The Scots College

HSC Mathematics Extension 2

Trial

August 2010

Name:______________________

General Instructions

- Working time : 3 hours + 5 minutes Reading time.
- Write using blue or black pen
- Board approved calculators may be Used (Non Graphic)
- All necessary working should be shown in every question
- Standard Integrals Table attached

TOTAL MARKS: 120

Attempt Questions 1 - 8
All questions are of equal value

WEIGHTING: 40 %

Answer each question in a SEPARATE writing booklet.

Question 1 (Marks 15) Use a SEPARATE writing booklet.

a) Evaluate [2]

$$\int_0^1 \frac{2\,dx}{\sqrt{2-\ x^2}}$$

b) Find [2]

$$\int \frac{\tan x}{\ln(\cos x)}\ dx$$

c) Evaluate [3]

$$\int_0^1 sin^{-1}x\ dx$$

d) [4]

i) Find the real numbers $a,\ \ b,\ \ and\ \ c$ such that

$$\frac{1+\ 4\,x}{(x^2+\ 1)(4-\ \ x)} \equiv \frac{a}{4-x} + \frac{bx+c}{x^2+\ 1}$$

ii) Hence evaluate

$$\int_0^2 \frac{1+4x}{(x^2+1)(4-x)}\ dx$$

e) By using the substitution $x = 2\cos\theta$ or otherwise evaluate [4]

$$\int_{-2}^0 \sqrt{4-\ x^2}\ dx$$

Question 2 (Marks 15) Use a SEPARATE writing booklet.

a) Express $\left(\frac{1+i}{1-i}\right)^3$ in the form $a + ib$; where a and b are real numbers [2]

b) i) Write $\frac{5-i}{2-3i}$ in the form $x + iy$, where x and y are real numbers. [2]

ii) Write $\frac{5-i}{2-3i}$ in the modulus-argument form. [3]

iii) Calculate $\left(\frac{5-i}{2-3i}\right)^4$. [1]

c) Sketch the region on the Argand diagram where the inequalities [3]

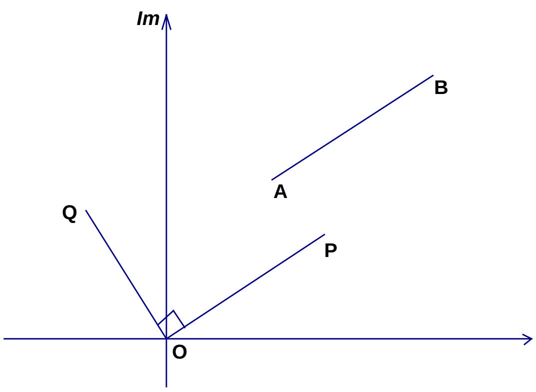
$$|z - \bar{z}| < 4 \quad \text{and} \quad |z + 1| > 1$$

hold simultaneously.

d) In the Argand diagram below, $AB = OP = OQ$, $OP \parallel AB$ and $OP \perp OQ$. [4]
A represents the complex number $5 + 3i$ and B represents $11 + 5i$. Copy this diagram into your answer scripts and find the complex numbers represented by the point :

i) P

ii) Q



Question 3 (Marks 15) Use a SEPARATE writing booklet.

a) i) Determine whether $f\,(x) = \frac{x}{x^2 - 1}$ is an odd or even function. [6]

ii) Sketch the graph of $y = f(x)$.

iii) Using the graph of $y = f(x)$, sketch on separate axes, the graphs of

(α) $y = f(-x)$

(β) $y = |f(x)|$

(γ) $y = [f(x)]^2$

b) Tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b} = 1$ at points $P(x_1, y_1)$ and $Q(x_2, y_2)$ intersect at T. M is the mid-point of PQ. [6]

i) Given that the tangent to the ellipse at P has equation $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ write down the equation of the tangent to the ellipse at Q.

ii) Show that the line $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{xx_2}{a^2} + \frac{yy_2}{b^2}$ passes through T and M.

iii) Deduce that $O, T\ and\ M$ are collinear.

iv) If $\angle PTQ$ is a right angle, show that $\frac{x_1x_2}{a^4} + \frac{y_1y_2}{b^4} = 0$

c) i) If $a > b > 0$, sketch the curve and shade the region $\int_b^a \sqrt{a^2 - x^2}\ dx$. [3]

ii) By using your diagram, or otherwise, show that

$$\int_b^a \sqrt{a^2 - x^2}\ dx = \frac{a^2}{2} cos^{-1}\left(\frac{b}{a}\right) - \frac{b}{a}\sqrt{a^2 - x^2}$$

Question 4 (Marks 15) Use a SEPARATE writing booklet.

a) i) Sketch the graph of the curve $y = x + e^{-x}$ showing clearly the coordinates of any turning points and the equations of any asymptotes. [2]

ii) The region in the first quadrant between the curve $y = x + e^{-x}$ and the line $y = x$ and bounded by the line $x = 1$ is rotated through one complete revolution about the y-axis. Use the method of cylindrical shells to show that the volume V of the solid of revolution is given by [3]

$$V = 2\pi \int_0^1 xe^{-x}\ dx$$

iii) Find the volume of the solid of revolution. [3]

b) Let $f(x) = x^2(x^2 - 2)$. The tangent to the curve $y = f(x)$ at the point A with x coordinate α meets the curve again at B. [7]

i) Show the tangent AB has equation $y = 4\alpha(\alpha^2 - 1)x + \alpha^2(2 - 3\alpha^2)$.

ii) Deduce that $x^2(x^2 - 2) = 4\alpha(\alpha^2 - 1)x + \alpha^2(2 - 3\alpha^2)$ has real roots $\alpha,\ \alpha,\ \beta,\ \gamma$ for some $\beta,\ \gamma$.

iii) For $\alpha \neq 0$, find $\beta + \gamma$ and $\beta\gamma$ in terms of α and write down a quadratic equation with roots $\beta,\ \gamma$.

iv) Find the possible values of α.

Question 5 (Marks 15) Use a SEPARATE writing booklet.

a) If $u_1 = 12,\ u_2 = 30$ and $u_n = 5u_{n-1} - 6u_{n-2}$ for $n \geq 3$. [5]

(i) Determine u_3 and u_4 .

(ii) Show that $u_n = 2 \times 3^n + 3 \times 2^n$ for $n = 1$ **and** $n = 2$.

