THE SCOTS COLLEGE

2009
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS EXTENSION 2

General Instructions

- All questions are of equal value
- 5 minutes reading time
- Working time - 3 hours
- Write using blue or black pen
- Board approved calculators may be used
- Start a new booklet for each question
- All necessary working should be shown in every question

Standard Integrals Table is attached

TOTAL MARKS: 120

WEIGHTING: 40 %

QUESTION 1 **[15 MARKS]** **START A NEW BOOKLET** **MARKS**

a. If $z_1 = 1+3i$, $z_2 = 1-i$,

(i) Find in the form $a+ib$, where a and b are real, the numbers $z_1 \times z_2$ and $\dfrac{z_1}{z_2}$. **[2]**

(ii) On an Argand Diagram the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ represent the complex numbers $z_1\ z_2$ and $\dfrac{z_1}{z_2}$ respectively (where z_1 and z_2 are given above). **[1]**

Show this on an Argand Diagram, giving the coordinates of A and B.

(iii) From your diagram, deduce that $\dfrac{z_1}{z_2} - z_1\, z_2$ is real. **[1]**

b. Given that $z = \sqrt{3} - i$,

(i) Express z in modulus-argument form. **[1]**

(ii) Hence, evaluate the following in the form $x+iy$:

(α) z^5 **[1]**

(β) $\left(\overline{z}\right)^5$ **[1]**

(γ) $\dfrac{z^5}{\left(\overline{z}\right)^5}$ **[1]**

c. On an Argand diagram, sketch the locus of z if:

(i) $|z+3| < |z-1-4i|$ **[2]**

(ii) $4\arg\dfrac{z-1}{z+3} = \pi$ **[2]**

d. Solve the equation $\dfrac{1}{z+1} + \dfrac{1}{1-2i} = 1$, **[3]**

QUESTION 2 **[15 MARKS]** **START A NEW BOOKLET** **MARKS**

a. Let $f(x)=\frac{4}{x}-x$. Provide separate half page sketches of the graphs of the following:

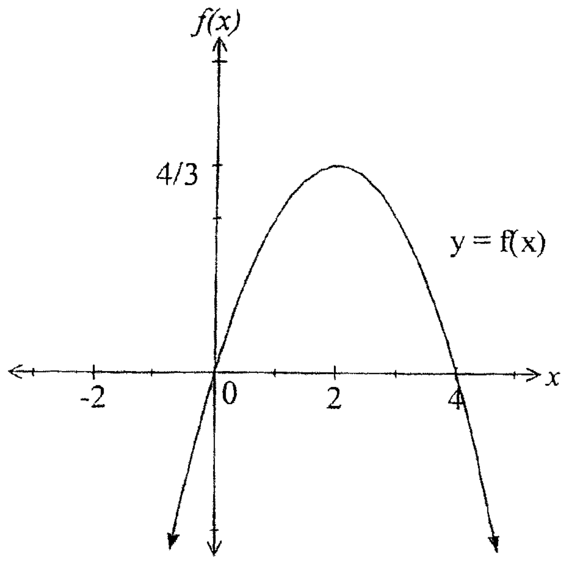
(i) $y=f(x)$ **[2]**

(ii) $y=\sqrt{f(x)}$ **[2]**

(iii) $y=e^{f(x)}$ **[2]**

Label each graph carefully.

b.



(i) Use the diagram to find the values of a,b,c given $f(x)=ax^2+bx+c$ **[2]**

(ii) Solve $-1\le f(x)\le 1$ **[3]**

(iii) Hence or otherwise sketch

α $y=\ln[f(x)]$ **[2]**

β $y=\cos^{-1}\left[f(x)\right]$ **[2]**

QUESTION 3 **[15 MARKS]** **MARKS**

a. A particle moves in a straight line. Prove that its acceleration at any instant can be expressed as: **[2]**

(i) $v\dfrac{dv}{dx}$

(ii) $\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)$

where x denotes its position and v its velocity at any instant.

b. A particle moving in a straight line from the origin is subject to a resisting force which produces a retardation of kv^3 where v is the speed at time t and k is a constant. If u is the initial speed, x is the distance moved in time t,

(i) Show that $v = \dfrac{u}{kux+1}$ **[3]**

(ii) Deduce that $kx^2 = 2t - \dfrac{2x}{u}$. **[2]**

c. A particle of mass m kg is projected upwards with initial speed U m/s. The particle is subject to gravity and a resistive force of magnitude mkv Newtons, where v m/s is the speed of the particle at any instant and k is a constant, k>0.

(i) Explain why $\ddot{\ }\quad \_ \quad \cdot kv$, where g is the acceleration due to gravity. **[2]**

(ii) Prove that the particle will reach a maximum height after T seconds **[3]**

given by $T = \dfrac{1}{k}\ln,\left(\dfrac{g+kU}{g}\right)$.

(iii) Prove that the maximum height reached is $\dfrac{1}{k}\left(U - gT\right)$ metres. **[3]**

QUESTION 4 **[15 MARKS]** **START A NEW BOOKLET** **MARKS**

a. The area bounded by $y = e^{2x}$, the x-axis, the y-axis, and the line $x = 2$, is rotated about the y-axis. Use the method of cylindrical shells to find the volume of the solid formed. **[5]**

b. The area bounded by the parabola $y = 2x^2$ and the line $y = 8$ is rotated about the line $y = 10$. Use the slice (washer) method to find the volume of the solid formed. **[4]**

c. The horizontal base of a solid is the circle $x^2 + y^2 = 16$. Vertical cross-sections perpendicular to the x-axis are regions bounded by a parabola and its latus rectum, with the latus rectum lying on the base and the vertex of the parabola vertically above the latus rectum. **[6]**

(i) Prove that the area bounded by a parabola of focal length a units and its latus rectum is $\frac{8a^2}{3}$ unit^2.

(ii) Hence find the volume of the solid.

