THE SCOTS COLLEGE

YEAR 12 MATHEMATICS EXTENSION 2

HSC TRIAL

AUGUST 2008

General Instructions

- All questions are of equal value
- Working time - 3 hours + 5 minutes reading time.
- Write using blue or black pen
- Board approved calculators may be used
- Start a new booklet for each question
- All necessary working should be shown in every question
- A Standard Integrals Table is attached

TOTAL MARKS: 120

WEIGHTING: 40 %

QUESTION 1 **[15 MARKS]** **MARKS**

a. If $z = 1 + i$, find:

(i) $|z|$ **[1]**

(ii) $\arg z$ **[1]**

(iii) z^{-6} in the form $x + iy$ **[2]**

b. Solve the equation for z **[3]**

$$z\overline{z} + 2iz = 12 + 6i$$

c. What is the locus in the Argand Diagram of the point Z which represents the complex number z where: **[2]**

$$z\overline{z} - 2\left(z + \overline{z}\right) = 5$$

d. The origin and the points representing the complex numbers z, $\frac{1}{z}$ and $z + \frac{1}{z}$ are joined to form a quadrilateral. Write down the conditions for z so that the quadrilateral will be a:

(i) rhombus **[1]**

(ii) square **[2]**

e. Prove by induction that, for all integers $n \geq 1$, **[3]**

$$\left(\cos\theta - i\sin\theta\right)^n = \cos\left(n\theta\right) - i\sin\left(n\theta\right)$$

QUESTION 2 **[15 MARKS]** **START A NEW BOOKLET** **MARKS**

a. If $f(x)=(x+2)(x-1)$, sketch the graphs of the following functions on separate diagrams.

(i) $y=f(x)$ **[1]**

(ii) $y=\dfrac{1}{|f(x)|}$ **[2]**

(iii) $y=\log_e(f(x))$ **[2]**

(iv) $y^2=f(x)$ **[2]**

b. **(i)** By using implicit differentiation, state where $\dfrac{dy}{dx}$ is undefined for $y^2=-x^2(x+2)(x-1)$. **[2]**

(ii) Hence or otherwise, sketch the curve. **[2]**

c. Let $f(x)=x-2+\dfrac{3}{x+2}$

(i) Find the points for which $f(x)=0$. **[1]**

(ii) Find the asymptotes. **[2]**

(iii) Sketch the curve. Show **all** asymptotes and the x and y intercepts. (There is no need to find or label stationary points.) **[1]**

QUESTION 3 **[15 MARKS]** **START A NEW BOOKLET** **MARKS**

a. Evaluate $\int_0^{\frac{\pi}{4}} x\sin 2x\, dx$ **[3]**

b. Find $\int x\sqrt{1-x}\, dx$ **[2]**

c. Find $\int \frac{1}{x\left(1+x^2\right)}\, dx$ **[3]**

d. By completing the square and using the table of Standard Integrals, find $\int \frac{dx}{\sqrt{x^2-4x+1}}$ **[2]**

e. Explain why the following integral cannot be evaluated. **[1]**

$$\int_0^5 \frac{1}{3-x}\, \mathrm{dx}$$

f. Evaluate $\int_0^{\pi/3} \frac{\tan x}{1+\cos x}\, dx$, using the substitution $t=\tan\frac{x}{2}$. **[4]**

QUESTION 4 [15 MARKS] **START A NEW BOOKLET** **MARKS**

a. For the ellipse $x^2+4y^2=100$

(i) Write down the eccentricity, the co-ordinates of the foci and the equations of the directrices. **[3]**

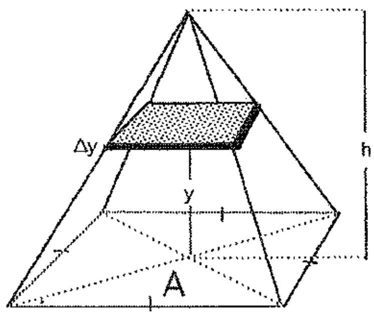
(ii) Sketch the graph of the ellipse showing the above features. **[1]**

(iii) Find the equation of the tangent and normal to the ellipse at the point P(8,3). **[3]**

(iv) The normal at P meets the major axis at G. A point K lies on the tangent which passes through the point P(8,3). A perpendicular from K passes through the origin O. Prove that PG × OK is equal to the square of the length of the semi-minor axis. **[2]**

b. A particle is projected from a point on a straight line with velocity $u\ ms^{-1}$ and moves in such a way that when it has travelled a distance of x metres it has a velocity of $v=\dfrac{u}{4+ux}ms^{-1}$. Prove that the acceleration of the particle is $-v^3ms^{-2}$. **[2]**

c. One of the largest pyramids in Egypt is approximately 150m high and has a square base with a base area of approximately 50,000m^2. The diagram below shows a square based pyramid with a base area A and height h. The thickness of the cross section at height y is Δy.



(i) Show that the area of the cross section at height y can be represented as:

$$A\times\left(\frac{h-y}{h}\right)^2$$ **[1]**

(ii) Find the volume of the pyramid by using the slicing technique. **[3]**

QUESTION 5 **[15 MARKS]** **START A NEW BOOKLET** **MARKS**

a. Let $I_n = \int_0^{\pi} x^n \sin x \, dx$, where n is a positive integer.