(iii) If $u_k = 2 \times 3^k + 3 \times 2^k$ and $u_{k+1} = 2 \times 3^{k+1} + 3 \times 2^{k+1}$,
where k is a positive integer, prove that

$$u_{k+2} = 2 \times 3^{k+2} + 3 \times 2^{k+2}.$$

b) (i) Show that $\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$ for all real values of θ . [7]

(ii) Use the result above to :

(α) find in surd form the values of $\cot\frac{\pi}{8}$ and $\cot\frac{\pi}{12}$.

(β) show without using calculators that

$$\operatorname{cosec}\frac{4\pi}{15} + \operatorname{cosec}\frac{8\pi}{15} + \operatorname{cosec}\frac{16\pi}{15} + \operatorname{cosec}\frac{32\pi}{15} = 0$$

c) i) Show that for $a > 0$ and $n \neq 0$, [3]

$$\log_{a^n}(x) = \frac{1}{n}\log_a x \ .$$

ii) Hence evaluate

$$\log_2 3 + \log_4 3 + \log_{16} 3 + \log_{256} 3 + \cdots \ .$$

Question 6 (Marks 15) Use a SEPARATE writing booklet.

a) The base of a certain solid is the region between the x-axis and the curve $y = \sin 2x$ between $x = 0$ and $= \frac{3\pi}{8}$. [6]

Each plane section of the solid perpendicular to the x-axis is an equilateral triangle with one side in the base of the solid.

Find the volume of the solid.

b) In the expansion of $(ax - bx^{-2})^8$ the coefficient of x^2 and x^{-1} are equal. [4]

Show that $a + 2b = 0$.

c) A particle of mass m kg falls from rest in a medium where the resistance to motion is mkv when the particle has velocity $v\ ms^{-1}$. [5]

i) Draw a diagram showing the forces acting on the particle.

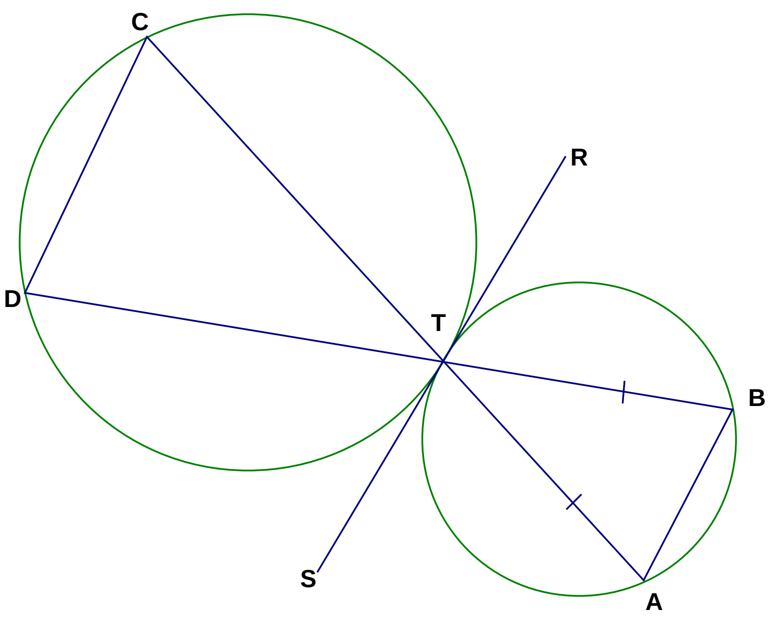
ii) Show that the equation of motion of the particle is $\ddot{x} = k(V - v)$ where $V\ ms^{-1}$ is the terminal velocity of the particle in this medium, and x metres is the distance fallen in t seconds.

iii) Find the time T seconds taken for the particle to attain 50% of its terminal **velocity**, and the **distance** fallen in time in terms of V and k .

Question 7 (Marks 15) Use a SEPARATE writing booklet.

a) Two circles touch externally at a point T. [8]

A and B are points on the first circle such that $AT = BT$, and AT and BT produced meet the second circle at C and D respectively. RS is the common tangent at T. Let $\angle BAT = \alpha$.



i) Copy the diagram and include the information above.

ii) Prove that $\angle BAC = \angle ACD$.

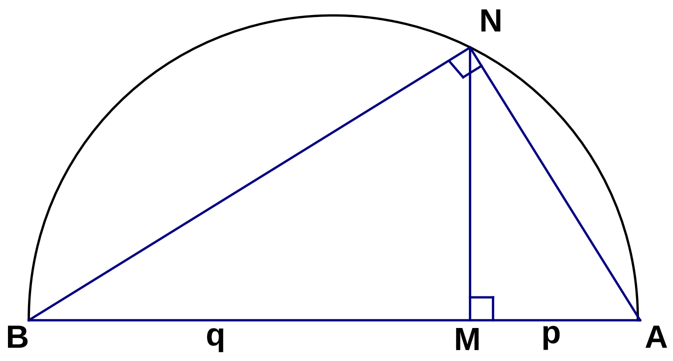
iii) Prove that $ABCD$ is a trapezium with two equal sides.

The line BC cuts the first circle in V and the second circle again in , and the line AD cuts the first circle in U and the second circle again in X.

iv) Prove that the points U, V, W and X are concyclic.

Question 7 continued...........

b) In the diagram, AB is the diameter of a semicircle. The angle ANB is $90°$ and M is a point on AB such that NM is perpendicular to AB. [7]



If $AM = p$ and $MB = q$,

i) Show that $NM = \sqrt{pq}$.

ii) Deduce, using the diagram, that $\sqrt{pq} \leq \frac{p + q}{2}$.