QUESTION 5 **[15 MARKS]** **START A NEW BOOKLET**

a. Prove that the 5 roots of $z^5 = -1$ are:

(i) $z = cis\frac{(2k+1)\pi}{5}$ for $k = 0,1,2,3,4$. **[3]**

(ii) Hence show that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$ **[2]**

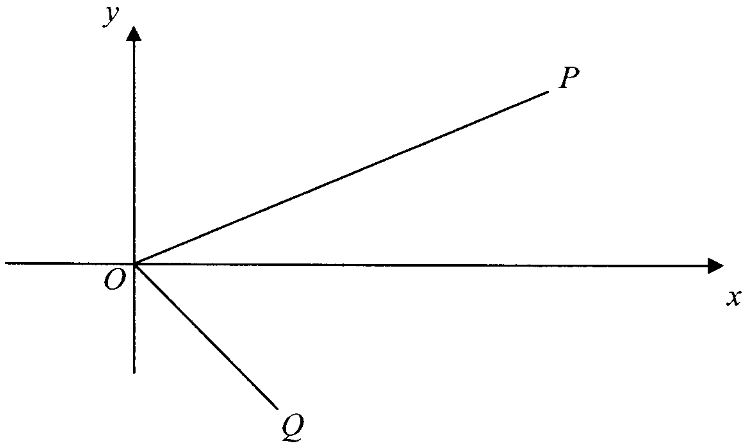
b. If the polynomial $P(x) = x^3 + qx + r$ has roots α, β and γ form an equation with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. **[3]**

c. If α, β and γ are the roots of $x^3 - 4x^2 + 5x + 3 = 0$ evaluate $\alpha^3 + \beta^3 + \gamma^3$. **[3]**

d. If the three roots of the equation $8x^3 - 36x^2 + 38x - 3 = 0$ are in arithmetic progression, find these roots in ascending order of magnitude. **[4]**

QUESTION 6 **[15 MARKS]** **START A NEW BOOKLET** **MARKS**

a.



In the Argand Diagram above, point P corresponds to complex number z. The triangle OPQ is a right-angled triangle and $OP = 3OQ$.

(i) What is the complex number that corresponds to point Q? **[1]**

(ii) $QOPR$ is a rectangle. Write down the complex number that corresponds to R. **[2]**

b. **(i)** Prove the identity $\cos(a-b)x - \cos(a+b)x = 2\sin ax \sin bx$ **[2]**

(ii) Hence find $\int \sin 3x \sin 2x dx$ **[2]**

c. If $u_1 = 8$, $u_2 = 20$ and $u_n = 4u_{n-1} - 4u_{n-2}$ for $n \geq 3$.

(i) Determine u_3 and u_4. **[1]**

(ii) Prove by induction that $u_n = (n+3)2^n$ for $n \geq 1$. **[4]**

d. If $ax^3 + bx^2 + d = 0$ $(a, b, d \neq 0)$ has a double root, show that $27a^2d + 4b^3 = 0$. **[3]**

QUESTION 7 **[15 MARKS]** **START A NEW BOOKLET** **MARKS**

a. $\int \frac{dx}{\sqrt{4-9x^2}}$ **[2]**

b. Find: **[4]**

(i) $\int xe^x dx$

(ii) $\int \cos^5 x \sin^3 x dx$

c. Find the value of: $\int_0^{\sqrt{2}} \frac{x^3}{x^2+4} dx$ **[2]**

d. Decompose $\frac{2(x+1)}{(x-1)(2x-1)}$ into partial fractions and hence find $\int_2^5 \frac{2(x+1)}{(x-1)(2x-1)} dx$ **[3]**

e. **[4]**

(i) Prove that if $u_n = \int_0^{\frac{x}{2}} x^n \sin x \, dx,\ n \geq 0,$ then $u_n + n(n-1)u_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$, $n \geq 2$, and

(ii) deduce the value of $\int_0^{\frac{n}{2}} x^4 \sin x \, dx$.

QUESTION 8 [15 MARKS] **START A NEW BOOKLET** **MARKS**

a. Use mathematical induction to show that:

$\sqrt{1}+\sqrt{2}+\sqrt{3}+$ .... $+\sqrt{n} \leq \frac{4n+3}{6}\sqrt{n}$ for all integers $n \geq 1$. **[5]**

b. The normal at $P\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ meets the curve again at Q.

(i) Prove that the equation of the normal is $t^3x - ty = ct^4 - c$ **[2]**

(ii) Find the coordinates of Q. **[2]**

(iii) A line from P through the origin meets the hyperbola again at R. Prove that PR is perpendicular to QR. **[3]**

(iv) If M is the midpoint of PQ, find the equation of the locus M. **[3]**

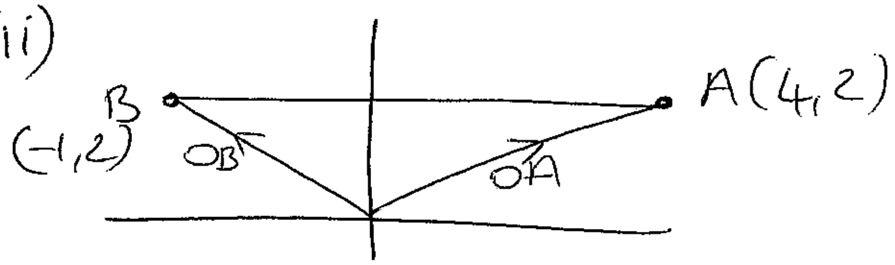
Question 4

a) i) $z_1 z_2 = (1+3i)(1-i)$

$= 4+2i$

iii) $\frac{z_1}{z_2} = \frac{1+3i}{1-i} \times \frac{1+i}{1+i} = -1+2i$

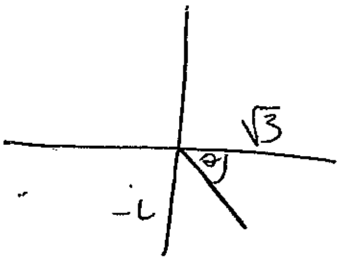
ii)



$\frac{z_1}{z_2} - z_1 \times z_2$ is the same as BA

$= -5$

b)



$\theta = \frac{\pi}{6}$ $(\sqrt{3})^2 + 1^2 = 4$

$2\text{cis}\left(-\frac{\pi}{6}\right)$

ii) $z^5 = 2^5 \text{cis}\left(-\frac{5\pi}{6}\right)$

$= 32\,\text{cis}\left(\frac{-5\pi}{6}\right)$

$= 32\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right]$

$= -16(\sqrt{3}+i)$

$\bar{z} = 2\text{cis}\left(\frac{\pi}{6}\right)$

$(\bar{z})^5 = 32\left[\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right]$

