Student Number: . . . .

Set: 12ME2-1 (FES)

SHORE

2008

Trial HSC Examination

Mathematics Extension 2

General Instructions

- Reading time − 5 minutes
- Working time − 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided on the back page of this question paper
- All necessary working should be shown in every question

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Total marks − 120

- Attempt Questions 1 − 8
- All questions are of equal value
- Start each question in a new writing booklet
- Write your examination number on the front cover of each booklet to be handed in
- If you do not attempt a question, submit a blank booklet marked with your examination number and "N/A" on the front cover

Blank Page

Total Marks – 120
Attempt Questions 1 – 8
All Questions are of equal value

Begin each question on a NEW BOOKLET, writing your name and question number at the top of the page. Extra booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find $\int (e^x + e^{-x})^2 \, dx$ **2**

(b) Find $\int \frac{2x^2 - 2x + 1}{(x-2)(x^2+1)} \, dx$ **3**

(c) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{2 + \sin x} \, dx.$ **5**

(d) Let $I_n = \int_0^1 \frac{1}{(1+x^2)^n} \, dx$; n=1,2,3,…

(i) Show that $I_{n+1} = \frac{2n-1}{2n} I_n + \frac{1}{n.2^{n+1}}$; n=1,2,3,… **3**

(ii) Hence evaluate $\int_0^1 \frac{1}{(1+x^2)^3} \, dx.$ **2**

P.T.O……………………..

Question 2 **Use a SEPARATE writing booklet** **Marks**

(a) $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$.

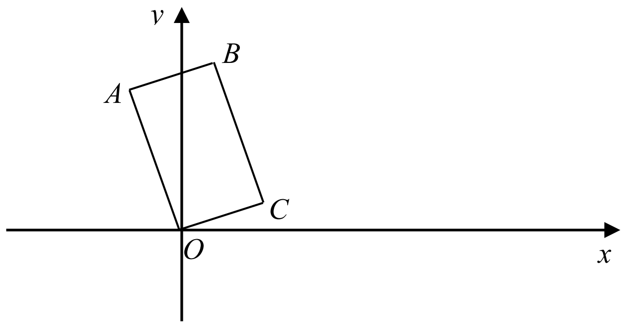
(i) Find $\frac{z_1}{z_2}$ in the form $a+ib$ where a and b are real. **1**

(ii) Write z_1 and z_2 in modulus – argument form. **2**

(iii) By equating equivalent expressions for $\frac{z_1}{z_2}$, write $\cos\frac{5\pi}{12}$ as a surd. **1**

(iv) Explain why there is no positive integer n such $z_1 z_2^n$ is real. **2**

(b)



The points $OABC$ are the vertices of a rectangle on the Argand diagram with $|OA| = 2|OC|$. If OC represents the complex number $p+iq$, write down the complex numbers represented by:

(i) $\overrightarrow{OA}$ **1**

(ii) $\overrightarrow{OB}$ **1**

(iii) $\overrightarrow{BC}$ **1**

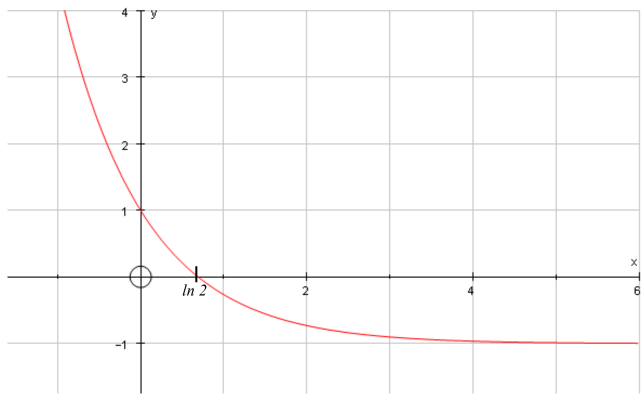
(iv) $\overrightarrow{AC}$ **1**

(c) (i) If $z = \cos\theta + i\sin\theta$, explain why $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2i\sin n\theta$ for positive integers n. **2**

(ii) By considering the Binomial expansions of $(z + z^{-1})^3$ and $(z - z^{-1})^3$, show that $4(\cos^3\theta + \sin^3\theta) = (\cos 3\theta - \sin 3\theta) + 3(\cos\theta + \sin\theta)$. **3**

Question 3 **Use a SEPARATE writing booklet** **Marks**

(a)



The diagram shows the graph of $f(x) = 2e^{-x} - 1$. On separate diagrams sketch the following graphs, showing the intercepts on the axes and the equations of any asymptotes:

(i) $y = |f(x)|$. **1**

(ii) $y = \{f(x)\}^2$. **1**

(iii) $y = \dfrac{1}{f(x)}$. **2**

(iv) $y = ln\ \{\text{f(x)}\}$. **1**

(b) Consider the curve $y^2 = x^4(4 + x)$

(i) Sketch the curve. **2**

(ii) Find the area of the loop of the curve from $x = -4$ to $x = 0$. **3**

(c) The roots of $x^3 - 3x^2 - 2x + 4 = 0$ are α, β and γ. Answer the following without finding the actual values of α, β and γ.

(i) Find a cubic equation whose roots are α^2, β^2 and γ^2. **2**

(ii) Hence or otherwise find the value of $\alpha^2 + \beta^2 + \gamma^2$. **1**

(iii) Determine the value of $\alpha^3 + \beta^3 + \gamma^3$. **2**

P.T.O……………………..