(i) Show that $I_n = \pi^n - n(n-1)I_{n-2}$, for $n \geq 2$ **[3]**

(ii) Hence evaluate I_5 **[3]**

b. Find, by the method of cylindrical shells, the volume of the solid generated when the region bounded by the curve $y = x^2 + 1$, the line $x = 2$ and the co-ordinate axes is rotated about the line $x = 3$. **[5]**

c. Let θ be a real number and consider $(\cos\theta + i\sin\theta)^3$

(i) Prove $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ **[3]**

(ii) Find a similar expression for $\sin 3\theta$ **[1]**

QUESTION 6 **[15 MARKS]** **START A NEW BOOKLET**

a. Let α, β, δ be the roots of the equation $x^3 + qx + r = 0$, where q and r are integers. Write down, in terms of q and r, the cubic equation whose roots are:

(i) α^{-1}, β^{-1}, δ^{-1} **[2]**

(ii) α^2, β^2, δ^2 **[2]**

b. Consider the following statements about a polynomial $P(x)$.

Indicate whether each of the following statements is true or false. Give reasons for your answer.

(i) If $P(x)$ is even, then $P'(x)$ is odd. **[2]**

(ii) If $P'(x)$ is even, then $P(x)$ is odd. **[2]**

c. **(i)** Evaluate $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$ **[2]**

(ii) Show that for $n \geq 2$ and $0 \leq x \leq \frac{1}{2}$, then $1 \geq 1 - x^n \geq 1 - x^2$ **[2]**

(iii) If $n \geq 2$, explain carefully why $\frac{1}{2} \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^n}} \leq \frac{\pi}{6}$ **[3]**

QUESTION 7 **[15 MARKS]** **START A NEW BOOKLET** **MARKS**

a. Consider a sequence of numbers $a_1, a_2, a_3 \ldots$ where $a_1 = 2,\ a_2 = 3$ and $a_n = 3a_{n-1} - 2a_{n-2}$ for all $n \geq 3$. Use mathematical induction to prove that $a_n = 2^{n-1} + 1$ for all $n \geq 1$. **[5]**

b. A particle is projected from a height H above a horizontal plane with speed V at an angle of elevation θ to the horizontal.

(i) If the range of the particle in the horizontal plane is R, show that $gR^2 \sec^2\theta = 2V^2\left(R\tan\theta + H\right)$. **[4]**

(ii) If R_1 is the maximum value of R and θ_1 is the corresponding value of θ, prove that $R_1 = \frac{v}{g}\sqrt{v^2 + 2gH}$ and $\theta_1 = \tan^{-1}\left(\frac{v^2}{gR}\right)$ **[4]**

(iii) Show that $\tan 2\theta_1 = \frac{R_1}{H}$ **[2]**

QUESTION 8 **[15 MARKS]** **START A NEW BOOKLET**

a. Let $m = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

(i) Prove that $1 + m + m^2 + \ldots + m^6 = 0$ **[2]**

(ii) The complex number $\alpha = m + m^2 + m^4$ is a root of the quadratic equation $x^2 + ax + b = 0$, where a and b are real. The second root of the quadratic equation $x^2 + ax + b = 0$ is β. Express β in terms of positive powers of m. Justify your answer. **[2]**

(iii) Find the values of the coefficients a and b. **[2]**

(iv) Deduce that $\sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}$ **[2]**

b. Given that $p + q + r = 1$ and $p + q + r \geq 3\sqrt[3]{pqr}$ (where p, q, r are positive real numbers):

(i) Prove that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 9$ **[4]**

(ii) Hence, or otherwise, show $\left(\frac{1}{p} - 1\right)\left(\frac{1}{q} - 1\right)\left(\frac{1}{r} - 1\right) \geq 8$ **[3]**

END OF EXAMINATION

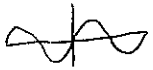
Question 1 Solutions Scots 08 Trial

1a) i) $|z| = \sqrt{2}$ ✓ ii) $\arg z = \frac{\pi}{4}$ ✓ iii) $z = \left[\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right]$

$z^{-6} = \frac{1}{8}\text{cis}\left(-\frac{3\pi}{2}\right)$ ✓

$= +\frac{1}{8}i$ ✓

b) $z\bar{z} + 2iz = 12 + 6i$, let $z = x + iy$



$(x+iy)(x-iy) + 2i(x+iy) = 12 + 6i$

$x^2 + y^2 + 2ix - 2y = 12 + 6i$ ✓

$2x = 6$ (equating imaginary parts)

$\boxed{x = 3}$ ✓

$\therefore 9 + y^2 - 2y = 12$

$y^2 - 2y - 3 = 0$ ✓ $\therefore z = 3 + 3i$ or $z = 3 - i$

$(y-3)(y+1) = 0 \therefore \boxed{y = 3 \text{ or } -1}$

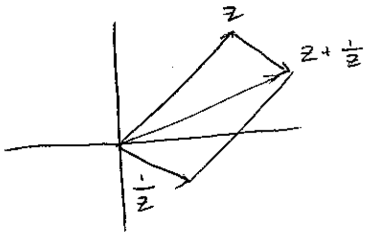
c) $z\bar{z} - 2(z + \bar{z}) = 5$, let $z = x + iy$

$x^2 + y^2 - 2(2x) = 5$

$x^2 - 4x + y^2 = 5$ ✓ ✓ $\therefore$ locus is a circle, centre 2, radius 3.