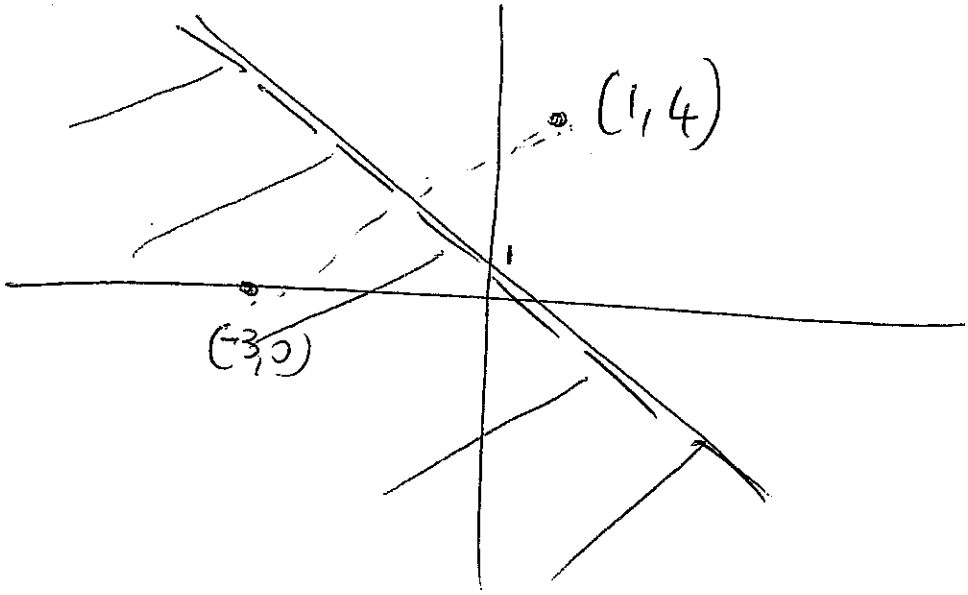
$= 16(\sqrt{3}+i)$

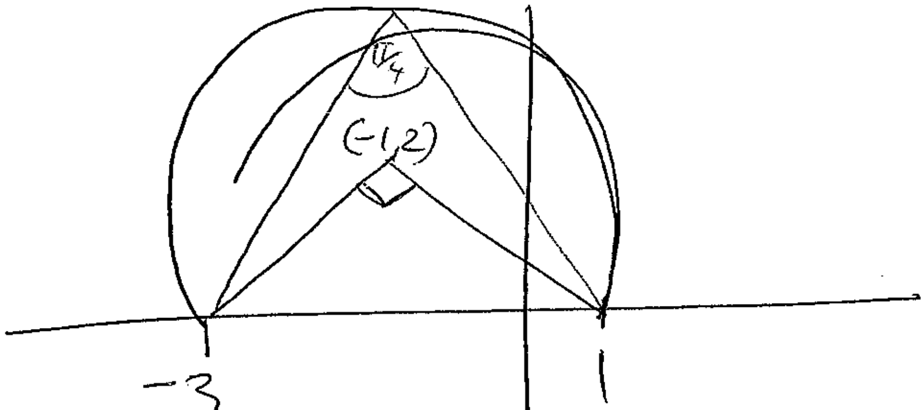
$\frac{z^5}{(\bar{z})^5} = \frac{-16(\sqrt{3}+i)}{16(-\sqrt{3}+i)}$ → $\frac{\sqrt{3}+i}{-\sqrt{3}+i} \times \frac{-\sqrt{3}-i}{-\sqrt{3}-i}$

$\frac{1}{2} + i\frac{\sqrt{3}}{2}$

Question 5

c)





d) $\frac{1}{z+1} + \frac{1}{1-2i} = 1$

$\frac{1}{(x+1)+yi} + \frac{1}{1-2i} = 1$

$\frac{1}{(x+1)+yi}\frac{(x+1)-yi}{(x+1)-yi} + \frac{1}{1-2i}\frac{(1+2i)}{(1+2i)} = 1$

$\frac{(x+1)-yi}{(x+1)^2+y^2} + \frac{(1+2i)}{5} = 1$

$$\frac{1}{z+1} + \frac{1}{(1-2i)} = 1$$

$$\frac{1}{z+1} = 1 - \frac{(1+2i)}{5}$$

$$= 1 - \tfrac{1}{5} + \tfrac{2}{5}i$$

$$\frac{1}{z+1} = \frac{-4+2i}{5}$$

~~$-\frac{5}{4}$~~

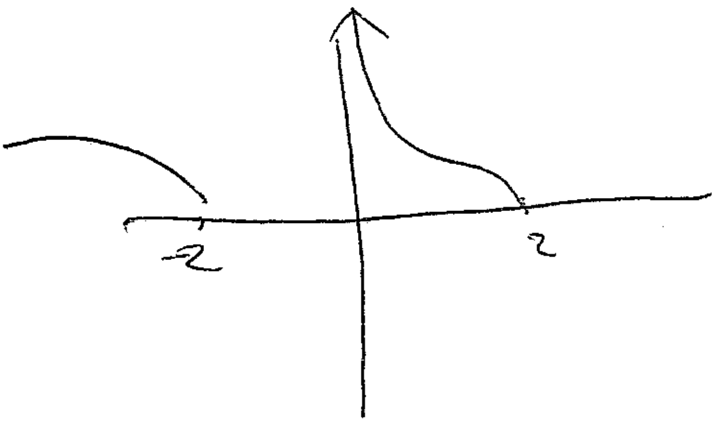
$$z+1 = \frac{5}{-4+2i} \times \frac{(-4-2i)}{(-4-2i)}$$