iii) Use (ii) to prove that if $p,\ q,\ x,\ y > 0$, then

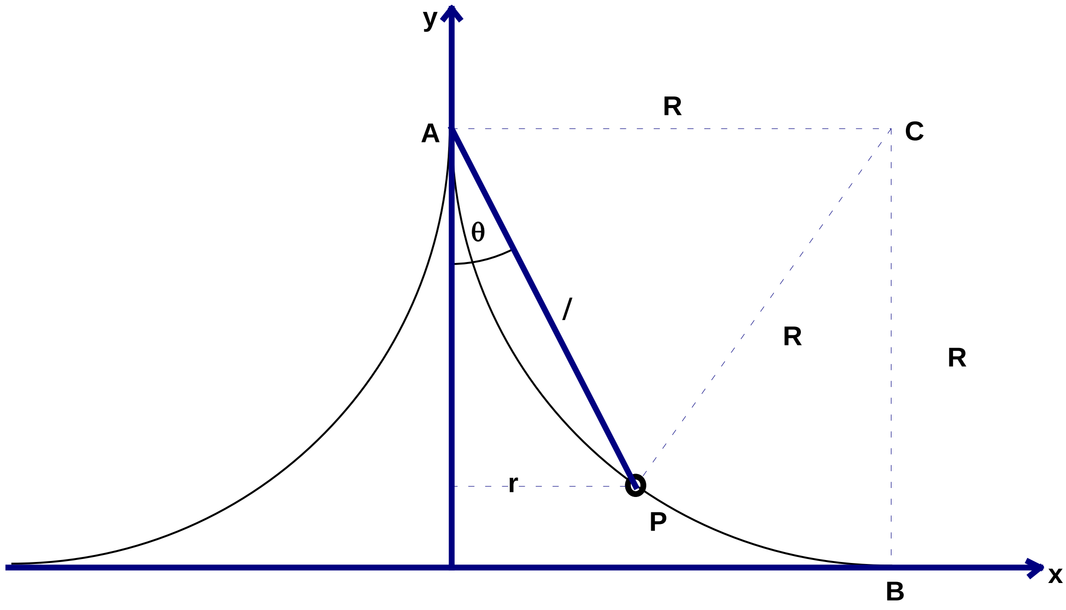
$$\frac{1}{4}(p + q + x + y) \geq (pqxy)^{\frac{1}{4}}$$

iv) Deduce that if $l,\ m,\ n,\ z > 0$, then $\frac{l}{m} + \frac{m}{n} + \frac{n}{z} + \frac{z}{l} \geq 4$

End of Question 7

Question 8 (Marks 15) Use a SEPARATE writing booklet.

a) [10]



AB is an arc of a circle centre C and radius R. A surface is formed by rotating the arc AB through one revolution about the y-axis. A light, inextensible string of length l, $l \le R$, is attached to point A, and a particle of mass m is attached to the other end. The particle is set in motion, tracing out a horizontal circle on the surface with constant angular velocity ω radians per second, while the string stays taught.

i) Explain why, when the particle is in position P shown on the diagram, the direction of the force N exerted by the surface on the particle is towards C.

ii) If the string makes an angle θ with the vertical, show that $\angle ACP = 2\theta$.

iii) Show on a diagram the tension force T, the force N and the weight force of magnitude mg acting on the particle, indicating their direction in terms of θ.

iv) Show that

$$T \cos\theta + N \sin 2\theta = mg$$

$$T \sin\theta - N \cos 2\theta = m\, l \sin\theta\, \omega^2$$

v) Show that

$$N = m\, l \sin\theta \left(\frac{g}{l}\sec\theta - \omega^2\right).$$

vi) Deduce that there is a maximum value ω for the motion to occur as described, and write down this maximum value.

Question 8 continued........

b) i) Show that $\tan^{-1}(n+1) - \tan^{-1}(n-1) = \tan^{-1}\frac{2}{n^2}$, where n is a positive integer. [5]

ii) Hence or otherwise show that for $n \geq 1$

$$\sum_{r=1}^{n} \tan^{-1}\frac{2}{r^2} = \frac{3\pi}{4} + \tan^{-1}\frac{2n+1}{1-n-n^2}$$

End of Assessment

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Q1

Tuesday, 10 August 2010
8:22 PM

Q1(a) $\int_0^1 \frac{2}{\sqrt{2-x^2}}\,dx = 2\int_0^1 \frac{1}{\sqrt{2-x^2}}\,dx$

$= 2\left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1$

$= 2\left[\sin^{-1}\frac{1}{\sqrt{2}} - \sin 0\right]$

$= 2\left(\frac{\pi}{4}\right)$

$= \frac{\pi}{2}$

(b) $\int \frac{\tan x}{\ln(\cos x)}\,dx$

Let $u = \ln(\cos x)$

$\frac{du}{dx} = \frac{1}{\cos x}(-\sin x)$

$du = -\tan x\,dx$

$-\frac{du}{\tan x} = dx$

$\Rightarrow \int \frac{\tan x}{u} \cdot \frac{-du}{\tan x}$

$-\int \frac{du}{u}$

$-\ln(\ln(\cos x)) + c$

(c) $\int_0^1 \sin^{-1}x\,dx = \int_0^1 1.\sin^{-1}x\,dx$

$= (\sin^{-1}x).x - \int x\frac{1}{\sqrt{1-x^2}}\,dx$

$= x\sin^{-1}x\Big|_0^1 + \frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx$

$= x\sin^{-1}x\Big|_0^1 + \frac{1}{2}\times\frac{1}{1/2}(1-x^2)^{1/2}\Big|_0^1$

$= \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^1$

$= \left[(1\cdot\sin^{-1}(1)+0) - (0+\sqrt{1})\right]$

$= \frac{\pi}{2} - 1$ ✓

(d) $\frac{1+4x}{(x^2+1)(4-x)} = \frac{A}{4-x} + \frac{Bx+C}{x^2+1}$

$1+4x = A(x^2+1) + (Bx+C)(4-x)$

$x=0 \quad A + 4C = 1 \quad$ — ①

$x=4 \quad 17A = 17$

$\quad A = 1 \quad$ — ②

$x=1 \quad 2A + 3B + C = 5 \quad$ — ③

from ①&② $\quad C = 0$

from ③ $\quad B = 1$

$\therefore \int_0^2 \frac{(1+4x)}{(4-x)(x^2+1)}\,dx = \int_0^2 \left[\frac{1}{4-x} + \frac{x}{x^2+1}\right]dx$