$(x-2)^2 + y^2 = 9$

d)



i) z must have $|z| = 1$

ii) z must have $|z| = 1$

and $\arg z = \frac{\pi}{4}$ or $-\frac{\pi}{4}$

$\left(\frac{3\pi}{4} \text{ or } -\frac{3\pi}{4}\right)$

i) $|z| = \left|\frac{1}{z}\right|$ ii) $z = \frac{1}{z}$ + $\left|z + \frac{1}{z}\right| = \left|z - \frac{1}{z}\right|$

$\frac{1}{2}$

✓×

Q1 cont...

e) when $n=1$, LHS = RHS

assume true for $n = k$

$(\cos\theta - i\sin\theta)^k = \cos(k\theta) - i\sin(k\theta)$

Prove true for $n = k+1$

$(\cos\theta - i\sin\theta)^k(\cos\theta - i\sin\theta)$

$(\cos k\theta - i\sin k\theta)(\cos\theta - i\sin\theta)$ by assumption ✓

$\cos k\theta\cos\theta - i\sin\theta\cos k\theta - i\sin k\theta\cos\theta - \sin\theta\sin k\theta$

$\cos k\theta\cos\theta - \sin\theta\sin k\theta - i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$ ✓

$\cos(k\theta + \theta) - i(\sin(k\theta + \theta))$

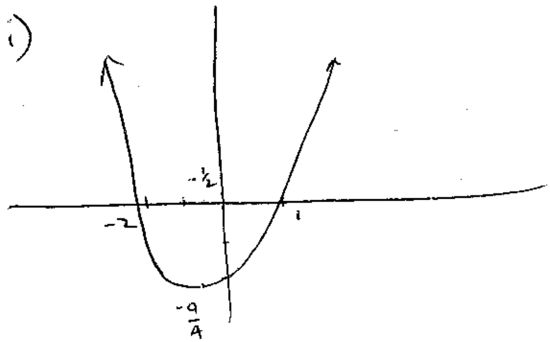
$\cos(k+1)\theta - i\sin(k+1)\theta$ as required. ✓

If result true for $n = k$, then true for $n = k+1$. Since true for $n = 1$, it is true for $n = 1 + 1 = 2$ and so on. Hence true for all positive integers.

Question 2

⊛ Note:

a) i)



Qu2

$-x^2(x^2+x-2)$
$= -x^4 - x^3 + 2x^2$

b) i) $y^2 = -x^2(x+2)(x-1)$

$y^2 = (-x^3 - 2x^2)(x-1)$

$= -x^4 + x^3 - 2x^3 + 2x^2$

$= -x^4 - x^3 + 2x^2$

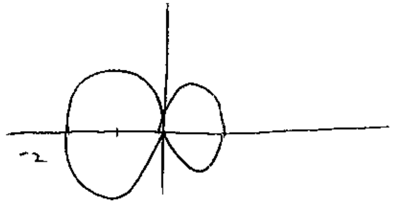
$2y \cdot \frac{dy}{dx} = -4x^3 - 3x^2 + 4x$

$\frac{dy}{dx} = \frac{-4x^3 - 3x^2 + 4x}{2y}$ ✓

$\frac{dy}{dx}$ is undefined for $y=0$

ie, $x = 0, 1, -2$ ✓

ii)



vertical tangents at $x = -1, 2$

tangent undefined at $x = 0$

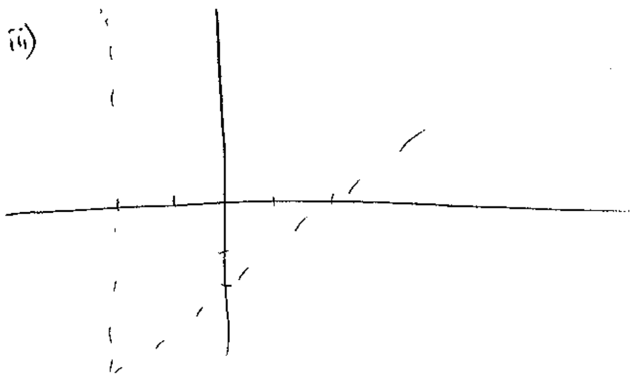
c) i) $f'(x) = x - 2 + \frac{3}{x+2}$

$= \frac{x^2 - 4 + 3}{x+2} = \frac{x^2-1}{x+2} = 0$ when $x = \pm 1$ ✓

ii) $x = -2$ (vertical asymptote) ✓

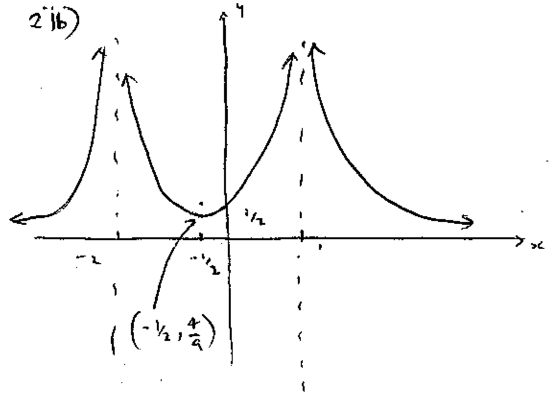
as $x \to \pm\infty$, $y \to x - 2$ $\therefore y = x - 2$, (oblique asymptote) ✓

iii)

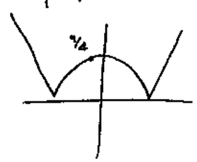


PTO.

