosec} 2x + \cot 2x}$

$\Rightarrow -\ln(\operatorname{cosec} 2x + \cot 2x)$

3

Now consider, using partial fractions, $\frac{8}{4-x^2}$

$\frac{8}{(2-x)(2+x)} = \frac{a}{2-x} + \frac{b}{2+x}$

$8 \equiv a(2+x) + b(2-x)$

for $x = -2 \Rightarrow 4b = 8 \quad b = 2$

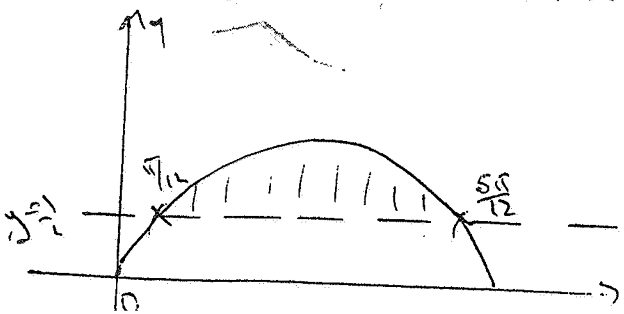
for $x = 2 \Rightarrow 4a = 8 \quad a = 2$

$\Rightarrow -\int_{-1}^{1} 1\,dx + \int_{-1}^{1} \frac{2}{2-x}\,dx + \int_{-1}^{1} \frac{2}{2+x}\,dx$

$\Rightarrow \left[-x - 2\ln(2-x) + 2\ln(2+x)\right]_{-1}^{1}$

$\Rightarrow 4\ln 3 - 2$

c) Sketch region bounded by $y = \sin 2x$, $y = \frac{1}{2}$ & domain $0 \le x \le \frac{\pi}{2}$



Question Six

a) Determine real values for k in equation

$$\frac{x^2}{19-k} + \frac{y^2}{7-k} = 1$$

for an ellipse

$19-k>0$ and $7-k>0$

$\therefore$ $k<19$ and $k<7$.

$\therefore$ $k<7$

for hyperbola

$19-k>0$ and $7-k<0$

$k<19$ and $k>7$

$\therefore$ $7<k<19$.

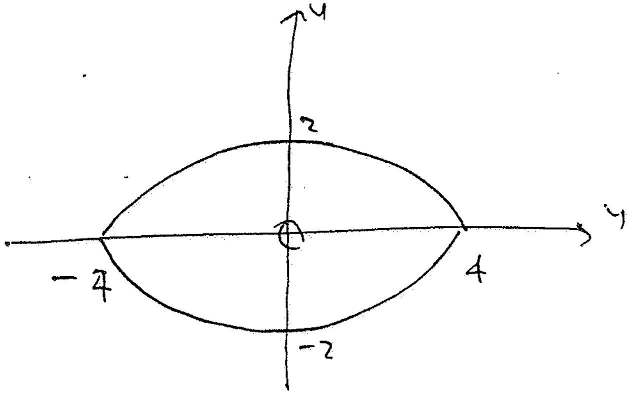
for $k=3$ equation becomes

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$a=4 \quad b=2$



b) Evaluate

$$\int_{-1}^{1} \frac{4+x^2}{4-x^2}\,dx$$

by long division

$4-x^2 \overline{)\,4+x^2}$

$$= -1 + \frac{8}{4-x^2}$$

(ii) $$\int \frac{\sec^2 x \cdot dx}{\tan^2 x - 3\tan x + 2}$$

let $u = \tan x$.

$\therefore \frac{du}{dx} = \sec^2 x$.

$du = \sec^2 x \cdot dx$

$$\Rightarrow \int \frac{du}{u^2 - 3u + 2}$$

$$\int \frac{1}{(u-2)(u-1)}\,du.$$

Now consider partial fractions.

$$\frac{1}{(u-2)(u-1)} = \frac{a}{u-2} + \frac{b}{u-1}.$$

$$1 \equiv a(u-1) + b(u-2)$$

for $u=1 \quad b=-1$

for $u=2 \quad a=1$

3

$$\Rightarrow \int \left(\frac{1}{u-2} - \frac{1}{u-1}\right) du.$$

$\ln(u-2) - \ln(u-1) + C.$

but $u = \tan x$.