$= -\ln(4-x) + \frac{1}{2}\ln(x^2+1)\Big]_0^2$

$= \ln\frac{\sqrt{x^2+1}}{(4-x)}\Big]_0^2$

$= \ln\frac{\sqrt{5}}{2} - \ln\frac{1}{4}$

$= \ln(2\sqrt{5})$

(e) $x = 2\cos\theta$

$dx = -2\sin\theta\,d\theta$

$x=-2 \quad \theta = \pi$

$x=0 \quad \theta = \pi/2$

$\int_\pi^{\pi/2} \sqrt{4-4\cos^2\theta}\cdot(-2\sin\theta)\,d\theta$ ✓

$-2\int_\pi^{\pi/2} 2\sin\theta\sin\theta\,d\theta$

$= -2\int_\pi^{\pi/2} 2\sin^2\theta\,d\theta$

$= -2\int_\pi^{\pi/2}(1-\cos 2\theta)\,d\theta$ ✓

$= -2\left[(\pi/2 - 0) - (\pi - 0)\right]$

$= \pi$ ✓

Q2(a) Consider $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i}$

$= \frac{1+2i+i^2}{1-i^2}$

$= \frac{1+2i-1}{2}$

$= i = 0+1i$

$\therefore \left(\frac{1+i}{1-i}\right)^3 = i^3 = i \cdot i^2 = -i$

$= 0 - 1i$

(b)i) $\frac{5-i}{2-3i} = \frac{5-i}{2-3i} \times \frac{2+3i}{2+3i}$

$= \frac{10+15i-2i+3}{2^2-(3i)^2} = \frac{13+13i}{4+9}$

$= 1+i$

(ii) $\sqrt{1^2+1^2} = \sqrt{2}$

let $1+i = r(\cos\theta + i\sin\theta)$

$r\cos\theta = 1 \quad r\sin\theta = 1$

$\tan\theta = 1$

$\theta = 45^\circ, 225^\circ$

Since $\cos\theta$ and $\sin\theta$ are positive, θ lies in 1^{st} Quad.

Hence $\theta = 45^\circ$

$\frac{5-i}{2-3i} = \sqrt{2}\left(\cos 45^\circ + i\sin 45^\circ\right)$ ✓

(iii) $\left(\frac{5-i}{2-3i}\right)^4 = \left(\sqrt{2}\left(\text{cis}\,\frac{\pi}{4}\right)\right)^4$

$= 4\ \text{cis}\,\pi$ ✓

(c) $|z-\bar{z}| < 2$ and $|z-1| \geq 1$

$z = x+iy, \quad \bar{z} = x-iy$

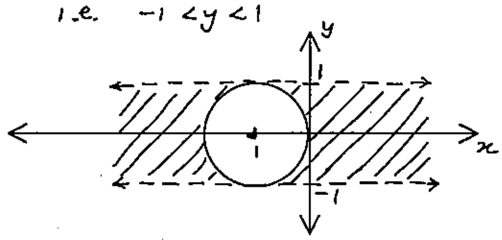
$\therefore z-\bar{z} = 2iy$

hence $|z-\bar{z}| = |2iy|$

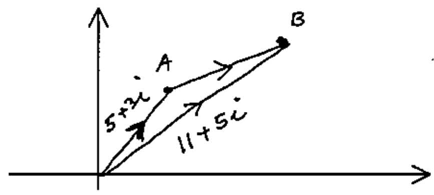
$= |2y| < 2$

$\Rightarrow |y| < 1$

i.e. $-1 < y < 1$



(d)



$\vec{AB} = (11+5i) - (5+3i)$

$= 6+2i$ (2)

$P = \vec{AB} = 6+2i$

$Q = i \times P$

$= i(6+2i)$ (2)

$= -2+6i$

Q3 a

Tuesday, 10 August 2010
7:48 PM

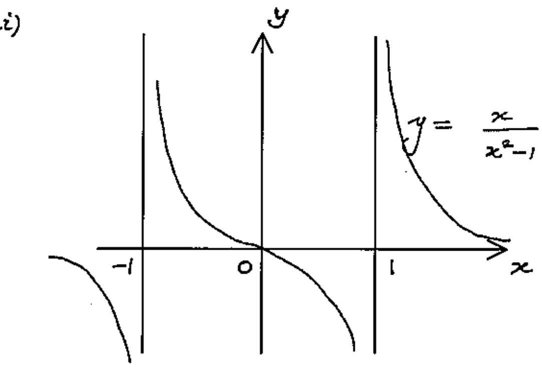
3(a) i) $f(x) = \dfrac{x}{x^2-1}$

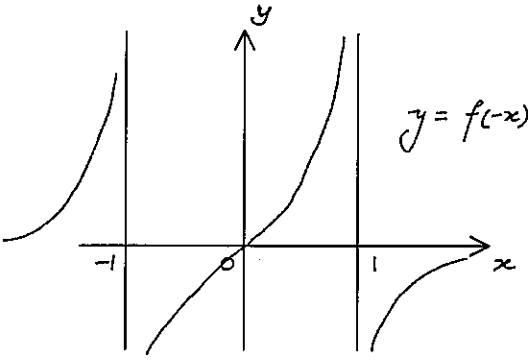
$f(-x) = \dfrac{-x}{x^2-1}$

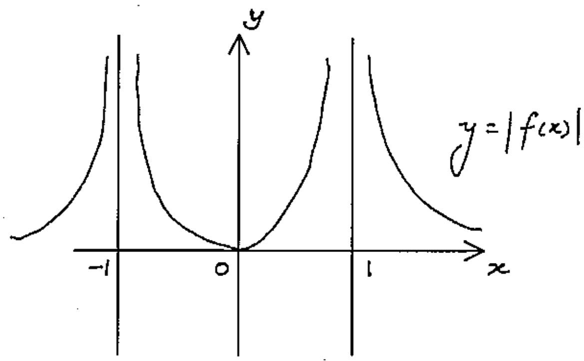
$= -f(x)$

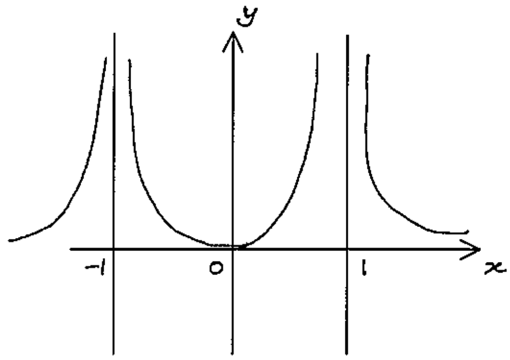
$\therefore$ odd function

(ii)









Q3 b

Tuesday, 10 August 2010
7:48 PM

3(b) i) Tangent at P: $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ ①

" Q: $\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1$ ②

(ii) T lies on both tangents

$\therefore \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{xx_2}{a^2}+\frac{yy_2}{b^2}$ — ③