Question 4 **Use a SEPARATE writing booklet** **Marks**

(a) Given $|z+i| \leq 2$ and $0 \leq arg(z+1) \leq \frac{\pi}{4}$. Sketch the region in an Argand diagram which contains the point P representing z. **3**

(b) Consider the five 5^{th} roots of unity.

(i) Solve $z^5 - 1 = 0$ over the complex field giving your answers in modulus-argument form. **3**

(ii) Hence express $z^5 - 1$ as the product of real linear and quadratic factors. **3**

(iii) Write down the complex roots of $z^4 + z^3 + z^2 + z + 1 = 0$ giving your answers in modulus-argument form. **1**

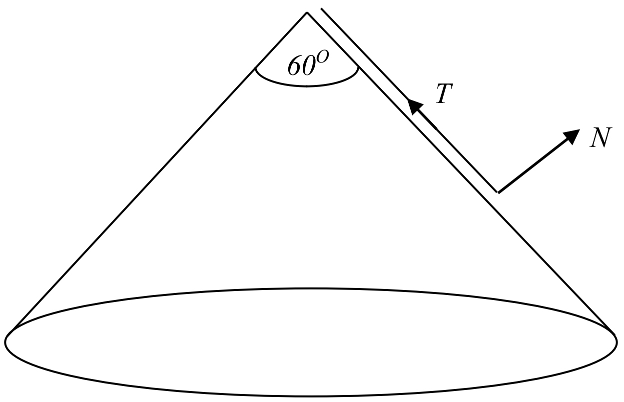
(iv) Hence prove that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$. **2**

(c) If $x > 0$ and $y > 0$ prove that $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$. **3**

Question 5 **Use a SEPARATE writing booklet** **Marks**

(a) A mass of 3kg is attached to the vertex of a cone of vertical angle 60^0 by an elastic string of length 1 metre. The mass is moving in a horizontal circle on the curved, frictionless surface of the cone. Acceleration due to gravity is 10 m s^{-2}.

T = Tension in string
N = Normal reaction of the cone surface on the mass.



Not to Scale

(i) If the mass is moving at a speed of 1 m s^{-1}, by resolving forces vertically and horizontally find the values of T and N. **4**

(ii) What is the maximum speed of the particle for it to just remain on the cone's surface and what will be the string's tension at this time? **4**

(b) A mass of 1 kg is allowed to fall under gravity from rest at the surface of a medium in which the retardation on the mass is proportional to the distance fallen (x). In other words, the net force for this motion is $g - kx$ Newtons with the downward direction as positive.

(i) Show that it falls $\dfrac{2g}{k}$ metres before it becomes stationary. **2**

(ii) Show that the displacement equation in terms of t is given by: **5**

$$x = \frac{g}{k}\left(\sin\left(\frac{2\sqrt{k}\ t\ - \pi}{2}\right) + 1\right)$$

P.T.O……………………..

Question 6 **Use a SEPARATE writing booklet** **Marks**

(a) Consider the complex number z which satisfies $|z| = 1$.

(i) Using double angle trigonometric identities show that: **2**

$$1 + \cos\alpha + i\sin\alpha \equiv 2\cos\frac{1}{2}\alpha(\cos\frac{1}{2}\alpha + i\sin\frac{1}{2}\alpha).$$

(ii) If $z = \cos\theta + i\sin\theta, -\pi < \theta \le \pi,$ write $1 + z^2$ in terms of $\cos\theta$ and $\sin\theta$. Hence deduce that if in an Argand diagram, points A and B represent z and $1 + z^2$ respectively, then A, B and O are collinear, where O is the origin. State the values of θ such that B lies on the interval OA. **3**

(b) A solid shape has as its base the parabola $y = x^2$ in the XY plane. Sections taken perpendicular to the axis of the parabola (i.e. perpendicular to the y-axis) are equilateral triangles. Using the method of slicing determine the volume of the solid, if the length of the axis of the parabola is 16cm. **5**

(c) With θ increasing, the point P given by $(r\cos\theta, r\sin\theta)$ is moving in a circular motion about O at a distance of r units from O. Starting with the two equations $x = r\cos\theta$ and $y = r\sin\theta$ prove that the Normal acceleration of P towards O is $r\omega^2$ where ω is the rate of change of θ with respect to time. **5**

Question 7 **Use a SEPARATE writing booklet** **Marks**

(a) Consider the polynomial $P(x) = x^4 - 4x^3 + 5x^2 - 2x - 2$.

(i) Show that the curve $y = P(x)$ has a maximum turning point at (1, -2) and minimum turning points at $x = 1 \pm \frac{1}{2}\sqrt{2}$. Hence deduce from a sketch of the curve that the equation $P(x) = 0$ has two real roots and two non-real roots. **3**

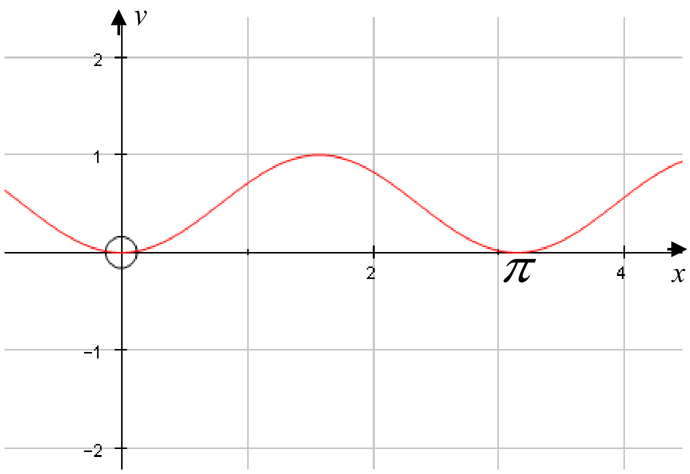
(ii) Explain why the real roots cannot be rational. What do you know about the nature of the non-real roots? **2**

(iii) Given that $1 + i$ is a root of the equation $P(x) = 0$, factor $P(x)$ into two quadratic factors with rational coefficients. Hence find the x-intercepts of the curve $y = P(x)$ and show them on your graph. **3**

(b) (i) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **3**

Hence show $\int_0^\pi x\cos 2x\,dx = 0$.