2 i b)



$y = |f(x)|$



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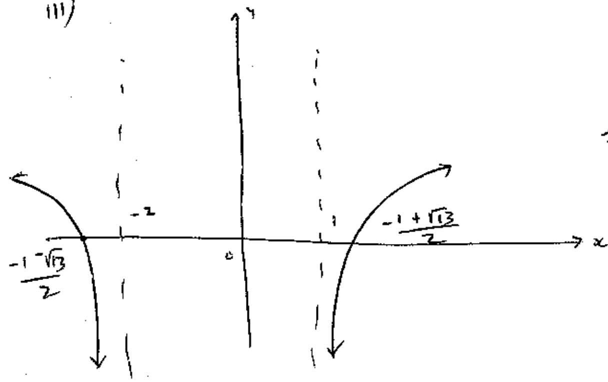
iii)



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show x intercepts

$(x+2)(x-1) = 1$

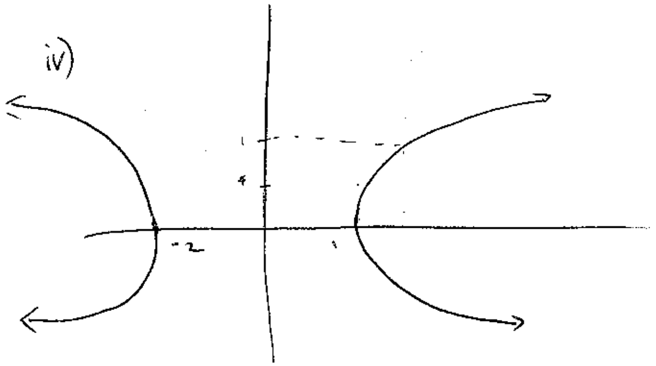
$x^2 + 2x - x - 2 = 1$

$x^2 + x - 3 = 0$

$x = \frac{-1 \pm \sqrt{1 - 4 \times -3}}{2}$

$x = \frac{-1 \pm \sqrt{13}}{2}$

iv)



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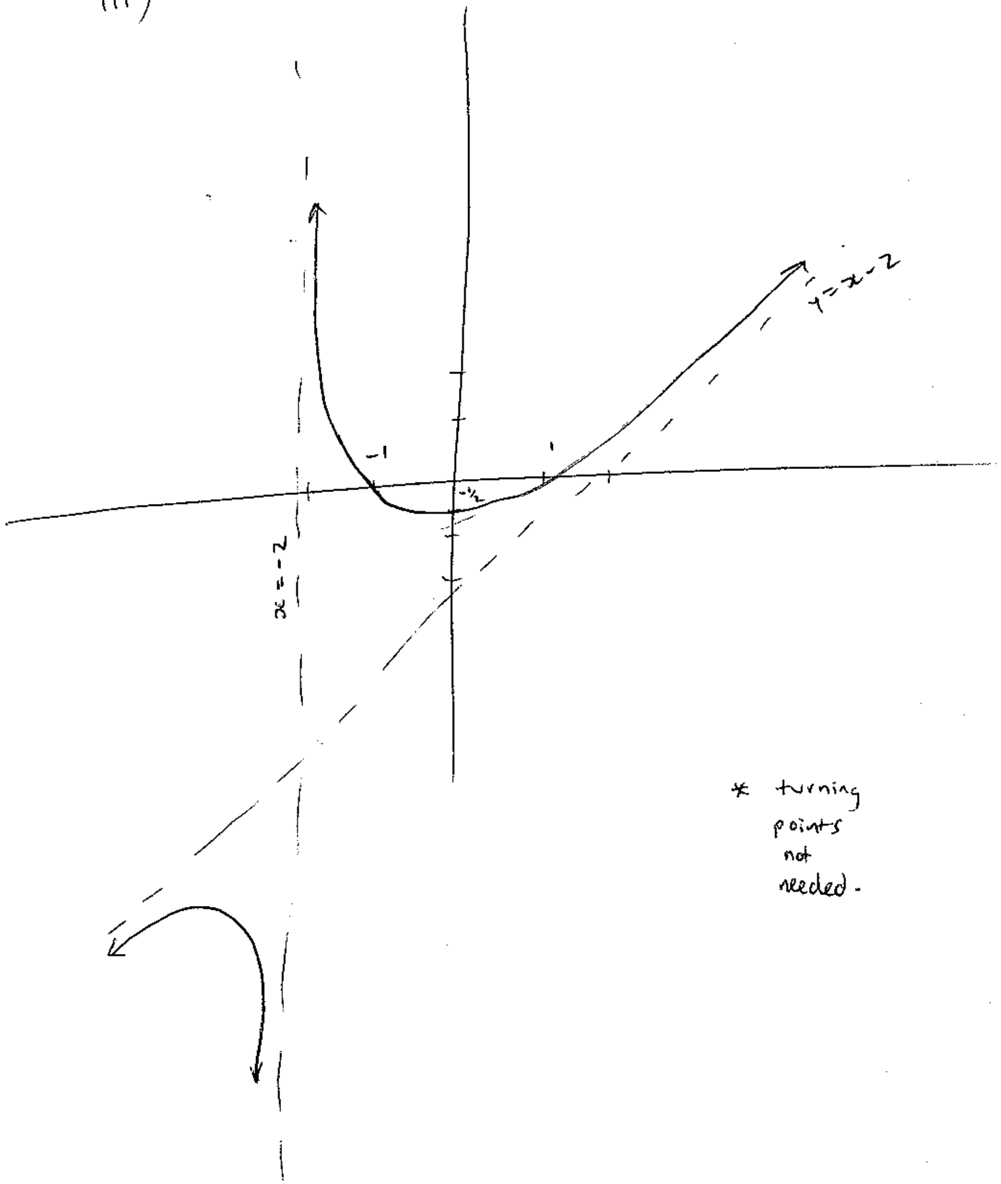
iii)