$\Rightarrow \frac{x(x_1-x_2)}{a^2}+\frac{y(y_1-y_2)}{b^2}=0$ — ④

at $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

④ $\Rightarrow$ L.H.S $=\frac{(x_1+x_2)(x_1-x_2)}{2a^2}+\frac{(y_1+y_2)(y_1-y_2)}{2b^2}$

$=\frac{1}{2}\left\{\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)-\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}\right)\right\}$

$=\frac{1}{2}\{1-1\}$

$=0$

i.e. coordinates of M also satisfy ③

iii) ③ is the equation of Line TM and $(0,0)$ satisfies this eqn.
Hence O, T, M are collinear

(iv) using ④ of eqn TM

m_1 of TM is $-\frac{b^2(x_1-x_2)}{a^2(y_1-y_2)}$

m_2 of PQ is $\frac{(y_1-y_2)}{(x_1-x_2)}$

Product $m_1m_2=-\frac{b^2}{a^2}$

PT has gradient $m_3=-\frac{b^2x_1}{a^2y_1}$

QT " $m_4=-\frac{b^2x_2}{a^2y_2}$

$PT\perp QT \Rightarrow m_1m_2=-1$

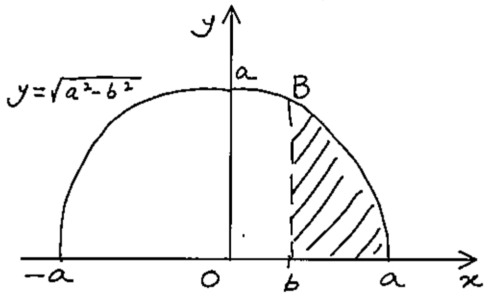
$\therefore \frac{b^4x_1x_2}{a^4y_1y_2}=-1$

$\therefore \frac{x_1x_2}{a^4}+\frac{y_1y_2}{b^4}=0$

Q3 c

Tuesday, 10 August 2010
7:48 PM

3c i) $a > b > 0$, $y = \sqrt{a^2 - x^2}$



ii) $\displaystyle\int_b^a \sqrt{a^2-b^2}\,dx = \text{sector } OBA - \triangle OBC$

$\cos \angle BOC = \dfrac{b}{a} \Rightarrow \angle BOC = \cos^{-1}\left(\dfrac{b}{a}\right)$

Area of sector $BOA = \dfrac{1}{2} a^2 \cos^{-1}\left(\dfrac{b}{a}\right)$

$BC = \sqrt{a^2-b^2}$

Area $\triangle BOC = \dfrac{b}{2}\sqrt{a^2-b^2}$

$\therefore \displaystyle\int_b^a \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\cos^{-1}\left(\frac{b}{a}\right) - \frac{b}{2}\sqrt{a^2-b^2}$

<u>OR</u>

Let $x = a\cos\theta \Rightarrow \dfrac{dx}{d\theta} = -a\sin\theta\, d\theta$

$$\int_b^a \sqrt{a^2-x^2}\,dx = \int \sqrt{a^2 - a^2\cos^2\theta}\,(-a\sin\theta\, d\theta)$$

$$= -a^2 \int \sin^2\theta\, d\theta$$

$$= -a^2 \int \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= -\frac{a^2}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + c$$

$$= -\frac{a^2}{2}\left(\theta - \sin\theta\cos\theta\right) + c$$

$\cos\theta = \dfrac{x}{a}$, $\sin\theta = \dfrac{\sqrt{a^2-x^2}}{a}$, $\theta = \cos^{-1}\left(\dfrac{x}{a}\right)$

$$= -\frac{a^2}{2}\left[\cos^{-1}\left(\frac{x}{a}\right) - \frac{x\sqrt{a^2-x^2}}{a^2}\right]_b^a$$

$$= \frac{a^2}{2}\left[-\cos^{-1}\left(\frac{a}{a}\right) + 0 - \left\{-\cos^{-1}\left(\frac{b}{a}\right) + \frac{b^2\sqrt{a^2-b^2}}{a^2}\right\}\right]$$

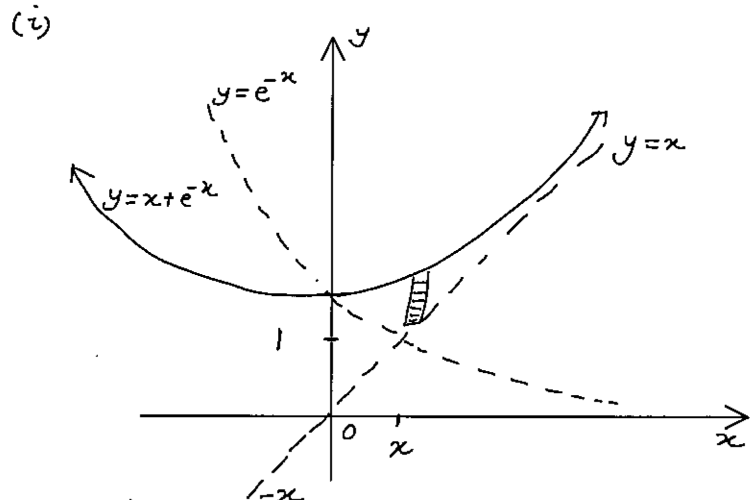
$$= \frac{a^2}{2}\left[0 + \cos^{-1}\left(\frac{b}{a}\right) - \frac{b^2\sqrt{a^2-b^2}}{a^2}\right]$$

$$= \frac{a^2}{2}\cos^{-1}\left(\frac{b}{a}\right) - \frac{b^2\sqrt{a^2-b^2}}{2}$$

Q4 a

Wednesday, 11 August 2010
8:23 PM

(i)



$$y' = 1 - e^{-x} = 0$$

$$e^{-x} = 1$$

$x = 0$ at Turning Pt.

