Question 1 (15 marks) The Scots College Ext2 Maths trial 2003 **Marks**

a Find $\int \frac{\ln x}{x} dx$ 1

b By completing the square find $\int \frac{dx}{x^2+4x+8}$ 2

c Use integration by parts to find $\int \sin^{-1} x dx$ 2

d Find $\int \frac{x^2}{x^2-9} dx$ 3

e Evaluate $\int_0^{\frac{\pi}{4}} \sin 5x \cos 4x dx$ 3

f Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2\sin x + \cos x + 1}$ 4

Question 2 (15 marks)

a By first writing $1+i\sqrt{3}$ in mod-arg form, express $(1+i\sqrt{3})^7$ in the form $a+ib$ 2

b Find the square roots of $40+42i$ 2

c Let $z = \frac{3+4i}{5}$ and $w = \frac{12+5i}{13}$ so that $|z| = |w| = 1$

(i) Write zw and $z\overline{w}$ in the form $x+iy$ 2

(ii) Hence by considering the square of the moduli find different pairs of positive integers p and q such that $p^2+q^2 = 65^2$ 2

d On separate Argand Diagrams sketch

(i) $|z-4| = |z+4i|$ 1

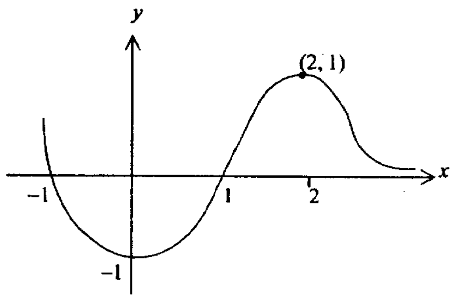
(ii) $\arg(z-4) = \arg(z+4i)$ 1

e In an Argand Diagram, OABCDE (in clockwise order) is a regular hexagon, where O is the origin and A represents the number 4i.

(i) Sketch the figure stating the numbers represented by B, C and E. 2

(ii) The figure is rotated anticlockwise through 90° about A to give a figure AB'C'D'E'O'. Sketch the figure stating the numbers represented by B', C' and E'. 3

Question 3 (15 marks)



a The diagram shows the graph of $y = f(x)$. Draw separate one-third page sketches of the graphs of the following:

(i) $y = \dfrac{1}{f(x)}$ 2

(ii) $y^2 = f(x)$ 2

(iii) $y = \ln f(x)$ 2

(iv) $|y| = f(|x|)$ 2

b Sketch the following graphs showing features such as asymptotes, intercepts and turning points.

(i) $y = \dfrac{(x-5)(x+2)}{x-1}$ 4

(ii) $y = \ln(\sin e^x)$ 3

Question 4 (15 marks)

a Given that $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 - 2x^3 - 5x^2 + x + 7 = 0$, find

(i) $\Sigma\alpha$ (ii) $\Sigma\alpha\beta$ (iii) $\Sigma\alpha\beta\gamma$ (iv) $\alpha\beta\gamma\delta$ 2

(v) $\Sigma\alpha^2$ (vi) $\Sigma\alpha^3$ (vii) $\Sigma\alpha^4$ 1,2,1

(viii) the equation with roots $\dfrac{2}{\alpha}, \dfrac{2}{\beta}, \dfrac{2}{\gamma}, \dfrac{2}{\delta}$ 2

b If ω is a complex cube root of unity, show that

(i) ω^2 is the other complex root 1

(ii) $1 + \omega + \omega^2 = 0$ 1

(iii) $(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega) = a^3 + b^3 + c^3 - 3abc$ 2

c Factorise $x^6 - 7x^2 + 6$ over the

(i) rational numbers

(ii) real numbers

(iii) complex numbers 1,1,1

Question 5 (15 marks)

a (i) On the same axes sketch $y = \sin 2x$ and $y = \tan x$ for $|x| \le \pi/2$ 2

(ii) Find the area bounded by these curves 2

(iii) Find the volume generated when these regions are rotated about the x axis. 3

(iv) Use the foregoing results to find the volume if these regions are rotated about the line $x = -1$ 1

b A solid has an elliptical base with equation $16x^2 + 25y^2 = 400$. Each vertical cross section perpendicular to the x axis is in the shape of a parabola (vertex uppermost) with its latus rectum in the base. Clearly explaining your method, find the volume of the solid. 7

Question 6 (15 marks)

a If P is any point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, show that the difference of the distances of P from the two foci is a constant and thus independent of the position of P. Give the value of this constant. 2

b The tangent at $P(a\cos\theta, b\sin\theta)$ on an ellipse meets the coordinate axes at Q and R.

(i) Find the area of the triangle OQR and deduce its least value. 3

(ii) Hence state the minimum area of a quadrilateral which circumscribes the ellipse and sketch TWO quadrilaterals with that property. 2

(iii) Formulate a statement which relates the ratio of the area of an ellipse and that of a circumscribing parallelogram to the ratio of the area of a circle and that of the circumscribing square. 1

c (i) Find the equation of the normal to the curve $xy = c^2$ at $P(cp, \dfrac{c}{p})$ 2

(ii) This normal meets the x axis at Q. Show that the coordinates of M, the midpoint of PQ, are $(\dfrac{c(2p^4-1)}{2p^3}, \dfrac{c}{2p})$ 2

(iii) Find the equation of the locus of M. 3

Question 7 (15 marks)

a Two light rigid rods AB and BC, each of length 2m, are smoothly jointed at B and the rod AB is smoothly jointed at A to a smooth vertical rod. The joint at B has a mass of 3 kg attached. A ring of mass 2 kg is smoothly jointed to BC at C and can slide on the vertical rod below A. The ring rests on a smooth horizontal table fixed to the rod $2\sqrt{3}$ m below A. The system rotates about the rod with an angular velocity ω.

(i) Find the forces in the rods and the force exerted on the ring by the table. (Draw a clear diagram) 5

(ii) What value must ω exceed for the ring to rise above the table? 2

b A body of mass m falling under gravity experiences a resistance per unit mass of kv where v is its velocity. Leaving your answer in terms of g :-

(i) Find the terminal velocity of the body. 1

(ii) Find the time taken and how far it falls to attain half of this terminal velocity. (Clearly define your notation) 3,4

Question 8 (15 marks)

a Two equal circles touch at A. AB is a diameter of one circle. BR is the tangent from B to the other circle and cuts the first circle at Q. Find the ratio of BQ:QR. 3

b (i) Sketch the quadrilateral $|x-2|+|y|=1$ and calculate its area 2

(ii) Use the method of cylindrical shells to find the volume generated when this area is rotated around the y axis 3

c Let $I_n = \int_0^{\pi/2} \cos^n x\,dx$

(i) Show that $I_n = \frac{n-1}{n} I_{n-2}$, for $n \geq 2$. 3

(ii) Hence show that $\int_0^{\pi/2} \cos^{2n} x\,dx = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$ 4