* turning points not needed.

Qu 3

a) $\int_0^{\frac{\pi}{4}} x \sin 2x \, dx = \left[\frac{-x}{2} \cdot \cos 2x \right]_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\pi/4} \cos 2x \, dx$ ✓

$u = x$

$u' = 1$

$v = -\frac{1}{2}\cos 2x$

$v' = \sin 2x$

$uv - vu'$

$= 0 + \frac{1}{2}\left[\frac{\sin 2x}{2} \right]_0^{\frac{\pi}{4}}$ ✓

$= \frac{1}{4}$ ✓

b) $\int x\sqrt{1-x} \, dx$

let $u = 1-x$

$du = -dx$

$\int -(1-u)(u^{\frac{1}{2}}) \, du$ ✓

$= \int -u^{1/2} + u^{3/2} \, du$

$= -\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2} + C$

$= -\frac{2}{3}\sqrt{(1-x)^3} + \frac{2}{5}\sqrt{(1-x)^5} + C$ ✓

c) $\int \frac{1}{x(1+x^2)} \, dx = \frac{a}{x} + \frac{bx+c}{1+x^2}$

$\therefore 1 = a(1+x^2) + x(bx+c)$

$0 = a+b$, $0 = c$, $1 = a$ ✓

$\therefore b = -1$ ✓

$\therefore \int \frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$

$= \ln x - \frac{1}{2}\ln(1+x^2) + C$ ✓

d) $\int \frac{dx}{\sqrt{x^2-4x+1}} = \int \frac{dx}{\sqrt{(x-2)^2-3}}$ (let $u = x-2$) $\frac{du}{dx} = 1$

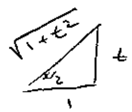
$= \int \frac{du}{\sqrt{u^2-3}}$

$= \ln\left(u + \sqrt{u^2-3}\right) + C$ (from $\int$ table)

$= \ln\left[x-2 + \sqrt{(x-2)^2-3}\right] + C$

e) The function does not exist at $x = 3$, which is within the range of the integration.

f) $\int_0^{\pi/3} \frac{\tan x}{1+\cos x}\,dx$, $t = \tan\frac{x}{2}$



$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2} \times 1 + t^2$

$dx = \frac{2}{1+t^2}\,dt$

$= \int_0^{1/\sqrt{3}} \frac{\frac{2t}{1-t^2}}{1 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$

$= \int_0^{1/\sqrt{3}} \frac{\frac{4t}{1-t^2}}{1+t^2+1-t^2}\,dt$

$= \int_0^{1/\sqrt{3}} \frac{\frac{4t}{1-t^2}}{2}\,dt = \int_0^{1/\sqrt{3}} \frac{2t}{1-t^2}\,dt$

$= \left[-\ln(1-t^2)\right]_0^{\frac{1}{\sqrt{3}}} = -\ln\left(\frac{2}{3}\right) = \ln\frac{3}{2}$

$\boxed{Q4}$

a) i) $x^2 + 4y^2 = 100$ or $\frac{x^2}{100} + \frac{y^2}{25} = 1$ $\therefore a = 10, b = 5$

$b^2 = a^2(1-e^2)$

$25 = 100(1-e^2)$

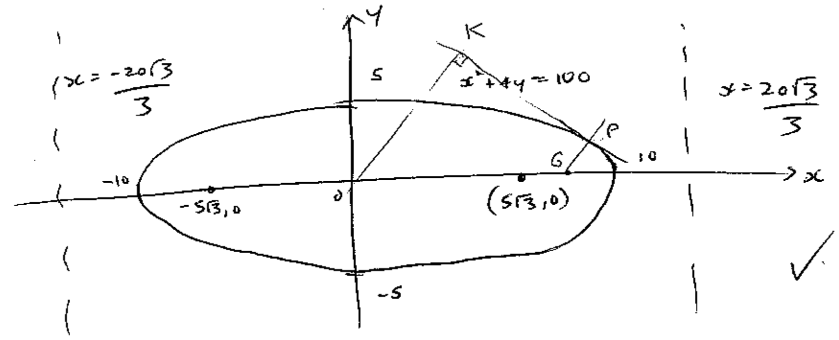
$\boxed{e = \sqrt{3}/2}$

Foci $= (\pm ae, 0)$

$= (\pm 5\sqrt{3}, 0)$

Directrices: $\pm\frac{a}{e} = x = \pm\frac{20\sqrt{3}}{3}$

ii)