$(0,1)$

$$y'' = e^{-x} = 1 > 0 \quad \therefore \text{min}$$

ii) $$V = 2\pi \int r\, h\, dx$$

$$= 2\pi \int_0^1 x(x + e^{-x} - x)\, dx$$

$$= 2\pi \int_0^1 x e^{-x}\, dx \qquad \checkmark$$

iii) $$V = 2\pi \int_0^0 x \frac{d}{dx}(-e^{-x})\, dx$$

$$= 2\pi \left[(-x e^{-x})_0^1 + \int_0^1 e^{-x}\, dx \right]$$

$$= 2\pi \left[-e^{-1} + [-e^{-x}]_0^1 \right]$$

$$= 2\pi \left[-e^{-1} + (-e^{-1} + 1) \right]$$

$$V = 2\pi \left(1 - \frac{2}{e}\right) \quad u^3$$

Q4 b

Wednesday, 11 August 2010
8:37 PM

4b i) $f(x) = x^2(x^2-2)$

$\Rightarrow f'(x) = 4x(x^2-1)$

$\therefore$ Gradient of Tangent at $A(\alpha, f(\alpha))$ is $4\alpha(\alpha^2-1)$

Equation of Tangent

$y - \alpha^2(\alpha^2-2) = 4\alpha(\alpha^2-1)(x-\alpha)$

$y = 4\alpha(\alpha^2-1)x + \alpha^2(2-3\alpha^2)$

(ii) The tangent meets the curve where

$x^2(x^2-2) = 4\alpha(\alpha^2-1)x + \alpha^2(2-3\alpha^2)$ — (1)

Tangent touches at A

$\Rightarrow$ α is a double root of (1)

Since Tangent meets curve again at B, equation (1) has another real root β, where β is the x coordinate of B.

Since equation (1) has real coeffs., the fourth root γ cannot be non real as such roots come in complex conjugate pairs.

Hence eqn (1) has real roots $\alpha, \alpha, \beta, \gamma$.

iii) Rearranging (1) we get:

$x^4 - 2x^2 - 4\alpha(\alpha^2-1)x + \alpha^2(3\alpha^2-2) = 0$ — (2)

For $\alpha \neq 0$

$2\alpha + \beta + \gamma = 0 \Rightarrow \beta + \gamma = -2\alpha$

$\alpha^2\beta\gamma = \alpha^2(3\alpha^2-2) \Rightarrow \beta\gamma = (3\alpha^2-2)$

Hence β & γ are the roots of the quadratic equation

$x^2 + 2\alpha x + (3\alpha^2-2) = 0$ — (3)

(iv) For $\alpha \neq 0$, since (3) has real roots β, γ

$\Delta = 4\alpha^2 - 4(3\alpha^2-2)$

$= 8(1-\alpha^2) \geq 0$ and $\alpha^2 \leq 1$

For $\alpha = 0$, tangent at A is the x-axis, meeting the curve again in $(\sqrt{2}, 0), (-\sqrt{2}, 0)$.

Hence $-1 \leq \alpha \leq 1$

Q5 a

Wednesday, 18 August 2010
7:48 PM

Q5 a i) $u_3 = 5(30) - 6(12) = 78$
$u_4 = 5(78) - 6(30) = 210$

ii) $n = 1$,
$2.3^n + 3.2^n = 2.3^1 + 3.2^1$
$= 6 + 6 = 12 = u_1$

$n = 2$,
$2.3^n + 3.2^n = 2.3^2 + 3.2^2$
$= 18 + 12 = 30 = u_2$

so $u_n = 2.3^n + 3.2^n$ for $n = 1, 2$

iii) $u_{k+2} = 5u_{k+1} - 6u_k$
$= 5(2.3^{k+1} + 3.2^{k+1}) - 6(2.3^k + 3.2^k)$
$= 10.3^{k+1} + 15.2^{k+1} - 12.3^k - 18.2^k$
$= 10.3^{k+1} + 15.2^{k+1} - 4.3^{k+1} - 9.2^{k+1}$
$= 6.3^{k+1} + 6.2^{k+1}$
$= 2.3^{k+2} + 3.2^{k+2}$

Q5 b

Wednesday, 11 August 2010
8:49 PM

Q5b i) $\operatorname{cosec} 2\theta + \cot 2\theta = \dfrac{1}{\sin 2\theta} + \dfrac{\cos 2\theta}{\sin 2\theta}$

$$= \frac{1 + \cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta}$$

$$= \frac{2\cos^2\theta}{2\sin\theta\cos\theta}$$

$$= \cot\theta$$

(ii) (α) $\cot\dfrac{\pi}{8} = \operatorname{cosec}\dfrac{\pi}{4} + \cot\dfrac{\pi}{4}$

$$= \sqrt{2} + 1$$

$$\cot\frac{\pi}{12} = \operatorname{cosec}\frac{\pi}{6} + \cot\frac{\pi}{6}$$

$$= 2 + \sqrt{3}$$

(iii) (β)

$$\operatorname{cosec}\frac{4\pi}{15} + \operatorname{cosec}\frac{8\pi}{15} + \operatorname{cosec}\frac{16\pi}{15} + \operatorname{cosec}\frac{32\pi}{15}$$

$$= \left[\cot\frac{2\pi}{15} - \cot\frac{4\pi}{15}\right] + \left[\cot\frac{4\pi}{15} - \cot\frac{8\pi}{15}\right]$$

$$+ \left[\cot\frac{8\pi}{15} - \cot\frac{16\pi}{15}\right] + \left[\cot\frac{16\pi}{15} - \cot\frac{32\pi}{15}\right]$$

$$= \cot\frac{2\pi}{15} - \cot\frac{32\pi}{15} \quad ✓$$

$$= \cot\frac{2\pi}{15} - \cot\left[2\pi + \frac{2\pi}{15}\right]$$

$$= \cot\frac{2\pi}{15} - \cot\frac{2\pi}{15}$$

$$= 0 \quad ✓$$

5(c)(i)

$$\log_{a^n} x = \frac{\log_a x}{\log_n a^n}$$

$$= \frac{\log_a x}{n}$$

ii) $\log_{2^n} 3 = \frac{1}{n}\log_2 3$

$\Rightarrow \log_2 3 + \log_4 3 + \log_{16} 3 + \log_{256} 3 + \cdots$

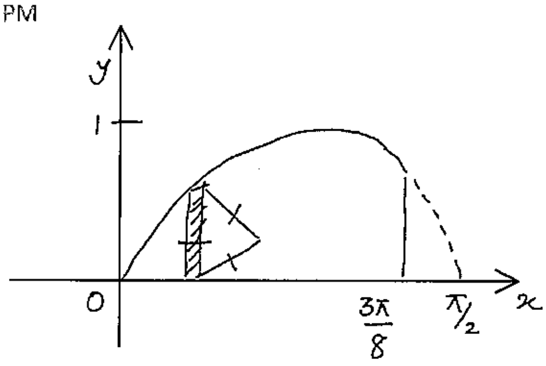
$\Rightarrow \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)\log_2 3$