(ii)



The area bounded by the curve $y = \sin^2 x$ and the x-axis between $x = 0$ and $x = \pi$ is rotated through one revolution about the y-axis. By taking the limiting sum of the volumes of cylindrical shells, show that the volume of the solid of revolution is given by $V = 2\pi\int_0^\pi x\sin^2 x\,dx$. Hence by using your answer in part (i) or otherwise find the volume of this solid. **4**

P.T.O……………………..

Question 8 **Use a SEPARATE writing booklet** **Marks**

(a) A particle of mass M is projected vertically upward under gravity with speed U in a medium in which the resistance is Mk times the speed, here k is a positive constant. If the particle reaches its greatest height H in time T, show that $U=gT+kH$. (You may assume that the net force is given by $-Mg-Mkv$ with upward direction positive.) **4**

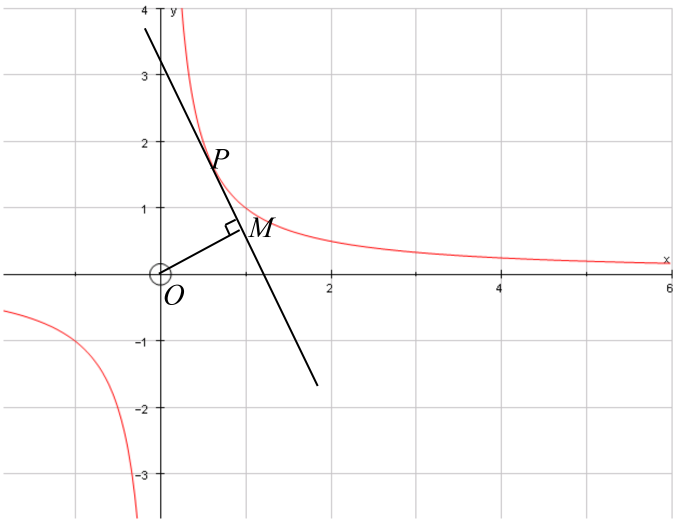
(b) $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. PQ is a diameter of the ellipse. The tangent to the ellipse at P meets the vertical through Q at R and the Y-axis at V. O is the origin.

(i) Prove that the tangent at P has equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. **2**

(ii) Show that the area of ΔPQR $=4$ times the area of ΔPOV. **1**

(iii) Show that the area of ΔPQR is $\frac{2ab}{|\tan\theta|}$ square units. **2**

(c) $P(t,\frac{1}{t})$ is a variable point on the rectangular hyperbola $xy=1$. M is the foot of the perpendicular from the origin to the tangent to the hyperbola at P.



(i) Show that the tangent to the hyperbola at P has equation $x+t^2y=2t$. **2**

(ii) Find the equation of OM. **1**

(iii) Show that the locus of M as P varies has equation $x^2+y^2=2\sqrt{xy}$. **3**

End of Paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \; dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \; dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \; dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \; dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \; dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \; dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{\mathrm{x}^2 - a^2}} \; dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{\mathrm{x}^2 + a^2}} \; dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \qquad x > 0$

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1(a) $I = \int (e^{2x} + 2 + e^{-2x})\,dx$

$= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$ (2)

(b) Let $\frac{2x^2-2x+1}{(x-2)(x^2+1)} = \frac{a}{x-2} + \frac{bx+c}{x^2+1}$

For $x \doteqdot 2$, $\frac{5}{5} = a \Rightarrow a = 1$

Also $2x^2-2x+1 = x^2+1 + (bx+c)(x-2)$

$= (1+b)x^2 - 2bx + cx + 1$

$\therefore$ $b = 1$ and $c = 0$

$\therefore I = \int \left(\frac{1}{x-2} + \frac{x}{x^2+1}\right)dx$

$= \ln|x-2| + \frac{1}{2}\ln|x^2+1| + c$

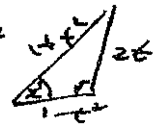
or $\ln\left|(x-2)\sqrt{x^2+1}\right| + c$ (3)

(c) Let $t = \tan\frac{x}{2}$

$\therefore 2\,dt = \sec^2\frac{x}{2}\,dx$

$= \left(1+\tan^2\frac{x}{2}\right)dx$

$= (1+t^2)\,dx$



$\therefore dx = \frac{2}{1+t^2}\,dt$

$\tan x = \frac{2t}{1-t^2}$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

$\therefore I = \int_0^1 \frac{1}{2+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$

$= \int_0^1 \frac{1}{1+t^2+t}\,dt$

$= \int_0^1 \frac{1}{t^2+t+\frac{1}{4}+\frac{3}{4}}\,dt$

$= \int_0^1 \frac{1}{(t+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\,dt$

$= \left[\frac{2}{\sqrt{3}}\tan^{-1}\left[(t+\tfrac{1}{2})\cdot\tfrac{2}{\sqrt{3}}\right]\right]_0^1$

