Shore School

4 unit mathematics

Trial HSC Examination 1987

1. **(i)** Find the indefinite integrals: **(a)** $\int \frac{9+x^2}{9-x^2}\,dx$ **(b)** $\int \frac{2x+8}{x^2+4x+8}\,dx$
(ii) Evaluate $\int_1^2 x\sqrt{3x-2}\,dx$
(iii) Let n be an integer greater than 1 and let $I_n = \int_0^1 \frac{dx}{(x^2+1)^n}$. Prove that $2nI_{n+1} = 2^{-n} + (2n-1)I_n$. Hence, deduce the value of $\int_0^1 \frac{dx}{(x^2+1)^3}$.

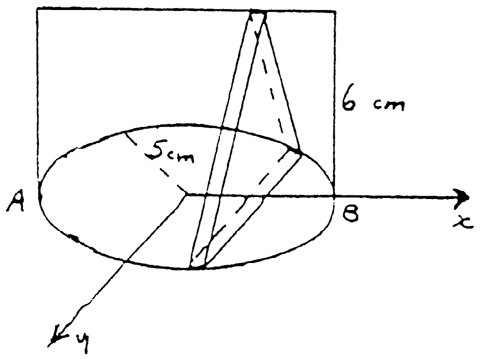
2. **(i)** Given that $w = \frac{2-3i}{1+i}$, determine **(a)** $|w|$ **(b)** $\overline{w}$ **(c)** $w + \overline{w}$
(ii) Describe, in geometric terms, the locus (in the Argand plane) represented by $2|z| = z + \overline{z} + 4$.
(iii) Mark clearly on an Argand diagram the region in the complex plane satisfied by $-\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{3}$
(iv) Find the locus of $|\frac{z+1}{z+2}| = 1$

3. **(i)** Show, by geometrical considerations or otherwise, that, if the complex numbers z_1 and z_2 are such that $|z_1| = |z_2|$, then $\frac{z_1+z_2}{z_1-z_2}$ is purely imaginary.
(ii) Prove that $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$. Hence, show that $x = \cos(\frac{2r+1}{14})\pi$ for $r = 0, 1, 2, 4, 5, 6$ are the roots of the equation $64x^6 - 112x^4 + 56x^2 - 7 = 0$. Deduce that $\cos\frac{\pi}{14}.\cos\frac{3\pi}{14}.\cos\frac{5\pi}{14} = \frac{\sqrt{7}}{8}$

4. **(i)** Find the factors of $x^3 - 1$ and hence find the complex cube roots of unity.
(ii) Show that $1 + i$ is a zero of the polynomial $x^3 - x^2 + 2$ and find its other two zeros in the complex field.
(iii) Find the polynomial of least degree with integral coefficients all of which have no common factor, but having $1 - i$ as a two-fold root.
(iv) If α, β, γ are the zeros of the polynomial $x^3 - x^2 + 15x + 3$ in the field of complex numbers, find the values of
(a) $(\alpha - 1)(\beta - 1)(\alpha - 1)$ **(b)** $\alpha^2 + \beta^2 + \gamma^2$.

5. **(i)** Derive the gradient of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) and hence find the equation of the tangent at $(a\cos\theta, b\sin\theta)$.
(ii) Show that the normal to the hyperbola $xy = c^2$ at the point $(cp, \frac{c}{p})$ cuts the hyperbola again at the point $(-\frac{c}{p^3}, -cp^3)$.
(iii) The tangents at two points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ on the parabola $y^2 = 4ax$ intersect at T. The normals at P and Q intersect at N. If angle PTQ is a right angle prove that TN is parallel to the x-axis.

6. (i) The base of the solid shown in the diagram is a circle of radius 5cm and each section of the solid formed by a plane perpendicular to a fixed diameter AB is an isosceles triangle with height 6cm. Find the volume of the solid.



(ii) Sketch, on the same graph, the circles $x^2 + y^2 = 16$ and $x^2 + y^2 = 8x$. Find the equation of their common chord. Shade in the area common to both circles. By using a slicing technique, find the volume of the solid formed when this area is revolved about the common chord.

7. (i) Find all x such that $\sin x = \cos 7x$ and $0 < x < \pi$.
(ii) Show that $\cos^2(\theta + \frac{7\pi}{6}) + \cos^2(\theta + \frac{5\pi}{6}) - \cos^2\theta$ is independent of θ and find its value.
(iii) If a, b, c are real and unequal show that $a^2 + b^2 > 2ab$ and hence deduce that $a^2 + b^2 + c^2 > ab + bc + ca$. If $a + b + c = 6$ show that $ab + bc + ca < 12$.

8. (i) Show that if $0 < A < 1$ and $0 < B < 1$, then $\tan^{-1} A + \tan^{-1} B = \tan^{-1}(\frac{A+B}{1-AB})$
(ii) A relation is defined implicitly by $x^2 + xy - 2y^2 = 0$. Show that $\frac{dy}{dx}$ has only two possible values: 1 and $-\frac{1}{2}$. Hence, or otherwise, sketch the relation.
(iii) In a triangle ABC, b, c and B are given such that two distinct triangles ABC are possible. Show that the difference between the two possible values of the third sides of the two triangles is $2\sqrt{b^2 - c^2 \sin^2 B}$.