$\Rightarrow 2\log_2 3$

Q6 a

Tuesday, 10 August 2010
8:22 PM

6(a)



Vol. of Sphere $= \frac{1}{2} \cdot y \cdot y \cdot \sin 60° \cdot \delta x$

$$\therefore V = \frac{\sqrt{3}}{4} \int_0^{3\pi/8} y^2 \, dx$$

$$= \frac{\sqrt{3}}{4} \int_0^{3\pi/8} \sin^2 2x \, dx$$

$$= \frac{\sqrt{3}}{8} \int_0^{3\pi/8} 1 - \cos 4x \, dx$$

$$= \frac{\sqrt{3}}{8} \left[x - \frac{\sin 4x}{4} \right]_0^{3\pi/8}$$

$$= \frac{\sqrt{3}}{8} \left[\frac{3\pi}{8} - \frac{-1}{4} - 0 \right]$$

$$= \frac{\sqrt{3}\,(3\pi + 2)}{8}$$

$$\approx 0.31 \quad u^3$$

6(b) $(ax - bx^{-2})^8$

$$T_{k+1} = {}^8C_k \, (ax)^{8-k} (-bx^{-2})^k$$

$\therefore 8 - k - 2k = 2$

$6 = 3k$

$k = 2$

or $8 - 3k = -1$

$9 = 3k$

$k = 3$

$\therefore$ Coeff. of T_3 = Coeff. of T_4

$${}^8C_2 \, a^6 (-b)^2 = {}^8C_3 \, a^5 (-b)^3$$

$$\frac{\cancel{8} \times \cancel{7}}{\cancel{2}} a^6 b^2 = \frac{\cancel{8} \times \cancel{7} \times \cancel{6}^{2}}{\cancel{3} \times \cancel{2}} a^5 \cdot -b^3$$

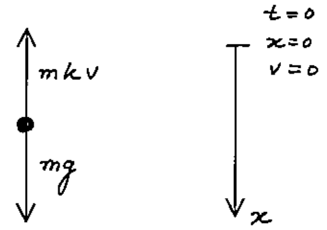
$a = -2b$

$a + 2b = 0$

Q6 c

Tuesday, 10 August 2010
8:29 PM

6c (i) Forces on particle



ii $\quad m\ddot{x} = mg - mkv$

$$\ddot{x} = k\left(\frac{g}{k} - v\right)$$

as $v \to \frac{g}{k}$, $\quad \ddot{x} \to 0$

hence terminal velocity $V = g/k$

$\therefore \quad \ddot{x} = k(V - v)$

iii) $\quad \dfrac{dv}{dt} = k(V-v)$

$$-k\frac{dt}{dv} = \frac{-1}{V-v}$$

$$-kt = \ln\{(V-v)A\}, \ A \text{ constant}$$

$\left.\begin{matrix} t=0 \\ v=0 \end{matrix}\right\} \Rightarrow A = \dfrac{1}{V}$

$$-kt = \ln\left(\frac{V-v}{V}\right)$$

$$t = \frac{1}{k}\ln\left(\frac{V}{V-v}\right)$$

Particle attains 50% of terminal velocity when

$$v = \tfrac{1}{2}V, \quad t = T = \frac{\ln 2}{k}$$

and $\quad x = \dfrac{V}{2k}(2\ln 2 - 1)$

$$v\frac{dv}{dx} = k(V-v)$$

$$\frac{dv}{dx} = k\left(\frac{V-v}{v}\right)$$

$$-k\frac{dx}{dv} = \frac{-v}{V-v}$$

$$= 1 + V\frac{-1}{V-v}$$

$$-kx = v + V\ln\{(V-v)B\}, \ B \text{ const}$$

$\left.\begin{matrix} x=0 \\ v=0 \end{matrix}\right\} \Rightarrow B = \dfrac{1}{V}$

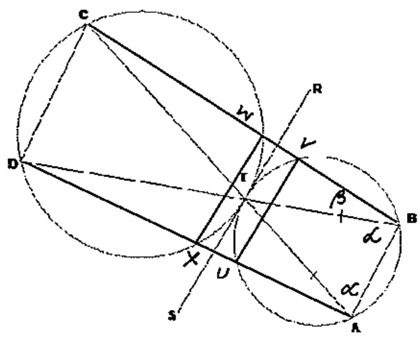
$$-kx = v + V\ln\left(\frac{V-v}{V}\right)$$

$$x = \frac{1}{k}\left\{-v + V\ln\left(\frac{V}{V-v}\right)\right\}$$

7 a

Tuesday, 10 August 2010
8:49 PM

7a i)



ii) $\angle BAC = \angle BTR$ (alt. seg. th.)

$\angle BTR = \angle STD$ (Vert. opp. $\angle$'s)

$\angle STD = \angle ACD$ (alt. seg. th.)

$\therefore \angle BAC = \angle ACD = \alpha$

iii) But these are alt. angles

$\therefore AB \parallel DC \Rightarrow ABCD$ is a trap.

Now $AT = BT$

$\therefore \angle TAB = \angle ABT = \alpha$

& $\angle ABT = \angle TDC = \alpha$ (equal alt. $\angle$'s)

$\therefore TC = TD$

In $\triangle BTC$ & $\triangle ATD$

$BT = AT$ (given)

$TC = TD$ (above)

$\angle BTC = \angle ATD$ (Vert. opp. $\angle$'s)

$\therefore \triangle BTC \equiv \triangle ATD$ (S.A.S)

$\therefore BC = AD$ (Corr. sides in cong. $\triangle$s)

$\therefore ABCD$ is a trap. with two equal sides

i.e. isos trap.

iv) Let $\angle TBV = \beta$

$\therefore \angle UAT = \beta$ (corresponding $\angle$'s in cong. $\triangle$'s)

$\therefore \angle UVW = \alpha + \beta$ (Ext. $\angle$ cyc. quad equals int. opp. $\angle$)

If $\angle WCT = \gamma$ then $\angle WCD = \alpha + \gamma$

$\therefore (\alpha + \beta) + (\alpha + \gamma) = 180^\circ$ (Co int. $\angle$s are suppl)

But $\angle WXD + (\alpha + \gamma) = 180^\circ$ (opp $\angle$s cyc. quad suppl)

$\therefore \angle WXD = \alpha + \beta$

$= \angle UVW$

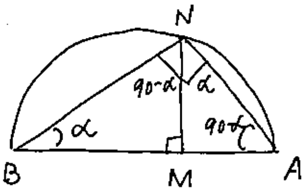
$\therefore UVWX$ is a cyclic quad. (Ext $\angle$ = int opp $\angle$)

i.e. U, V, W, X are concyclic.