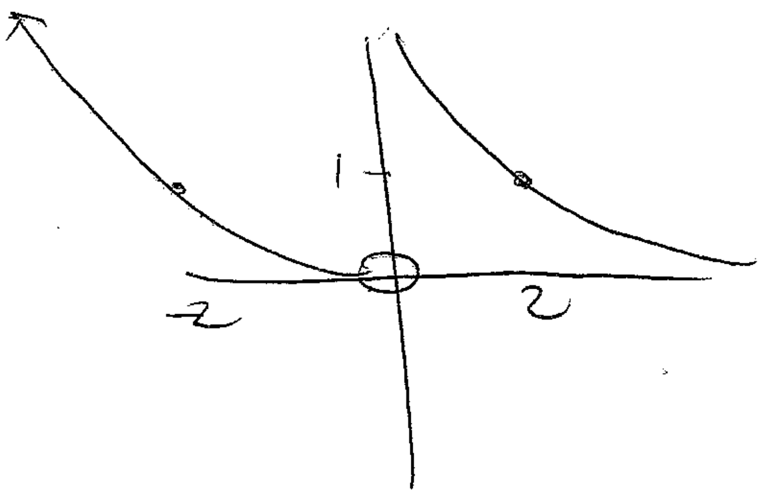
$$z+1 = \frac{-20-10i}{20}$$

$$z+1 = -1 - \tfrac{1}{2}i$$

$$\boxed{z = -2 - \tfrac{1}{2}i}$$

Question 8





Question 8

i) $y = ax^2 + bx + c$

$c = 0$ By observation

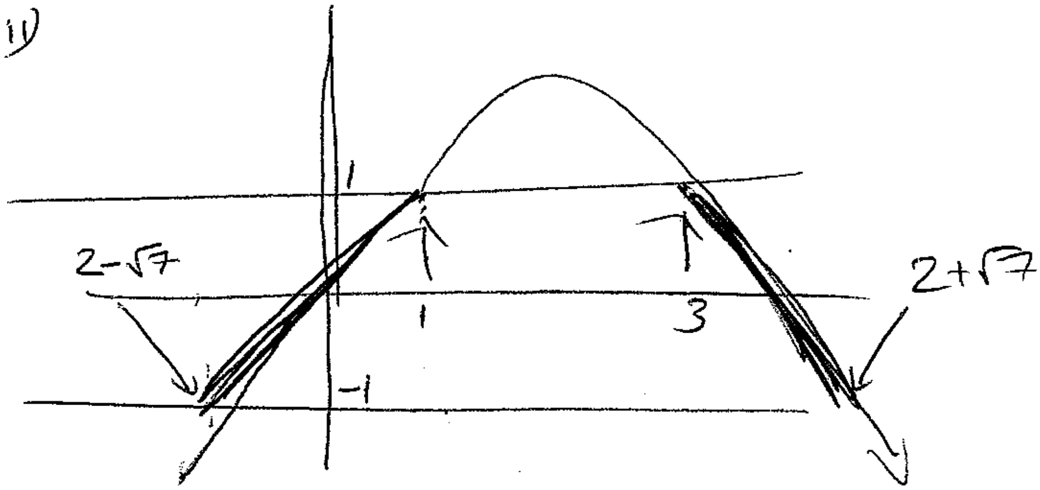
$x = 4 \quad y = 0 \quad 16a + 4b = 0$

$x = 2 \quad y = 4/3 \quad 2a + b = 2/3$

$a = -1/3 \qquad b = 4/3$

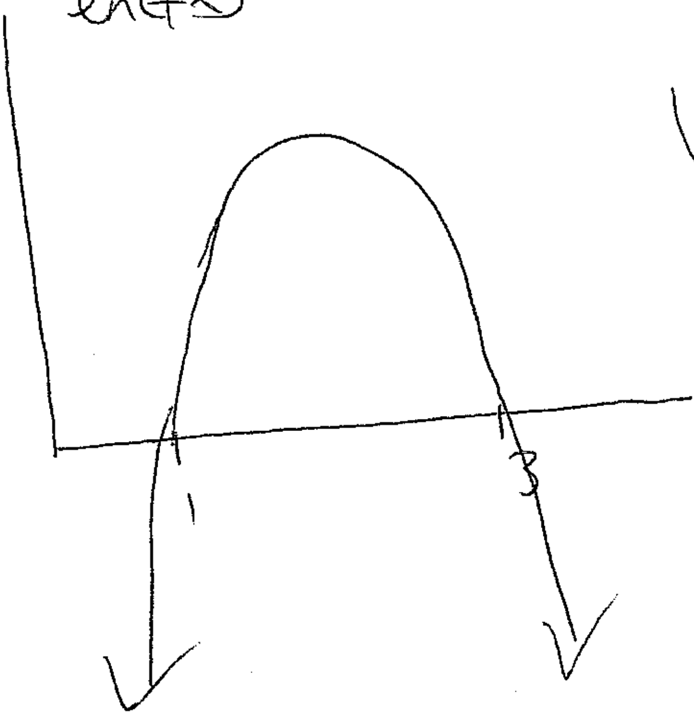
$y = -\frac{x}{3}(x-4)$

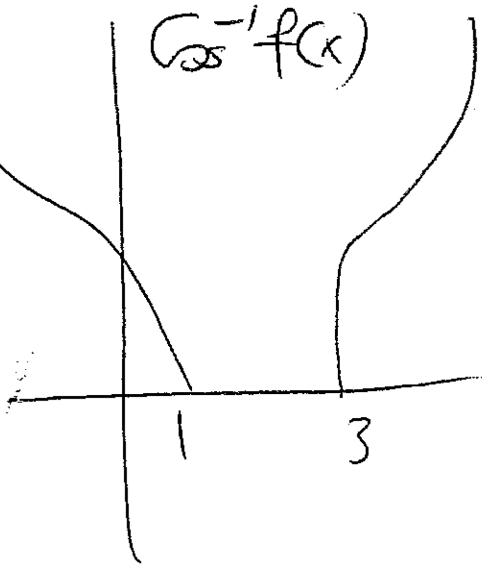
ii)



$2-\sqrt{7} \le x \le 1$ or $3 \le x \le 2+\sqrt{7}$

iii) $\ln[f(x)]$





Question B

a) $\frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt}$

$= \frac{dx}{dt}\frac{dv}{dx}$

$\ddot{x} = v\frac{dv}{dx}$

(1)

$\frac{d}{dx}\frac{1}{2}v^2 = \frac{1}{2}\times 2v\times\frac{dv}{dx}$

$= v\frac{dv}{dx}$

$= \ddot{x}$

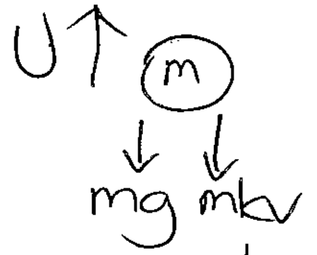
(1)

b)

i) Force $= m\ddot{x}$

$\therefore m\ddot{x} = -mg - mkv$ (2)

$= -g - kv$



ii) $\frac{dv}{dt} = -g - kv$

$\frac{dt}{dv} = \frac{-1}{g+kv}$

$t = -\frac{1}{k}\ln(g+kv) + C$ (4)

when $t=0$ $v=U$

$\therefore 0 = -\frac{1}{k}\ln(g+kU) + C$

$\therefore C = \frac{1}{k}\ln(g+kU)$

$\therefore t = \frac{1}{k}\left[\ln(g+kU) - \ln(g+kv)\right]$

$t = \frac{1}{k}\ln\left[\frac{g+kU}{g+kv}\right]$

At max height $v=0$ $T_{\text{MAX HEIGHT}} = \frac{1}{k}\ln\left(\frac{g+kU}{g}\right)$

(b) iii)

$$V\frac{dV}{dx} = -g - kV$$

$$\frac{dV}{dx} = \frac{-g-kV}{V}$$

$$\therefore \frac{dx}{dV} = \frac{V}{-g-kV}$$

$$-g-kV \;\overline{\big)\, V}\quad -1/k$$

$$V + \frac{g}{k}$$

$$-g/k$$

(4)

$$\therefore \frac{dx}{dV} = -1/k + \frac{g/k}{g+kV}$$

$$= 1/k\left[\frac{g}{g+kV} - 1\right]$$

$$~~V~~ = 1/k\left[g/k \ln(g+kV) - V\right] + C$$

$x=0$ $V=U$

$$\therefore C = -1/k\left[g/k \ln(g+kU) - U\right]$$

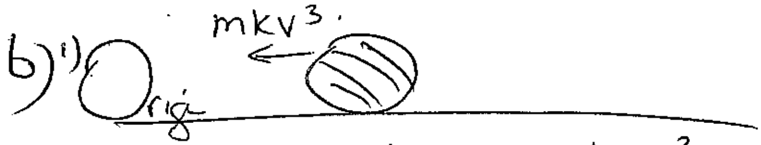
$$x = 1/k\left[g/k \ln(g+kV) - g/k \ln(g+kU) - V + U\right]$$

max height $V=0$ ↓ careful

$$\therefore X_{MAX} = 1/k\left[U - g/k \ln\left[\frac{g+kU}{g}\right]\right]$$

$$X_{MAX} = 1/k\left[U - gT\right]$$

b) i)