iii) $x^2 + 4y^2 = 100$

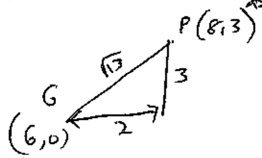
$2x + 8y\cdot\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x}{4y} = -\frac{2}{3}$ at $P(8,3)$

tangent: $y - 3 = -\frac{2}{3}(x-8)$ or $3y + 2x - 25 = 0$

normal: $y - 3 = \frac{3}{2}(x-8)$ or $3x - 2y - 18 = 0$

iv) Normal at meets x axis at $G(6,0)$



$\therefore PG = \sqrt{13}$

$OK = \frac{|-25|}{2^2+3^2} = \frac{25}{\sqrt{13}}$

$\therefore PG \cdot OK = 25 = 5^2 \Rightarrow$ (square of semi minor axis)

Qu4

b) $V = \dfrac{u}{4+ux}\ ms^{-1}$

$V^2 = \dfrac{u^2}{(4+ux)^2}$

$\frac{1}{2}V^2 = \frac{1}{2}u^2(4+ux)^{-2}$

$\ddot{x} = \dfrac{d}{dx}\left(\frac{1}{2}V^2\right) = -u^3(4+ux)^{-3}$

$= \dfrac{-u^3}{(4+ux)^3} = -V^3\ ms^{-2}$

c) i) $\dfrac{a(y)}{A} = \dfrac{(h-y)^2}{h^2}$

$\therefore a(y) = A \times \left(\dfrac{h-y}{h}\right)^2$

ii) $\Delta V = a(y)\ \Delta y$

$V = \dfrac{A}{h^2}\displaystyle\int_0^h (h-y)^2\ dy$

$= \dfrac{-A}{3h^2}\left[(h-y)^3\right]_0^h$

$= \dfrac{-A}{3h^2}(0-h^3)$

$= \dfrac{Ah}{3} = \dfrac{50\,000 \times 150}{3} = 25\,00{,}000\ m^3$

Qu5

a) i) $I_n = \displaystyle\int_0^{\pi} x^n \sin x\ dx$

$= \left[-x^n\cos x\right]_0^{\pi} + n\displaystyle\int_0^{\pi} \cos x \cdot x^{n-1}\ dx$

let $u = x^n$, $\dfrac{du}{dx} = nx^{n-1}$, $v' = \sin x$, $v = -\cos x$

$= \pi^n + n\left[\left[x^{n-1}\sin x\right]_0^{\pi} - \displaystyle\int_0^{\pi}(n-1)x^{n-2}\sin x\ dx\right]$

let $u = x^{n-1}$, $\dfrac{du}{dx} = (n-1)x^{n-2}$, $v' = \cos x$, $v = \sin x$

$= \pi^n + n\left[0 - (n-1)I_{n-2}\right]$

$= \pi^n - n(n-1)I_{n-2}$

ii) $I_5 = \pi^5 - 5(4)I_3$

$I_3 = \pi^3 - 3(2)I_1$

$u = x$, $u' = 1$, $v' = \cancel{\sin x}\ \cancel{\cos x}$, $v = -\cancel{\cos x}$

$I_1 = \displaystyle\int_0^{\pi} x\sin x\ dx$

$= [x\cos x]_0^{\pi} + \displaystyle\int_0^{\pi}\cos x$

$= \pi + [\sin x]_0^{\pi}$

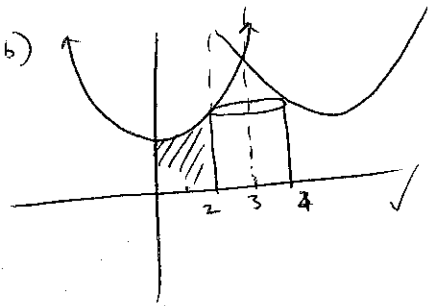
$= \pi$

$\therefore I_5 = \pi^5 - 20 \times [\pi^3 - 6(\pi)]$

$= \pi^5 - 20\pi^3 + 120\pi$

$r = 3 - x$ $\quad h = x^2+1$

b)



$V = 2\pi\displaystyle\int_0^2 (3-x)(x^2+1)\ \delta x$

$V = 2\pi\displaystyle\int_0^2 3x^2 + 3 - x^3 - x\ dx$

$V = 2\pi\left[x^3 + 3x - \dfrac{x^4}{4} - \dfrac{x^2}{2}\right]_0^2$

$V = 2\pi[8+6-4-2]$

$V = 16\pi\ u^3.$

Qu5

c) By de Moivre

$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta \cdot i\sin\theta + 3\cos\theta \cdot i^2\sin^2\theta + i^3\sin^3\theta$ ✓

$= \cos^3\theta - 3\cos\theta\sin^2\theta + i3\cos^2\theta\sin\theta - i\sin^3\theta$ ✓

Equating Real

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ ✓

ii) Equating imaginary

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ ✓

Qu6

a) i) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ will satisfy

$\left(\frac{1}{x}\right)^3 + q\left(\frac{1}{x}\right) + r = 0$ ✓

$1 + qx^2 + rx^3 = 0$ ✓

ii) $\alpha^2, \beta^2, \gamma^2$ will satisfy.