$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{3\sqrt{3}}$ (5)

1(d) $I_n = \int_0^1 \frac{1}{(1+x^2)^n}\cdot d(x)$ ($\frac{1}{2}$ start)

$= \left[x\cdot\frac{1}{(1+x^2)^n}\right]_0^1 - \int_0^1 x\,d\left(\frac{1}{(1+x^2)^n}\right)$

$= \frac{1}{2^n} - \int_0^1 x\cdot\frac{-n\cdot 2x}{(1+x^2)^{n+1}}\,dx$

$= \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$

$= \frac{1}{2^n} + 2n\int_0^1 \frac{(1+x^2)-1}{(1+x^2)^{n+1}}\,dx$

$= \frac{1}{2^n} + 2n\cdot I_n - 2n\cdot I_{n+1}$

$\therefore 2n I_{n+1} = \frac{1}{2^n} + (2n-1)I_n$

$\therefore I_{n+1} = \frac{1}{n\cdot 2^{n+1}} + \frac{2n-1}{2n}I_n$ (3)

$I_3 = \frac{1}{2\cdot 2^3} + \frac{3}{4}I_2$

& $I_2 = \frac{1}{1\cdot 2^2} + \frac{1}{2}I_1$

& $I_1 = \int_0^1 \frac{1}{1+x^2}\,dx$

$= \left[\tan^{-1}x\right]_0^1$

$= \frac{\pi}{4}$

$\therefore I_3 = \frac{1}{16} + \frac{3}{4}\left(\frac{1}{4} + \frac{\pi}{8}\right)$

$= \frac{1}{4} + \frac{3\pi}{32}$

$= \frac{1}{32}(3\pi + 8)$ (2)

2(a)(i) $\frac{z_1}{z_2} = \frac{1+i}{\sqrt{3}-i} \times \frac{\sqrt{3}+i}{\sqrt{3}+i}$

$= \frac{\sqrt{3}-1+(1+\sqrt{3})i}{3+1}$

$= \frac{\sqrt{3}-1}{4} + \frac{1+\sqrt{3}}{4}i$ (1)

(ii) 

$z_1 = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

$z_2 = 2\,\text{cis}\,\frac{-\pi}{6}$ (2)

(iii) $\frac{z_1}{z_2} = \frac{\sqrt{2}\,\text{cis}\frac{\pi}{4}}{2\,\text{cis}\frac{-\pi}{6}}$

$= \frac{1}{\sqrt{2}}\,\text{cis}\,\frac{5\pi}{12}$

Equating real parts in (i) & (iii)

$\therefore \frac{1}{\sqrt{2}}\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{4}$

$\therefore \cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{1}{4}(\sqrt{6}-\sqrt{2})$ (1)

(iv) $z_1(z_2)^n = \sqrt{2}\,\text{cis}\frac{\pi}{4} \cdot 2^n\,\text{cis}\frac{-n\pi}{6}$

$= \sqrt{2}\cdot 2^n\,\text{cis}\left(\frac{\pi}{4} - \frac{n\pi}{6}\right)$

For $z_1 z_2^n$ to be real, arg is $M\pi$, M integer.

But $\frac{\pi}{4} - \frac{n\pi}{6} = \frac{3-2n}{12}\pi$

$\therefore 3-2n$ has to be multiple of 12.

But $2n$ is always even $\therefore 3-2n$ is always odd

$\therefore$ it can never be a multiple of 12 (2)

(b)(i) $\vec{OA}$ is $2i(p+iq) = -2q+2pi$ (1)

(ii) $\vec{OB}$ is $-2q+2pi+p+iq = (p-2q)+(2p+q)i$ (1)

(iii) $\vec{BC}$ is $-(p-2q)-(2p+q)i+(p+iq) = +2q-2pi$

OR $\vec{BC} = -\vec{OA}$

$= 2q-2pi$ (1)

(iv) $\vec{AC}$ is $(p+iq)-(-2q+2pi) = p+2q+i(q-2p)$ (1)

(c)(i) $z^n = \text{cis}\,n\theta$ & $z^{-n} = \text{cis}(-n\theta)$

$\therefore z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$ (1)

$z^n - z^{-n} = \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$

$= 2i\sin n\theta$ (1)

(ii) $(z+z^{-1})^3 = z^3+3z+3z^{-1}+z^{-3}$

$(z-z^{-1})^3 = z^3-3z+3z^{-1}-z^{-3}$

$\therefore (z+z^{-1})^3+(z-z^{-1})^3 = 2z^3+6z^{-1}$

Let $z = \text{cis}\,\theta$ and using (i) above

$\therefore (2\cos\theta)^3 + (2i\sin\theta)^3 = 2\,\text{cis}\,3\theta + 6\,\text{cis}(-\theta)$

$\therefore 8\cos^3\theta - 8i\sin^3\theta = 2\cos 3\theta + 2i\sin 3\theta + 6\cos\theta - 6i\sin\theta$

Equating real parts:

$8\cos^3\theta = 2\cos 3\theta + 6\cos\theta$ ①

Equating imaginary parts:

$-8\sin^3\theta = 2\sin 3\theta - 6\sin\theta$ ②

① − ②:

$8(\cos^3\theta + \sin^3\theta) = 2(\cos 3\theta - \sin 3\theta) + 6(\cos\theta + \sin\theta)$

$\therefore 4(\cos^3\theta + \sin^3\theta) = (\cos 3\theta - \sin 3\theta) + 3(\cos\theta + \sin\theta)$