$t=0 \quad x=0$
$v=U$

$$\therefore \ddot{x} = -kv^3$$

$$v\frac{dv}{dx} = -kv^3$$

$$\therefore \frac{dv}{dx} = -kv^2$$

$$\therefore \frac{dx}{dv} = \frac{-1}{k}v^{-2}$$

$$\therefore x = +\frac{1}{k}v^{-1} + C$$

(2)

$$\therefore C = -\frac{1}{k}U^{-1}$$

$$x = \frac{1}{k}\left[\frac{1}{v} - \frac{1}{U}\right]$$

$$xk + \frac{1}{U} = \frac{1}{v}$$

$$\frac{kUx + 1}{U} = v$$

$$\therefore v = \frac{U}{kUx+1}$$

ii)

$$\frac{dx}{dt} = \frac{U}{kUx+1}$$

$$\frac{dt}{dx} = \frac{kUx}{U} + \frac{1}{U}$$

(2)

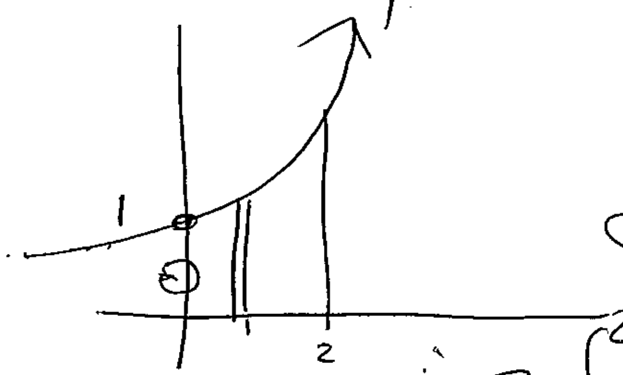
$$t = \frac{kx^2}{2} + \frac{x}{U} + C$$

$t=0 \quad x=0 \therefore C=0$

$$2t = kx^2 + \frac{2x}{U}$$

$$kx^2 = 2t - \frac{2x}{U}$$

Question 24



Height = y
Radius = x.

$$\delta V = 2\pi x . y \delta x$$

$$\therefore 2\pi \int_0^2 x y \, \delta x$$

$$I = 2\pi \int_0^2 x e^{2x} \, \delta x$$

$u = x \qquad dv = e^{2x}$

$du = 1 \qquad v = \frac{1}{2} e^{2x}$

$$I = 2\pi \left[\left[\frac{x e^{2x}}{2} \right]_0^2 - \int_0^2 \frac{e^{2x}}{2} \, \delta x \right]$$

$$= 2\pi \left[\frac{2e^4}{2} - \left[\frac{e^{2x}}{4} \right]_0^2 \right]$$

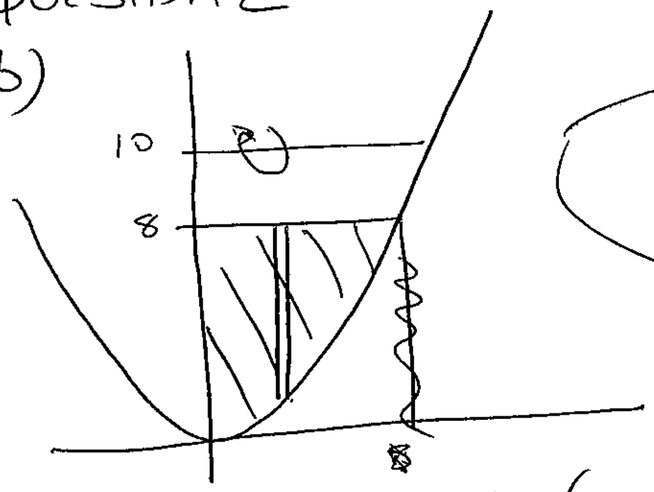
$$= 2\pi \left[e^4 - \frac{e^4}{4} + \frac{1}{4} \right]$$

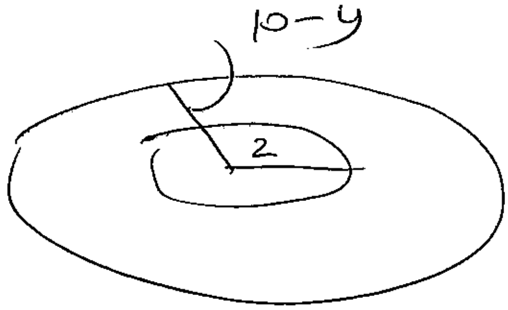
$$\text{Volume} = \frac{2\pi}{4} \left[3e^4 + 1 \right]$$

$$\text{Volume} = \frac{\pi}{2} \left[3e^4 + 1 \right]$$

Question 2

b)





$8 = 2x^2$

$4 = x^2$

$x = 2$

$$dV = \pi\left[(10-y)^2 - 2^2\right]dx$$

$$V = \int_0^2 \pi\left[100 - 20y + y^2 - 4\right]dx$$

$$V = \pi\int_0^2 96 - 20y + y^2\, dx$$

$$= \pi\int_0^2 96 - 40x^2 + 4x^4\, dx$$

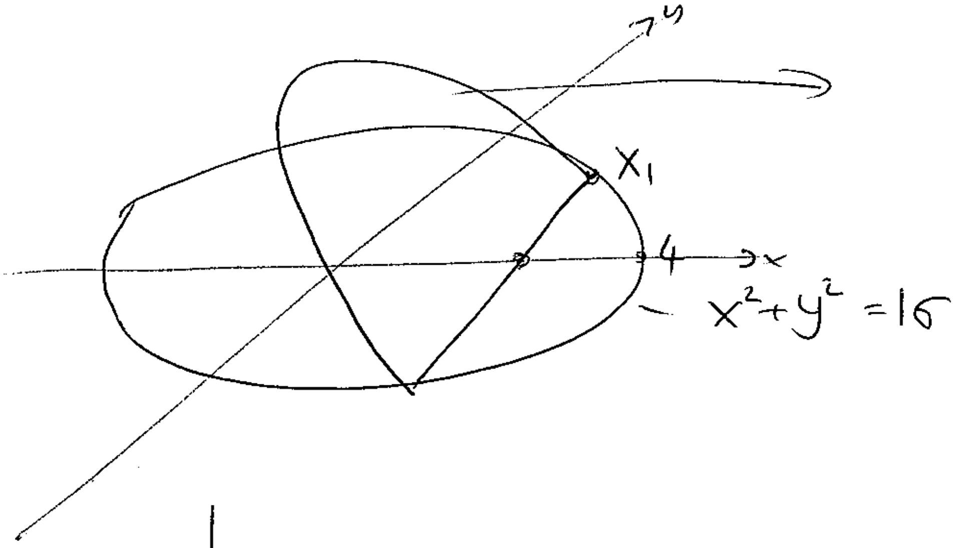
$$= \pi\left[96x - \frac{40}{3}x^3 + \frac{4}{5}x^5\right]_0^2$$

$$= \pi\left[96\times2 - \frac{40}{3}\times8 + \frac{4}{5}\times32\right]$$

$$= \frac{\pi}{15}\left[2880 - 1600 + 384\right]$$

$$\text{Volume} = \frac{\pi}{15}(1664)\ U^3$$

c)