$\therefore \ln(\tan x - 2) - \ln(\tan x - 1) + C.$

i) $\int_0 \frac{x}{x^2+8x+4}$ complete the

$x^2+8x=-4$

$x^2+8x+(4)^2=-4+(4)$

$(x+4)^2=12$

$(x+4)^2-12=0$

$\Rightarrow \int_0^1 \frac{dx}{(x+4)^2-12}$

$\Rightarrow \frac{1}{2\sqrt{12}}\left[\ln \frac{x+4-\sqrt{12}}{x+4+\sqrt{12}}\right]_0^1$

$\Rightarrow \frac{1}{4\sqrt{3}}\left[\ln \frac{5-\sqrt{12}}{5+\sqrt{12}} - \ln \frac{4-\sqrt{12}}{4+\sqrt{12}}\right]$

$\Rightarrow \frac{1}{4\sqrt{3}}\left[\ln\left(\frac{5-\sqrt{12}}{5+\sqrt{12}} \times \frac{4+2\sqrt{3}}{4-2\sqrt{3}}\right)\right]$

$\Rightarrow \frac{1}{4\sqrt{3}} \ln \frac{20+2\sqrt{3}-12}{20-2\sqrt{3}-12}$

$\Rightarrow \frac{1}{4\sqrt{3}} \ln \frac{4+\sqrt{3}}{4-\sqrt{3}}$

3

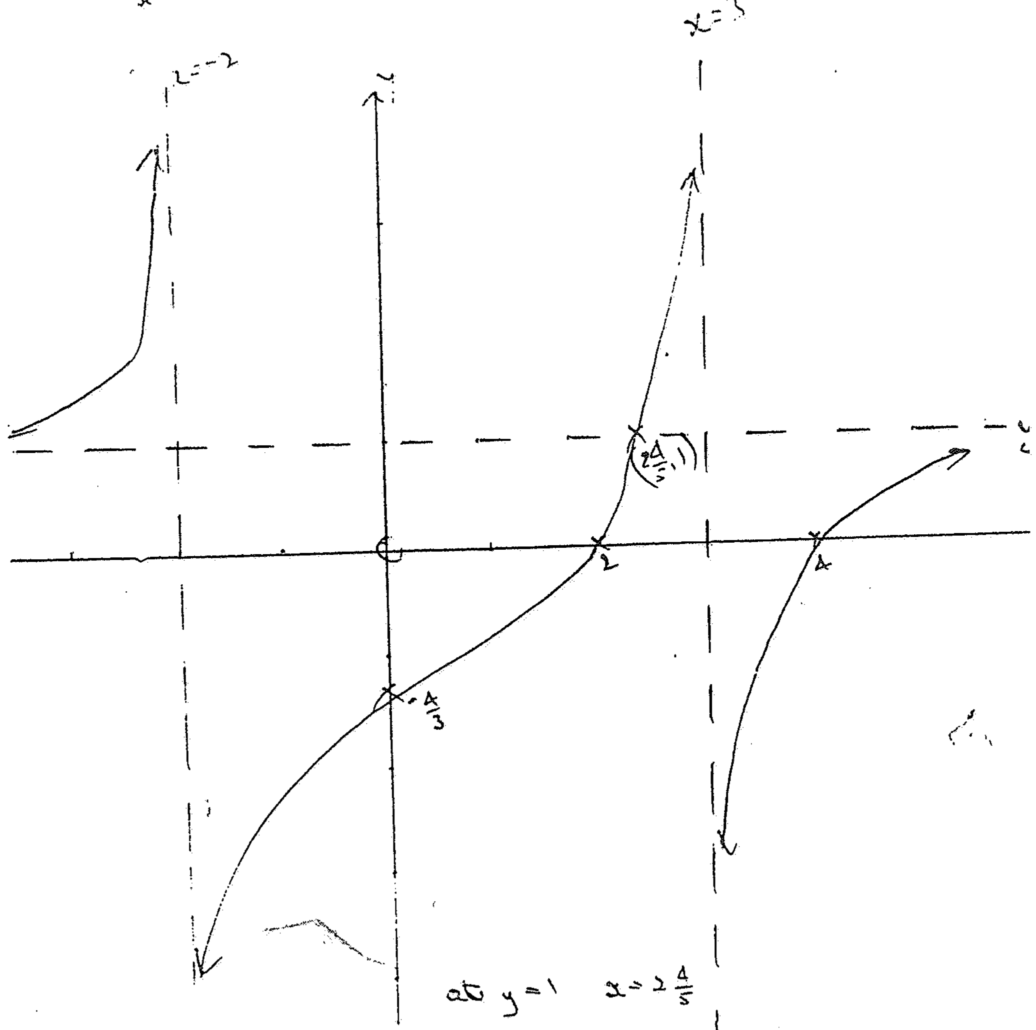
ii) $\int_0^1 x^2 e^{-x}\,dx$

$\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$

$u = x^2 \qquad \frac{dv}{dx} = e^{-x}$

horizontal asymptote

$y = \dfrac{\frac{x^2}{x^2} - \frac{6x}{x^2} + \frac{8}{x^2}}{\frac{x^2}{x^2} - \frac{x}{x^2} - \frac{6}{x^2}}$ $\therefore y = 1$