(3)

(b)(i) Assuming $|OA| = |OC|$

Then (i) $\vec{OA} = -q+ip$

(ii) $\vec{OB} = (p-q)+i(p+q)$

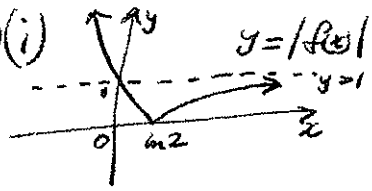
(iii) $\vec{BC} = q-ip$

(iv) $\vec{AC} = (p+q)+i(q-p)$

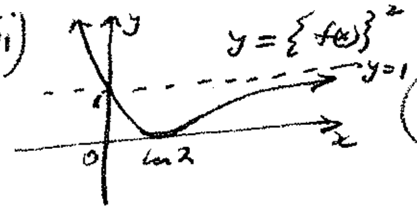
($3\frac{1}{2}$/4)

HSC Trial Maths Ext 2 2008 Solutions

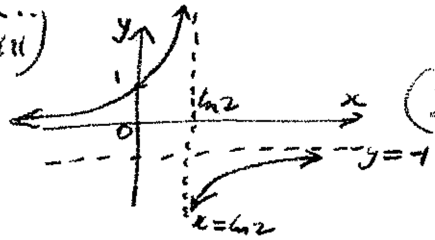
3(a)(i) (1)



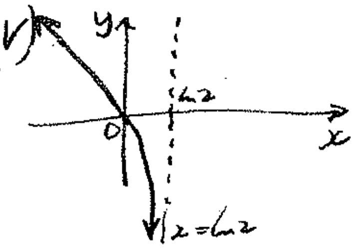
(ii) (1)



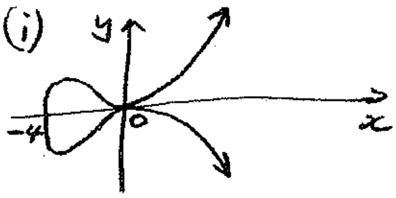
(iii) (2)



(iv) (1)



b)(i) (2)



(ii) $\text{Area} = 2\int_{-4}^{0} x^2\sqrt{4+x}\,dx$

Let $u = 4+x$ ∴ $x = u-4$

$du = dx$ $\quad x^2 = u^2 - 8u + 16$

$\therefore \text{Area} = 2\int_0^4 (u-4)^2\sqrt{u}\,du$

$= 2\int_0^4 \sqrt{u}\,(u^2 - 8u + 16)\,du$

$= 2\int_0^4 u^{5/2} - 8u^{3/2} + 16u^{1/2}\,du$

$= 2\left[\frac{2}{7}u^{7/2} - 8\cdot\frac{2}{5}u^{5/2} + 16\cdot\frac{2}{3}u^{3/2}\right]_0^4$

$= \frac{4096}{105}u^2$ (3) or $39\frac{1}{105}$

(c)(i) Let $y = x^2$ ∴ $x = \sqrt{y}$

$\therefore (\sqrt{y})^3 - 3(\sqrt{y})^2 - 2\sqrt{y} + 4 = 0$

$\therefore y\sqrt{y} - 3y - 2\sqrt{y} + 4 = 0$

$\therefore (y-2)\sqrt{y} = 3y - 4$

$\therefore (y-2)^2\cdot y = (3y-4)^2$

$\therefore y^3 - 4y^2 - 9y^2 + 4y + 24y - 16 = 0$

$\therefore y^3 - 13y^2 + 28y - 16 = 0$ (2)

(ii) $\therefore \alpha^2 + \beta^2 + \gamma^2 = -\frac{b}{a}$

$= 13$ (1)

(iii) Now $\alpha^3 - 3\alpha^2 - 2\alpha + 4 = 0$

$\beta^3 - 3\beta^2 - 2\beta + 4 = 0$

$\gamma^3 - 3\gamma^2 - 2\gamma + 4 = 0$

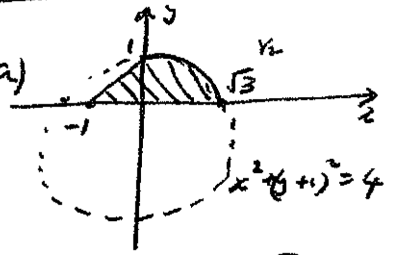
$\therefore \alpha^3 + \beta^3 + \gamma^3 = 3(\alpha^2 + \beta^2 + \gamma^2) + 2(\alpha + \beta + \gamma) - 12$

$= 3(13) + 2(3) - 12$

$= 33$ (2)

HSC Trial Maths Ext 2 2008 Solutions (4

4.(a)



(3)

(b)(i) Let $z = r\,\text{cis}\,\theta$, $z^5 = 1$

But $z^5 = r^5\,\text{cis}\,5\theta$ by deMoivre's Th

$\therefore r = 1$ & $\text{cis}\,5\theta = \text{cis}\,0, \text{cis}\,2\pi, \ldots$

$\therefore 5\theta = 0, 2\pi, \ldots$

$\therefore \theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \ldots$

$\therefore z = 1, \text{cis}\frac{2\pi}{5}, \text{cis}\frac{4\pi}{5}, \text{cis}\frac{6\pi}{5}, \text{cis}\frac{8\pi}{5}$ (3)

(ii)

$z^5 - 1 = (z-1)(z - \text{cis}\frac{2\pi}{5}) \ldots$

Now $z - \text{cis}\frac{8\pi}{5} = \left(z - \text{cis}\left(-\frac{2\pi}{5}\right)\right)$