$(\sqrt{x})^3 + q(\sqrt{x}) + r = 0$

$\sqrt{x}(x + q) = -r$ ✓

$x(x+q)^2 = r^2$

$x(x^2 + 2xq + q^2) - r^2 = 0$

$x^3 + 2x^2q + xq^2 - r^2 = 0$ ✓

b) I) true, $P(x)$ will be in form $P(x) = ax^{2n} \dots$

$P'(x) = 2n\,a\,x^{2n-1} \dots$

leaving odd powers

II) false, due to constant term.

all terms odd except constant term.

Qu6

c) i) $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1} x\right]_0^{\frac{1}{2}} = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$ ✓

ii) For $n \geq 2$ and $0 \leq x \leq \frac{1}{2}$

$0 \leq x^n \leq x^2$ ✓

$1 \geq 1-x^n \geq 1-x^2$ ✓

$1 \geq \sqrt{1-x^n} \geq \sqrt{1-x^2}$

$1 > x^n > 0$

$1 - x^n > 0$

$\therefore 1 \geq 1 - x^n$

$x^2 \geq x^n$

$(0 \leq x \leq \frac{1}{2})$

$\therefore 1 - x^n \geq 1 - x^2$

iii) $1 \leq \frac{1}{\sqrt{1-x^n}} \leq \frac{1}{\sqrt{1-x^2}}$ ✓

$\int_0^{\frac{1}{2}} 1 \, dx \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^n}} \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$ ✓

$\frac{1}{2} \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^n}} \leq \frac{\pi}{6}$ ✓

7a

step 1 Prove true when $n=1$ and $n=2$

when $n=1$, $a_1 = 2^{1-1}+1 = 2$ ✓

when $n=2$, $a_2 = 2^{2-1}+1 = 3$ $\therefore$ true for $n=1$ + $n=2$

step 2 Assume true for $n=k$ and when $n = k-1$.

ie, $a_k = 2^{k-1}+1$ (*) and $a_{k-1} = 2^{k-2}+1$ (**)

Step 3 Prove true for $n=k+1$

ie, $a_{k+1} = \boxed{2^k + 1}$ required to prove.

As $a_n = 3a_{n-1} - 2a_{n-2}$

$\therefore a_{k+1} = 3a_k - 2a_{k-1}$ ✓

$= 3(2^{k-1}+1) - 2(2^{k-2}+1)$ from assumption (*) (**) ✓

$= 3 \times 2^{k-1} + 3 - 2 \times 2^{k-2} - 2$

$= 3 \times 2^{k-1} - 2^{k-1} + 1$ ✓

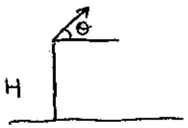
$= 2 \times 2^{k-1} + 1$

$= 2^k + 1$ ✓

$\therefore$ result true for $n=k+1$

Step 4 : conclusion.

Q7 b) i)



$\ddot{x} = 0$ $\qquad$ $\ddot{y} = -g$

$\dot{x} = v\cos\theta$ $\qquad$ $\dot{y} = v\sin\theta - gt$

$x = vt\cos\theta$

$y = vt\sin\theta - \frac{1}{2}gt^2 + H$ ✓ [for equations]

when $y = -H$

$\frac{1}{2}gt^2 - vt\sin\theta - H = 0$ ✓

Solving for t

$$t = \frac{v\sin\theta \pm \sqrt{v^2\sin^2\theta + 2gH}}{g} \quad (\text{disregard } -)$$

$$\therefore R = \frac{v\cos\theta\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2gH}\right)}{g} \quad ✓ \ (\text{sub } t \text{ into } x)$$

$$\frac{gR}{v\cos\theta} = v\sin\theta + \sqrt{v^2\sin^2\theta + 2gH}$$

$$\sqrt{v^2\sin^2\theta + 2gH} = \frac{gR}{v\cos\theta} - v\sin\theta$$

$$\cancel{v^2\sin^2\theta} + 2gH = \frac{g^2R^2}{v^2\cos^2\theta} - \frac{2gR\sin\theta}{\cos\theta} + \cancel{v^2\sin^2\theta}$$

$$2gH = \frac{g^2R^2\sec^2\theta}{v^2} - 2gR\tan\theta$$

$$g^{\cancel{2}}R^2\sec^2\theta = 2v^2\cancel{g}H + 2\cancel{g}R\tan\theta v^2$$

$$gR^2\sec^2\theta = 2v^2(H + R\tan\theta) \quad -(1) \quad ✓$$

ii)

$$2v^2(R\tan\theta + H) = gR^2\sec^2\theta$$

$$2v^2\left(R\sec^2\theta + \tan\theta \cdot R\frac{d}{d\theta}\right) = gR^2 2\sec^2\theta\tan\theta + g\sec^2\theta \cdot R^2\frac{d}{d\theta}$$

Note: $\left[\frac{d}{d\theta}(\cos\theta)^{-2} = -2(\cos\theta)^{-3} \cdot -\sin\theta = 2\sec^2\theta\tan\theta\right]$

$$2v^2\left(R\sec^2\theta + \tan\theta\frac{dR}{d\theta}\right) = gR^2 2\sec^2\theta\tan\theta + g\sec^2\theta\, 2R\frac{dR}{d\theta}$$

$$\frac{dR}{d\theta}\left(2v^2\tan\theta - 2gR\sec^2\theta\right) = 2R\sec^2\theta\left(gR\tan\theta - v^2\right)$$