7 b

Tuesday, 10 August 2010
8:49 PM

7b (i) Consider $\triangle ANM$ and $\triangle NBM$ (similar)



$$\frac{NM}{BM} = \frac{MA}{MN}$$

$$\frac{NM}{q} = \frac{p}{MN}$$

$$NM^2 = pq$$

$$NM = \sqrt{pq} \quad — ①$$

(ii) Now $AB \geqslant 2NM$ ($\because$ AB is a diameter)

& $AB = p+q$

$$\Rightarrow \quad p+q \geqslant 2\sqrt{pq}$$

$$\sqrt{pq} \leq \frac{p+q}{2} \quad — ②$$

(iii) $\frac{1}{4}(p+q+x+y)$

$$= \frac{1}{2}\cdot\frac{1}{2}(p+q+x+y)$$

$$= \frac{1}{2}\left(\frac{p+q}{2} + \frac{x+y}{2}\right)$$

$$\geqslant \frac{1}{2}\left(\sqrt{pq} + \sqrt{xy}\right) \quad \text{from } ②$$

$$\geqslant \sqrt{\sqrt{pq}\,\sqrt{xy}} \quad \text{from } ②$$

$$= (pqxy)^{1/4}$$

(iv) using part (iii)

$$\frac{\ell}{m} + \frac{m}{n} + \frac{n}{z} + \frac{z}{\ell} \geqslant 4\left(\frac{\ell}{m} + \frac{m}{n} + \frac{n}{z} + \frac{z}{\ell}\right)^{1/4}$$

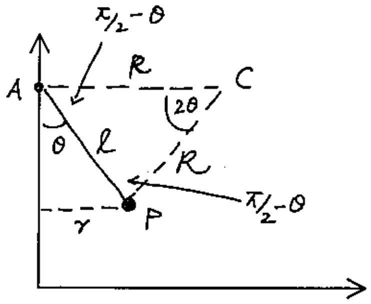
$$= 4(1)^{1/4}$$

i.e. $\frac{\ell}{m} + \frac{m}{n} + \frac{n}{z} + \frac{z}{\ell} \geqslant 4$

Q8 a

Wednesday, 11 August 2010
7:47 PM

i)



Dimensions

Force exerted by the surface on the particle is normal to the surface, and hence is directed along the radius towards C.

ii) ΔACP is isosceles

$\Rightarrow$ equal angles at A & P

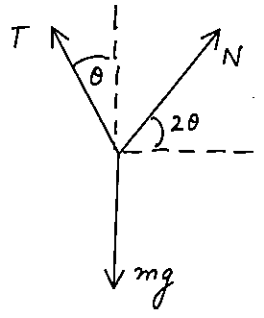
each $= \frac{\pi}{2} - \theta$

and angle sum is π

$\angle ACP = \pi - 2\left(\frac{\pi}{2} - \theta\right)$

$= 2\theta$

iii)



(iv) Resultant force is directed towards centre of circular path of particle and his magnitude $mr\omega^2$

$r = l \sin\theta$

Net Vert. Force is Zero

$T\cos\theta + N\sin 2\theta = mg$ ①

Net Hor. force is $mr\omega^2$ towards y-axis

$T\sin\theta - N\cos 2\theta = ml\sin\theta\,\omega^2$ — ②

(v) ① × $\sin\theta$ − ② × $\cos\theta$ (eliminate T)

$\Rightarrow$

$N\cos(2\theta - \theta) = m\sin\theta\left(g - l\cos\theta\,\omega^2\right)$

$N\cos\theta = ml\sin\theta\cos\theta\left(g\frac{\sec\theta}{l} - \omega^2\right)$

$N = ml\sin\theta\left(g\frac{\sec\theta}{l} - \omega^2\right)$

(vi) $\sin N \geqslant 0$

$g\frac{\sec\theta}{l} - \omega^2 \geqslant 0$

$\Rightarrow \quad \omega \leq \sqrt{\frac{g}{l}\sec\theta}$

Q8 b

Thursday, 12 August 2010
11:13 AM

Q8 (b) i) Let $\tan^{-1}(n+1) = \alpha$

& $\tan^{-1}(n-1) = \beta$

$\Rightarrow \tan\alpha = n+1$ & $\tan\beta = n-1$

Now

$$\tan(\alpha-\beta) = \frac{n+1-(n-1)}{1+(n+1)(n-1)} \checkmark$$

$$= \frac{2}{1+n^2-1} = \frac{2}{n^2}$$

$\therefore \alpha-\beta = \tan^{-1}\frac{2}{n^2}$ $\because \alpha-\beta$ & $\tan^{-1}\frac{2}{n^2}$ are $< 90^\circ$

i.e. $\tan^{-1}(n+1) - \tan^{-1}(n-1) = \tan^{-1}\frac{2}{n^2} \checkmark$

ii) $$\sum_{r=1}^{n} \tan^{-1}\frac{2}{r^2} = \tan^{-1}\frac{2}{1^2} + \tan^{-1}\frac{2}{2^2} + \tan^{-1}\frac{2}{3^2} + \cdots$$

$$\cdots + \tan^{-1}\frac{2}{(n-2)^2} + \tan^{-1}\frac{2}{(n-1)^2} + \tan^{-1}\frac{2}{n^2}$$

$$= (\tan^{-1}2 - \tan^{-1}0) + (\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}4 - \tan^{-1}2) + \cdots$$

$$\cdots + (\tan^{-1}(n-1) - \tan^{-1}(n-3)) + (\tan^{-1}n - \tan^{-1}(n-2)) + (\tan^{-1}(n+1) - \tan^{-1}(n-1))$$

$$= -\tan^{-1}0 - \tan^{-1}1 + \tan^{-1}(n+1) + \tan^{-1}n \checkmark$$

$$= -0 - \frac{\pi}{4} + \pi + \tan^{-1}\left(\frac{n+1+n}{1-(n+1)n}\right)$$

$$= \frac{3\pi}{4} + \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) \checkmark$$