at $y=1$ $x = 2\frac{4}{5}$

$(2\frac{4}{5}, 1)$

* Cuts x-axis at y=0

$x^2 - 6x + 8 = 0$

$(x-2)(x-4) = 0$

$(2,0)$ $(4,0)$

Cut y-axis at $x=0$

$\therefore y = -\frac{8}{6} = -\frac{4}{3}$

$\therefore \left(0, -\frac{4}{3}\right)$

* Stationary Values

$y = \frac{u}{v}$

$u = x^2 - 6x + 8 \quad v = x^2 - x - 6$

$u' = 2x - 6 \quad v' = 2x - 1$

$$dy/dx = \frac{(x^2-x-6)(2x-6) - (x^2-6x+8)(2x-1)}{(x^2-x-6)^2}$$

$$dy/dx = \frac{5x^2 - 28x + 44}{(x^2-x-6)^2}$$

$dy/dx = 0$ when $5x^2 - 28x + 44 = 0$

Consider $b^2 - 4ac$

$28^2 - 4 \times 5 \times 44$

$784 - 880$

$b^2 - 4ac < 0$

* asymptotes

vertical

1) when $x^2 - x - 6 = 0$

$(x+2)(x-3) = 0$

$x = -2 \quad x = 3$

approaching $x=3$

for $x<3$

$$y = \frac{(x-4)(x-2)}{(x-3)(x+2)}$$

$y = \frac{-+}{-+} = +$

y is large

$x<3 \quad y = \frac{+-}{++} = \frac{-}{+} = -$

y large -ve

$x>-2$

$y = \frac{--}{-+} = \frac{+}{-} = -$

y large -ve.

$x \cdot \int e^{-x} \cdot 2x \, dx$

$= \left[-x^2 \cdot e^{-x}\right]_0^1 + \int 2x \cdot e^{-x} \, dx$

2nd application.

$u = 2x. \quad \frac{dv}{dx} = e^{-x}.$

$\frac{du}{dx} = 2 \quad v = -e^{-x}$

$\equiv -2x e^{-x} + \int_0^1 2e^{-x} \, dx$

$\equiv -2x \cdot e^{-x} - 2(e^{-x})_0^1$

$\Rightarrow \left[-x^2 \cdot e^{-x}\right]_0^1 - \left[2x \cdot e^{-x}\right]_0^1 - 2\left[e^{-x}\right]_0^1$