$= \left(z - \overline{\text{cis}\frac{2\pi}{5}}\right)$

Let $\alpha = \text{cis}\frac{2\pi}{5}$ & consider

$(z-\alpha)(z-\bar{\alpha}) = z^2 - 2\text{Re}(\alpha)z + \alpha\bar{\alpha}$

where $|\alpha| = 1 \therefore \alpha\bar{\alpha} = 1$

$\therefore \left(z - \text{cis}\frac{2\pi}{5}\right)\left(z - \text{cis}\left(\frac{-2\pi}{5}\right)\right) = z^2 - 2\cos\frac{2\pi}{5}z + 1$

& $\left(z - \text{cis}\frac{4\pi}{5}\right)\left(z - \text{cis}\frac{-4\pi}{5}\right) = z^2 - 2\cos\frac{4\pi}{5}z + 1$ (3)

$\therefore z^5 - 1 = (z-1)\left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right)$

(iii) $z^5 - 1 = 0$

$\therefore (z-1)(z^4 + z^3 + z^2 + z + 1) = 0$ (1)

$\therefore z = 1$ OR $z = \text{cis}\frac{2\pi}{5}, \text{cis}\frac{4\pi}{5}, \text{cis}\frac{-4\pi}{5}, \text{cis}\frac{-2\pi}{5}$

(iv) Using sum of roots $= -\frac{b}{a}$

$\therefore \text{cis}\frac{2\pi}{5} + \text{cis}\frac{4\pi}{5} + \text{cis}\frac{-4\pi}{5} + \text{cis}\frac{-2\pi}{5} = -1$

Equating real parts:

$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{-4\pi}{5} + \cos\frac{-2\pi}{5} = -1$

But $\cos\theta = \cos(-\theta)$

$\therefore 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = -1$

$\therefore \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$ (2)

(c) RTP $\frac{1}{x} + \frac{1}{y} - \frac{4}{x+y} \geq 0$

Now $\frac{1}{x} + \frac{1}{y} - \frac{4}{x+y} = \frac{x+y}{xy} - \frac{4}{x+y}$

$= \frac{x^2 + 2xy + y^2 - 4xy}{xy(x+y)}$

$= \frac{x^2 - 2xy + y^2}{xy(x+y)}$

$= \frac{(x-y)^2}{xy(x+y)}$

Now $x, y > 0 \therefore xy > 0, x+y > 0$ & $(x-y)^2 \geq 0$

$\therefore \frac{1}{x} + \frac{1}{y} - \frac{4}{x+y} \geq 0$

$\therefore \frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$ (3)

Alternative proof:

Consider $(a-b)^2 \geq 0$

$\therefore a^2 + b^2 \geq 2ab$

$\therefore u + v \geq 2\sqrt{uv}$ where $u = a^2$, $v = b^2$

$\therefore \frac{1}{x} + \frac{1}{y} \geq \frac{2}{\sqrt{xy}}$ (A)

But $x + y \geq 2\sqrt{xy}$

$\therefore \frac{1}{x+y} \leq \frac{1}{2\sqrt{xy}}$

$\therefore \frac{2}{x+y} \leq \frac{1}{\sqrt{xy}}$

From (A) $\frac{1}{x} + \frac{1}{y} \geq 2 \times \frac{2}{x+y}$

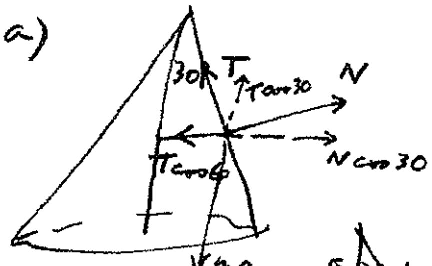
$\therefore \frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$

(b)(i) Ans only + circle diag. $1\frac{1}{2}/3$

HSC Trial Maths Ext 2 2008 Solutions

(5

5. (a)



(i)

Horizontally:

$F = T\cos 60° - N\cos 30°$

$\therefore \frac{mv^2}{r} = \frac{1}{2}T - \frac{\sqrt{3}}{2}N$ (3)

But $v = 1$ and $r = \frac{1}{2}$

$\therefore 6 = \frac{1}{2}T - \frac{\sqrt{3}}{2}N$

$\therefore 12 = T - \sqrt{3}N$ (1)

Vertically:

$F = T\cos 30° + N\cos 60° - mg$

$\therefore 0 = \frac{\sqrt{3}}{2}T + \frac{1}{2}N - 3\times 10$

$\therefore \sqrt{3}T + N = 60$

$\therefore 3T + \sqrt{3}N = 60\sqrt{3}$ (2)

Add (1) + (2)

$4T = 12 + 60\sqrt{3}$

$\therefore T = 3 + 15\sqrt{3} \doteqdot 28.98$

Subs back into (1):

$12 = 3 + 15\sqrt{3} - \sqrt{3}N$

$\therefore N = \frac{15\sqrt{3} - 9}{\sqrt{3}}$

$\therefore N = 15 - 3\sqrt{3} \doteqdot 9.8038$ (4)

(ii) Let $N = 0$ for critical value

Subs into (3) $\therefore \frac{3v^2}{1/2} = \frac{1}{2}T$

$\therefore T = 12v^2$

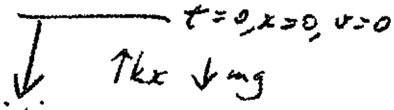
Subs into (2) $\therefore T = 20\sqrt{3}$