$\therefore \frac{dR}{d\theta} = 0$ when $gR\tan\theta - v^2 = 0$ or $\boxed{\tan\theta = \frac{v^2}{gR}}$ $-(2)$

sub (2) in (1) for max value

$$2v^2\left(\frac{v^2}{g} + H\right) = gR^2(1 + \tan^2\theta)$$

($\sec^2\theta$)

$$= gR^2 + \cancel{g}R^2\left[\frac{v^2}{\cancel{g}R}\right]^2$$

$$2v^2(v^2 + gH) = g^2R^2 + v^4$$

$$2v^4 + 2v^2gH = g^2R^2 + v^4$$

$$g^2R^2 = v^4 + 2v^2gH$$

$$R_1^2 = \frac{v^4}{g^2} + \frac{2v^2H}{g}$$

$$R_1^2 = \frac{v^2}{g^2}(v^2 + 2gH) \quad (*) \quad \& \quad \theta_1 = \tan^{-1}\left(\frac{v^2}{gR}\right)$$

$$R_1 = \frac{v}{g}\sqrt{v^2 + 2gH}$$ ✓

iii)

$$\tan 2\theta_1 = \frac{2\tan\theta_1}{1 - \tan^2\theta_1} = \frac{2\cdot\left(\frac{v^2}{gR_1}\right)}{1 - (v^2/gR_1)^2} = \frac{2gR_1v^2}{g^2R_1^2 - v^4}$$

$$= \frac{2gR_1v^2}{2v^2gH} = \frac{R_1}{H}$$

from (*)

Qu 8

a) i) $m = \text{cis}\,\frac{2\pi}{7}$

$m^7 = \left(\text{cis}\,\frac{2\pi}{7}\right)^7$

$m^7 = \text{cis}\,2\pi = 1$

$\therefore$ m is a root of $x^7 - 1 = 0$

$\therefore$ $(x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) = 0$

Since $m \neq 1$

m satisfies $x^6 + x^5 + x^4 + \dots + x + 1 = 0$

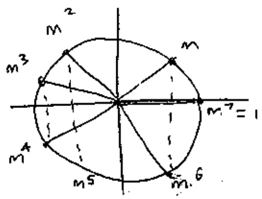
$\therefore$ $1 + m + m^2 + \dots + m^6 = 0$

ii) since coefficients of $x^2 + ax + b = 0$ are real and $\alpha = m + m^2 + m^4$ is a complex root, then $\bar{\alpha}$ is also a root.

$\therefore$ $\beta = \bar{\alpha} = \overline{m + m^2 + m^4}$

$= \bar{m} + \overline{m^2} + \overline{m^4}$

$= m^6 + m^5 + m^3$



iii) $\alpha + \beta = -a$

$a = -(\alpha + \beta)$

$a = -(m + m^2 + m^3 + m^4 + m^5 + m^6)$ from ii)

$\boxed{a = 1}$

$\alpha\beta = b$

$= (m + m^2 + m^4)(m^3 + m^5 + m^6)$

$= m^4(1 + m + m^3)(1 + m^2 + m^3)$

$= m^4(1 + m^2 + m^3 + m + m^3 + m^4 + m^3 + m^5 + m^6)$

$= m^4(1 + m + m^2 + m^3 + m^4 + m^5 + m^6 + 2m^3)$

$= m^4(0 + 2m^3)$

$= 2m^7$

$\boxed{b = 2}$

iv) from iii) $x^2 + x + 2 = 0$

$x = \frac{-1 \pm \sqrt{1-8}}{2}$

$x = \frac{-1 \pm i\sqrt{7}}{2}$

$\therefore$ $\text{Im}(\alpha) = \pm\frac{\sqrt{7}}{2}$

However, $\alpha = m + m^2 + m^4$

+ from ii) diagram $\text{Im}(m + m^2 + m^4) > 0$

$\therefore$ $\text{Im}(\alpha) = \frac{\sqrt{7}}{2}$

Now, $\text{Im}(\alpha) = \text{Im}(m + m^2 + m^4)$

$= \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}$

$= \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} - \sin\frac{\pi}{7}$

$= \frac{\sqrt{7}}{2}$ as required.

Qu 8 b)

i) $1 \geq 3\sqrt[3]{abc}$

$\frac{1}{3} \geq \sqrt[3]{abc}$

$\therefore$ $\boxed{\frac{1}{\sqrt[3]{abc}} \geq 3}$

Using 2nd given result $a + b + c \geq 3\sqrt[3]{abc}$

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3\sqrt[3]{\frac{1}{a}\frac{1}{b}\frac{1}{c}}$

$= 3\,\frac{1}{\sqrt[3]{abc}}$

$\geq 3 . 3$

≥ 9

ii) consider $\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right)$

$= \frac{1-a}{a} \times \frac{1-b}{b} \times \frac{1-c}{c}$

$= \frac{b+c}{a} \times \frac{a+c}{b} \times \frac{a+b}{c}$

$\geq \frac{2\sqrt{bc} \cdot 2\sqrt{ac} \cdot 2\sqrt{ab}}{abc}$

≥ 8

$b + c \geq 2\sqrt{bc}$

$a + c \geq 2\sqrt{ac}$

$a + b \geq 2\sqrt{ab}$

[note $= 8$ when $a = b = c = \frac{1}{3}$]