$\Rightarrow -\frac{1}{e} - 2e^{-1} - 2e^{-1} + 2$

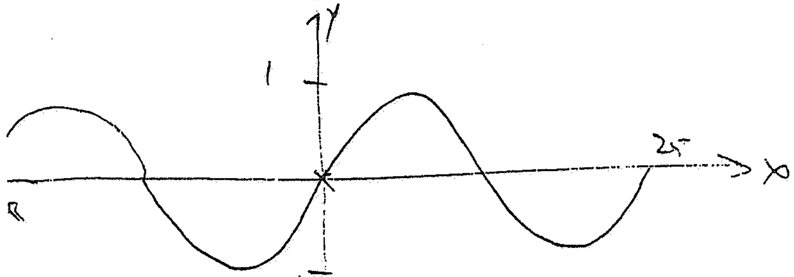
$= 2 - 5e^{-1}$

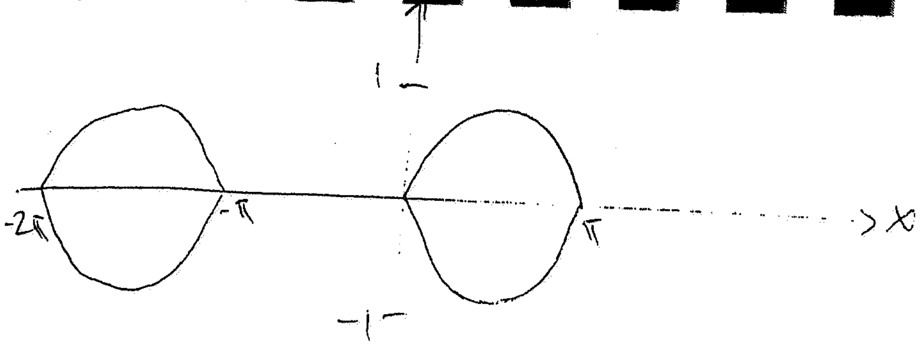
3

Question 5

$-2\pi \leq x \leq 2\pi$

a) Sketch $y = \sin x$ and hence $y^2 = \sin$





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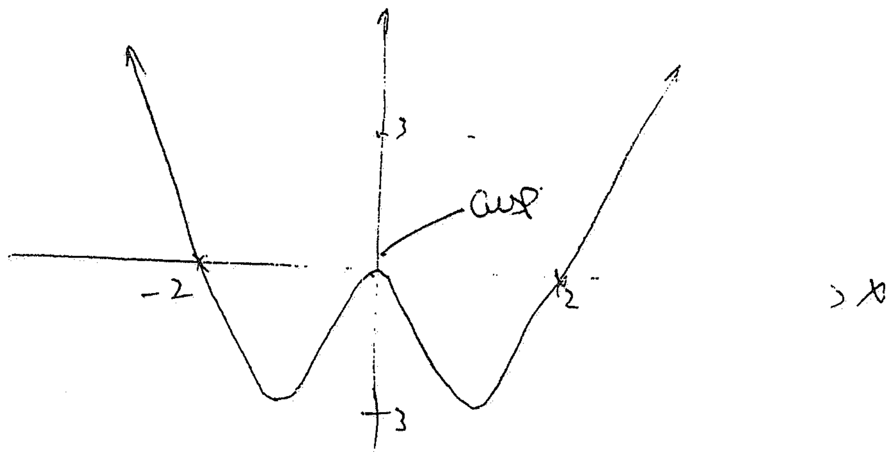

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b) Sketch $y = x^3 - 4x$.

Cut y-axis at $x = 0$ ∴ $y = 0$

Cut x-axis at $y = 0$

$0 = x(x^2 - 4)$

$x = 0 \quad x = \pm 2$

Stationary Values

$dy/dx = 3x^2 - 4 = 0$

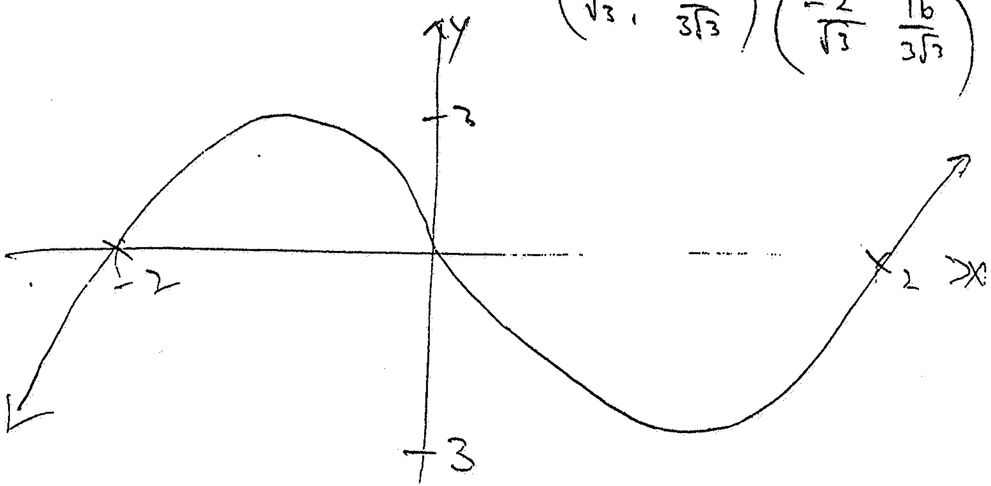
$3x^2 = 4$

$x^2 = \frac{4}{3}$

$x = \pm \frac{2}{\sqrt{3}}$

∴ $y = \mp \frac{16}{3\sqrt{3}}$

∴ S.V $\left(\frac{2}{\sqrt{3}}, -\frac{16}{3\sqrt{3}}\right) \left(\frac{-2}{\sqrt{3}}, \frac{16}{3\sqrt{3}}\right)$



or by subtraction of ordinates.

$y_1 = |x|^3 \qquad y^2 = 4|x|$

∴ $y_1 - y_2$

x	-3	-2	-1	0	1	2	3
y_1	27	8	1	0	1	8	27
y_2	12	8	4	0	4	8	12
$y_1 - y_2$	15	0	-3	0	-3	0	15

Sketch as shown above.

(c) Sketch $y = \dfrac{x^2 - 6x + 8}{x^2 - x - 6}$