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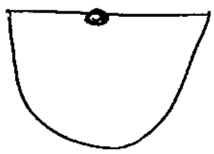
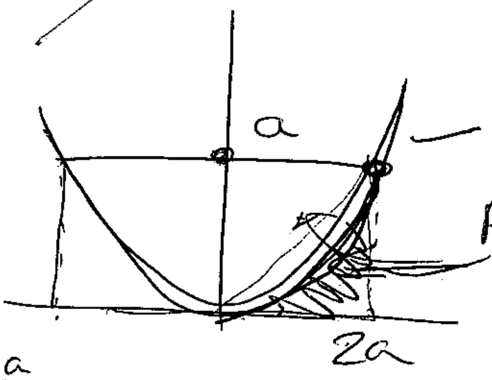


(6)



$x^2 = 4ay$

$\text{Area} = \int_0^{2a} y\,dx$

$= \int_0^{2a} \frac{x^2}{4a}\,dx$

$= \left[\frac{x^3}{12a}\right]_0^{2a} = \frac{2a^2}{3}$

$\int_0^a x\,dy$

$\int_0^a 2\sqrt{a}\,y^{1/2}\,dy$

$2\sqrt{a}\times\left[\frac{2}{3}y^{3/2}\right]_0^a$

$\frac{4}{3}\sqrt{a}\times a^{3/2}$

$\frac{4}{3}a^2$

$\therefore \text{Area} = \frac{8}{3}a^2\ u^3$

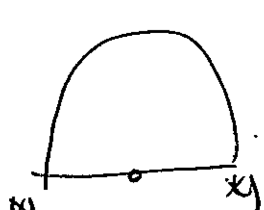
$a\ \boxed{2a^2} - \frac{2a^2}{3} = \frac{a^3}{3}$ (box width 2a)

$\therefore \text{Parabola} = \frac{2a^2}{3}$

ii)

~~$dA = \frac{8a^2}{3}$~~

$dV = \frac{8}{3}\frac{y^2}{4}\times dx$



$\therefore a = \frac{y}{2}$

$\therefore V = \frac{2\times 8}{12}\int_0^4 y^2\,dx$

$= \frac{4}{3}\int_0^4 16 - x^2\,dx$

$V = \frac{4}{3}\left[16x - \frac{x^3}{3}\right]_0^4$

$V = \frac{4}{3}\left[64 - \frac{64}{3}\right]$

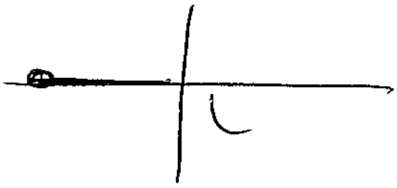
$= \frac{4}{9}\times 128$

$= \frac{512}{9}\ u^3$

Question 5

a)

i) $z^5 = 1$



$z^5 = 1 \operatorname{cis} \pi$

$\therefore z_1 = 1 \operatorname{cis} \pi/5$,

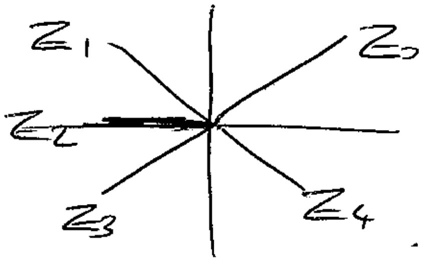
$\frac{2\pi}{5}$

$\therefore z_2 = 1 \operatorname{cis}\left(\frac{\pi}{5} + \frac{2\pi}{5}\right)$

$z_3 = 1 \operatorname{cis}\left(\frac{\pi}{5} + \frac{2\pi}{5} + \frac{2\pi}{5}\right)$

$\therefore z = \operatorname{cis} \frac{(2k+1)\pi}{5}$ for $k = 0, 1, 2, 3, 4$

ii) Roots of $z^5 + 1 = 0$ are z_0, z_1, z_2, z_3, z_4



$z_3 = -1$ $\quad z_0 + z_1 + z_2 + z_3 + z_4 = 0$

$z_0 + z_1 + z_3 + z_4 = 1$

~~z_0~~ $+ z_3 = \bar{z}_0 \quad \bar{z}_4 = \bar{z}_1$

$\therefore z_0 + z_1 + z_3 + z_4 = 1$

$\bar{z}_0 + z_0 + \bar{z}_1 + z_1 = 1$

$2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} = 1$

$\therefore \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$

Question 3

b)

$P(x) = x^3 + qx + r$

Roots α, β, γ

$\therefore P(\frac{1}{x}) = \left(\frac{1}{x}\right)^3 + q\left(\frac{1}{x}\right) + r$

$\frac{1}{x^3} + \frac{q}{x} = -r$

$rx^3 + qx^2 + 1 = 0$

c) $x^3 - 4x^2 + 5x + 3 = 0$

Roots are α, β, γ

d)

$\therefore (x^{1/3})^3 - 4(x^{1/3})^2 + 5(x^{1/3}) + 3 = 0$

$x - 4(x^{2/3}) + 5(x^{1/3}) + 3 = 0$

$x + 3 = 4(x^{2/3}) - 5(x^{1/3})$

$(x+3) = x^{1/3}(4x^{1/3} - 5)$

1
1 1
1 2 1
1 3 3 1

$x^3 + 9x^2 + 27x + 27 = x[64x +$

$(\alpha+\beta+\gamma)^3$

$(\alpha+\beta+\gamma)(\alpha+\beta+\gamma)$

$(\alpha^2+2\alpha\beta+2\alpha\gamma+\beta^2+2\beta\gamma+\gamma^2)(\alpha+\beta+\gamma)$

$\alpha^3+2\alpha^2\beta+2\alpha^2\gamma+\alpha\beta^2+2\alpha\beta\gamma+\alpha\gamma^2 \quad \beta^3+2\beta^2\gamma+$