$\therefore 20\sqrt{3} = 12v^2$

$\therefore v^2 = \frac{5\sqrt{3}}{3}$

$\therefore v = \sqrt{\frac{5\sqrt{3}}{3}}$ OR $\left(\frac{25}{3}\right)^{1/4}$ (4) $\doteqdot 1.699$

(b) (i)



$F = mg - kx$

$\therefore m v \frac{dv}{dx} = mg - kx$

Now $m = 1$

$\therefore v\frac{dv}{dx} = g - kx$

$\therefore \int_0^V v\,dv = \int_0^X (g - kx)\,dx$

$\therefore \left[\frac{v^2}{2}\right]_0^V = \left[gx - \frac{1}{2}kx^2\right]_0^X$

$\therefore \frac{V^2}{2} = gX - \frac{1}{2}kX^2$

$\therefore V^2 = X(2g - kX)$

$\therefore V = 0 \Rightarrow X = 0$ or

$X = \frac{2g}{k}$ (2)

(ii) from (i) $v^2 = x(2g - kx)$

$\therefore \frac{dx}{dt} = \sqrt{2gx - kx^2}, \ v > 0$

$\therefore \int dt = \int \frac{1}{\sqrt{2gx - kx^2}}\,dx$

$\therefore \int_0^T \sqrt{k}\,dt = \int \frac{dx}{\sqrt{-\left(x^2 - \frac{2g}{k}x + \frac{g^2}{k^2} - \frac{g^2}{k^2}\right)}}$

$= \int_0^X \frac{dx}{\sqrt{\left(\frac{g}{k}\right)^2 - \left(x - \frac{g}{k}\right)^2}}$

$\therefore \left[\sqrt{k}t\right]_0^T = \left[\sin^{-1}\left(\frac{\left(x - \frac{g}{k}\right)}{\frac{g}{k}}\right)\right]_0^X$

$\therefore \sqrt{k}T = \sin^{-1}\left(\frac{k}{g}\left(X - \frac{g}{k}\right)\right) - \sin^{-1}(-1)$

$\therefore \sqrt{k}T = \sin^{-1}\left(\frac{k}{g}\left(X - \frac{g}{k}\right)\right) + \frac{\pi}{2}$

$\therefore \sin\left(\sqrt{k}T - \frac{\pi}{2}\right) = \frac{k}{g}\left(X - \frac{g}{k}\right)$

$\therefore X = \frac{g}{k} + \frac{g}{k}\sin\left(\sqrt{k}T - \frac{\pi}{2}\right)$

$= \frac{g}{k}\left[\sin\left(\frac{2\sqrt{k}t - \pi}{2}\right) + 1\right]$ (5)

6 (a) (i)

LHS $= (1+\cos\alpha) + i\sin\alpha$

$= 2\cos^2\frac{\alpha}{2} + i\,2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$= 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$

= RHS (2)

(ii) $1+z^2 = 1+\operatorname{cis}2\theta$

$= 1+\cos2\theta + i\sin2\theta$

$= 2\cos\theta(\cos\theta + i\sin\theta)$ from (i) (1)

$\therefore 1+z^2 = 2\cos\theta . z$

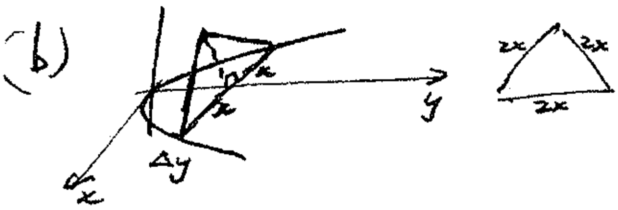
$\therefore$ O, A, B lie on the same straight line with $2\cos\theta$ as the scalar quantity

For B to lie on OA, $0 < 2\cos\theta < 1$

$\therefore 0 < \cos\theta < \frac{1}{2}$

$\therefore \frac{\pi}{3} < \theta < \frac{\pi}{2}$ & $-\frac{\pi}{2} < \theta < -\frac{\pi}{3}$ (2) $\frac{1}{2}$ $\frac{1}{2}$

(b)



$\Delta V = \left(\frac{1}{2}(2x)(2x)\sin\frac{\pi}{3}\right)\Delta y$

$= \sqrt{3}\,x^2\Delta y$

$\therefore V = \lim_{\Delta y\to 0}\sum \sqrt{3}\,x^2\Delta y$

$= \int_0^{16}\sqrt{3}\,x^2\,dy$

$= \int_0^{16}\sqrt{3}\,y\,dy$

$= \left[\frac{\sqrt{3}y^2}{2}\right]_0^{16}$

$= 128\sqrt{3}\ u^3$ (5)

(c) $x = r\cos\theta$

$\therefore \dot{x} = \frac{d}{d\theta}r\cos\theta . \frac{d\theta}{dt}$

$= -r\sin\theta . \omega$

$\therefore \ddot{x} = \omega . \left(\frac{d}{d\theta} - r\sin\theta . \frac{d\theta}{dt}\right) + -r\sin\theta . \dot{\omega}$