$\alpha\beta \qquad +2\alpha\beta^2+2\alpha\beta\gamma+$

$(A+B+C)(A+B+C)$

$A^2+AB+AC+BA+B^2+BC+CA+CB+C$

$[A^2+B^2+C^2+2AB+2BC+2AC]$

$\cancel{A^3}+AB^2+AC^2+\cancel{2A^2B}+2ABC+\cancel{2A^2C}$

$\cancel{A^2B}+\cancel{B^3}+C^2B+2AB^2+2B^2C+2ABC$

$\cancel{A^2C}+B^2C+\cancel{C^3}+2ABC+2BC^2+2AC^2$

$A^3+B^3+C^3+3A^2B+3A^2C+3$

$\lambda\rho \cdot \frac{\lambda\rho}{\Lambda\rho} =$

$\frac{\lambda\rho}{\Lambda\rho}\frac{\lambda\rho}{\cancel{\lambda}\rho} = \frac{\cancel{\lambda}\rho}{\Lambda\rho\Lambda} \quad (b$

Question 3

α, β, γ

or

$\alpha - d, \alpha, \alpha + d$

$\therefore$ Sum of roots $3\alpha = \frac{36}{8}$

$\therefore \alpha = \frac{3}{2}$

Pairs $(\alpha - d)\alpha + (\alpha + d)\alpha + (\alpha - d)(\alpha + d)$

$\alpha^2 - \alpha d + \alpha^2 + \alpha d + \alpha^2 - d^2$

$3\alpha^2 - d^2 = \frac{38}{8}$

$\frac{27}{4} - d^2 = \frac{38}{8}$

$d^2 = \frac{54}{8} - \frac{38}{8}$

$d^2 = \frac{16}{8}$

$d^2 = 2$

$\therefore d = \sqrt{2}$

$\frac{3}{2} - \sqrt{2}, \frac{3}{2}, \frac{3}{2} - \sqrt{2}$

Check

$\left(\frac{3}{2} - \sqrt{2}\right)\left(\frac{3}{2} + \sqrt{2}\right)$

$\frac{9}{4} - 2$

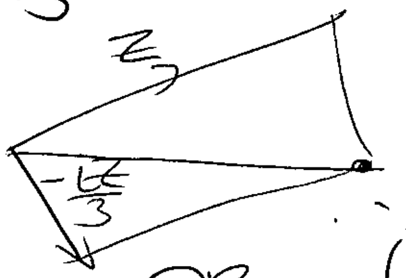
$\frac{1}{4} \times \frac{3}{2}$

$\frac{3}{8}$ ✓

Question 5

a) $OP \perp OQ$

$\therefore -\frac{1}{3} i z = OQ$



$OR = OP + OQ$

$OR = z - \frac{1}{3} i z$

$OR = (1 - \frac{1}{3} i) z$

b) i) $\cos(ax - bx) = \cos ax \cos bx + \sin ax \sin bx$

$\cos(ax + bx) = \cos ax \cos bx - \sin ax \sin bx$

Subtract. Cos cancel
Sin add

$\therefore$ $2 \sin ax \sin bx$

ii) $\int \sin 3x \sin 2x \, dx = \frac{1}{2} \int 2 \sin 3x \sin 2x \, dx$

$= \frac{1}{2} \int \cos(3-2)x - \cos(3+2)x \, dx$

$= \frac{1}{2} \int \cos x - \cos 5x \, dx$

$= \frac{1}{2} \left[\sin x - \frac{1}{5} \sin 5x \right]$

$= \frac{\sin x}{2} - \frac{\sin 5x}{10} + C$

Question 4

c) i) $U_3 = 4U_2 - 4U_1 = 48$

$U_4 = 4U_3 - 4U_2 = 112$

ii) Test for $n=1$, $n=2$

$n=1 \quad U_1 = 4\times 2 = 8$ ✓

$n=2 \quad U_2 = 5\times 4 = 20$ ✓

Assume true for k and k+1 $\quad U_k = (k+3)2^k \quad U_{k+1} = (k+4)2^{k+1}$

Prove $n = k+2 = (k+5)2^{k+2}$

$$U_{k+2} = 4U_{k+1} - 4U_k$$
$$= 4(k+4)2^{k+1} - 4(k+3)2^k$$
$$= 4.2^k\{2(k+4) - k+3\}$$
$$= 2^2.2^k\{k+5)$$
$$= 2^{k+2}(k+5)$$

If true f $n=k$, $n=k+1$ and true for $n=k+2$

If true for $n=1,2$.

∴ By induction true for $n=3$ . . .

for all integer $n \geq 1$.

Question 4

d)

$$ax^3 + bx^2 + d = 0$$

Double root

$\therefore$ $f'(x) = 0$ will be root

$$3ax^2 + 2bx = 0$$

$$x(3ax + 2b) = 0$$

Two options $x = 0$

$$x = \frac{-2b}{3a}$$

$$P(0) \neq 0 \therefore P\left(\frac{-2b}{3a}\right) = 0$$

$$-\frac{a \times 8b^3}{27a^3} + \frac{b\,4b^2}{9a^2} + d = 0$$

$\times 27a^2$

$$-8b^3 + 12b^3 + 27a^2d = 0$$

$$27a^2d + 4b^3 = 0$$

Question ~~6~~

a) $\int \frac{dx}{\sqrt{4-9x^2}}$

$\frac{1}{3}\int \frac{dx}{\sqrt{4/9 - x^2}}$

$I = \frac{1}{3}\sin^{-1}\frac{3x}{2}$

b) $\int xe^x dx$

$u = x \quad dv = e^x$

$du = 1 \quad v = e^x$

$I = [xe^x - \int e^x dx]$

$I = xe^x - e^x$

$\int \cos^5 x \sin^3 x dx$

$\cos^5 x(1-\cos^2 x)\sin x dx$

$\int \cos^5 x \sin x - \cos^7 x \sin x dx$

$I = \frac{1}{6}\cos^6 x - \frac{1}{8}\cos^8 x$

c) $\int_0^{\sqrt{2}} \frac{x^3}{x^2+4} dx$

$$\begin{array}{r} x \\ x^2+4 \overline{) x^3} \\ x^3 + 4x \\ \hline -4x \end{array}$$

$x - \frac{4x}{x^2+4}$

$I = \int_0^{\sqrt{2}} x - \frac{4x}{x^2+4} dx$

$= \left[\frac{x^2}{2} - 2\ln(x^2+4)\right]_0^{\sqrt{2}}$

$= 1 - 2\ln 6 + 2\ln 4$

$= 1 + 2\ln 2/3$

Question 6

d

$$\frac{2(x+1)}{(x-1)(2x-1)} = \frac{A}{(x-1)} + \frac{B}{(2x+1)}$$

$$2(x+1) = A(2x+1) + B(x-1)$$

$x=1 \qquad 4 = 3A \quad \therefore \quad A = 4/3$

$x = -1/2 \qquad 1 = -\frac{3}{2}B \qquad B = -2/3$

$$\int_2^5 \frac{2(x+1)}{(x-1)(2x-1)} = \int_2^5 \frac{4}{3(x-1)} - \int_2^5 \frac{2}{3(2x+1)}$$

$$= \left[\frac{4}{3}\ln(x-1) - \frac{1}{3}\ln(2x+1)\right]_2^5$$

$$= \frac{1}{3}\left[4\ln 4 - \ln(11) - 4\ln 1 + \ln 5\right]$$

$$= \frac{1}{3}\left[\ln\frac{256\times 5}{11}\right]$$

Question 6

e) $U_n = \int_0^{\frac{\pi}{2}} x^n \sin x$

$u = x^n \quad dv = \sin x$

$du = nx^{n-1} \quad v = -\cos x$

$U_n = \left[-x^n \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} nx^{n-1} \cos x \, dx$ (the bracket term is crossed out → 0)

$u = nx^{n-1} \quad dv = \cos x$

$du = n(n-1)x^{n-2} \quad v = \sin x$

$U_n = \left[nx^{(n-1)} \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n(n-1) x^{n-2} \sin x$ (arrow: $x^{n-2}\sin x \to U_{n-2}$)

$U_n + U_{(n-2)} = n\left(\frac{\pi}{2}\right)^{n-1} \quad n \geq 2$

$\int_0^{\frac{\pi}{2}} x^4 \sin x = 4\left(\frac{\pi}{2}\right)^3 - \int_0^{\frac{\pi}{2}} x^2 \sin x$

$= 4\left(\frac{\pi}{2}\right)^3 - 2\left(\frac{\pi}{2}\right) + \int_0^{\frac{\pi}{2}} \sin x \, dx$

$= \frac{\pi}{2} - \pi + \left[-\cos x\right]_0^{\frac{\pi}{2}}$

$I = 1 - \frac{\pi}{2}$

Question 8

$$\sqrt{1}+\sqrt{2}+\sqrt{3}+ \quad +\sqrt{n} \leq \frac{4n+3}{6}\sqrt{n} \quad n\geq 1$$

Step 1 $n \geq 1$ $\sqrt{1} \leq \frac{7}{6}\sqrt{1}$ ✓

Step 2 $\sqrt{1}+\sqrt{2}+ \sqrt{k} \leq \frac{4k+3}{6}\sqrt{k}$

Prove $\sqrt{1}+\sqrt{2}+\cdots\sqrt{k}+\sqrt{k+1} \leq \frac{4(k+1)+3}{6}\sqrt{k+1}$

$$\frac{4k+3}{6}\sqrt{k} + \sqrt{k+1} \leq \frac{4k+7}{6}\sqrt{k+1}$$

$$\frac{4\cancel{k}+7}{6}\sqrt{k+1} - \frac{4k+3}{6}\sqrt{k} - \sqrt{k+1} < 0$$

$$\frac{4k+1}{6}\sqrt{k+1} - \frac{4k+3}{6}\sqrt{k} \leq 0$$

Since $k \geq 1$ $\sqrt{k+1} > \sqrt{k}$.

$$\frac{4k+1}{6}\sqrt{k+1} - \frac{4k+3}{6}\sqrt{k+1}$$

$$-\frac{1}{3}\sqrt{k+1}$$

Since $\sqrt{k+1} > 0$

$$-\frac{1}{3}\sqrt{k+1} < 0.$$

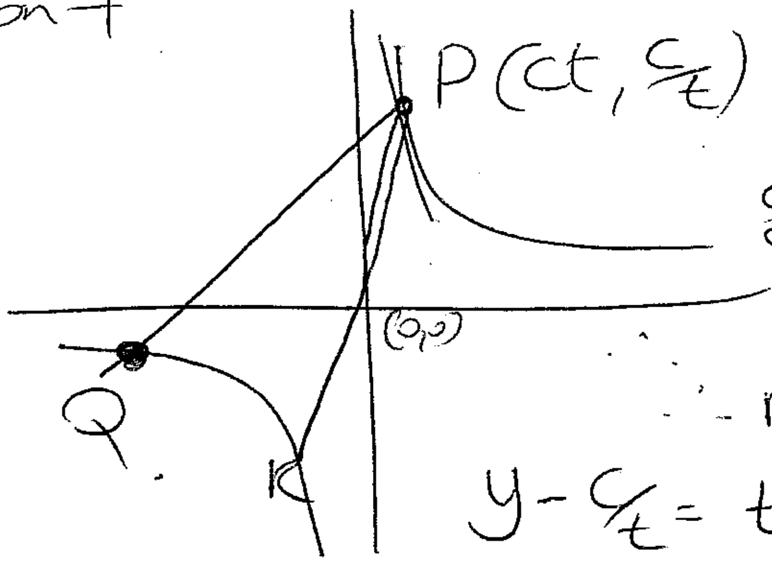
Statement is true.

$n = 1, 2$

True for $k = 1$

By Mathematical Induction

Question 7



$y = \frac{c^2}{x}$

$\frac{dy}{dx} = -\frac{c^2}{x^2}$

$m_{TAN} = -1/t^2$

$\therefore m_{NORMAL} \ t^2$

$y - c/t = t^2(x - ct)$

$yt - c = t^3x - ct^4$

$t^3x - ty = ct^4 - c$

ii) Sub $y = \frac{c^2}{x}$ into $t^3x - ty = ct^4 - c$

$t^3x - \frac{tc^2}{x} = ct^4 - c$

$t^3x^2 - (ct^4 - c)x - tc^2 = 0$

Product of roots $\frac{tc^2}{t^3} = ct \times Q_x$

$Q_x = -\frac{c}{t^3}$ $\quad Q_y = -ct^3$ $\quad [-c/t^3, -ct^3]$

iii By symmetry $R = (-ct, -c/t)$

$m_{QR} = \frac{-ct^3 - -c/t}{-\frac{c}{t^3} - -ct}$

$= \frac{\frac{c - ct^4}{t}}{\frac{ct^4 - c}{t^3}}$

$= \frac{-1}{t^2}$ $\perp$ (almost)!

$m_{PR} = \frac{c/t - -c/t}{ct - -ct}$

$= \frac{\frac{2c}{t}}{2ct}$

$= \frac{1}{t^2}.$

$P\left(ct, \frac{c}{t}\right) \quad Q\left(-\frac{c}{t^3}, -ct^3\right)$

$$\left(\frac{ct - \frac{c}{t^3}}{2}, \frac{\frac{c}{t} - ct^3}{2}\right)$$

$$\left(\frac{ct^4 - c}{2t^3}, \frac{c - ct^4}{2t}\right)$$

$$\left(\frac{ct^4 - c}{2t^3}, -\frac{(ct^4 - c)}{2t}\right)$$

$$x \qquad\qquad y$$

$$xy = t^2$$

Rectangular Hyperbola