$= \omega . -r\cos\theta . \omega - r\sin\theta\,\dot{\omega}$

$= -r(\cos\theta . \omega^2 + \sin\theta . \dot{\omega})$

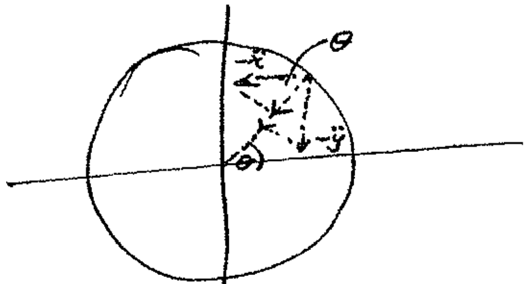
Also $y = r\sin\theta$

$\therefore \frac{dy}{dt} = \frac{d}{d\theta}r\sin\theta . \frac{d\theta}{dt}$

$\therefore \dot{y} = r\cos\theta . \omega$

$\therefore \ddot{y} = \omega\left(\frac{d}{d\theta}r\cos\theta . \frac{d\theta}{dt}\right) + r\cos\theta . \dot{\omega}$

$= -r\sin\theta . \omega^2 + r\cos\theta . \dot{\omega}$



towards the centre

$N = -\ddot{x}\cos\theta - \ddot{y}\sin\theta$

$= r(\cos^2\theta\,\omega^2 + \sin\theta\cos\theta\,\dot{\omega} + \sin^2\theta\,\omega^2 - r\cos\theta . \sin\theta . \dot{\omega})$

$= r\omega^2(\sin^2\theta + \cos^2\theta)$

$= r\omega^2$ as $\sin^2\theta + \cos^2\theta \equiv 1$.

(5)

7(a)(i) $P'(x) = 4x^3 - 12x^2 + 10x - 2$

Now

$$\begin{array}{r} 4x^2 - 8x + 2 \\ x-1 \overline{)\,4x^3 - 12x^2 + 10x - 2} \\ 4x^3 - 4x^2 \qquad\qquad \\ -8x^2 + 10x \qquad \\ -8x^2 + 8x \qquad \\ 2x - 2 \\ 2x - 2 \end{array}$$

Let $4x^2 - 8x + 2 = 0$

$\therefore 2x^2 - 4x + 1 = 0$

$\therefore x = \frac{4 \pm \sqrt{16-8}}{4}$

$= \frac{2 \pm \sqrt{2}}{2}$

$= 1 \pm \frac{1}{2}\sqrt{2}$

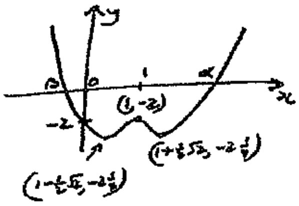
$\therefore$ stationary at 1, $1 \pm \frac{1}{2}\sqrt{2}$

$P''(x) = 12x^2 - 24x + 10$

Now $P''(1) = -2 < 0 \therefore x = 1$ max t.p.

& $P''(1 + \frac{1}{2}\sqrt{2}) = 4 > 0 \therefore$ min t.p.

& $P''(1 - \frac{1}{2}\sqrt{2}) = 4 > 0 \therefore$ min t.p.



from the sketch α, β shown are 2 real values.

As only 2, 2 must be unreal. as $P(x)$ is of degree 4 (3)

(ii) $P(x)$ has integer coeffs, coeff of x^4 is 1 & constant term is −2

$\therefore$ product of rational roots is −2

$\therefore$ test $x = 0, 1, -1, 2, -2$ into $P(x)$

$P(0) = -2$, $P(1) = 10$, $P(1) = -2$, $P(2) = 2$, $P(-2) = 70$

$\therefore$ none are zero

$\therefore$ no rational roots (1)

(ii) (Cont).

Complex roots are conjugates of each other as all the coeffs are real (1)

(iii) Roots are $1+i$, $1-i$, α, β

Now $(x-(1+i))(x-(1-i)) = ((x-1)-i)((x-1)+i)$

$= (x-1)^2 - i^2$

$= x^2 - 2x + 1 + 1$

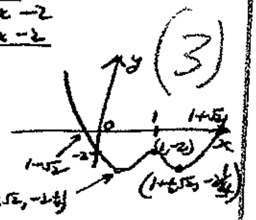
$= x^2 - 2x + 2$

$$\begin{array}{r} x^2 - 2x - 1 \\ x^2 - 2x + 2 \overline{)\,x^4 - 4x^3 + 5x^2 - 2x - 2} \\ x^4 - 2x^3 + 2x^2 \qquad\qquad \\ -2x^3 + 3x^2 - 2x \qquad \\ -2x^3 + 4x^2 - 4x \qquad \\ -x^2 + 2x - 2 \\ -x^2 + 2x - 2 \end{array}$$

Solving $x^2 - 2x - 1 = 0$

$\therefore x = \frac{2 \pm \sqrt{4+4}}{2}$

$= 1 \pm \sqrt{2}$ (3)



(b)(i) LHS $= [F(x)]_0^a$ since $F'(x) = f(x)$

$= F(a) - F(0)$

RHS $= [-F(a-x)]_0^a$

$= -F(0) + F(a)$ (1)

$\therefore$ LHS = RHS

Let $a = \pi$, $f(x) = x\cos 2x$, $f(\pi - x) = (\pi - x)\cos 2(\pi - x)$

by (i) $\int_0^\pi x\cos 2x\,dx = \int_0^\pi (\pi - x)\cos 2(\pi - x)\,dx$

$= \int_0^\pi \pi\cos(2\pi - 2x)\,dx - \int_0^\pi x\cos(2\pi - 2x)\,dx$

$= \int_0^\pi \pi\cos 2x\,dx - \int_0^\pi x\cos 2x\,dx$

$\therefore 2\int_0^\pi x\cos 2x\,dx = \pi\int_0^\pi \cos 2x\,dx$

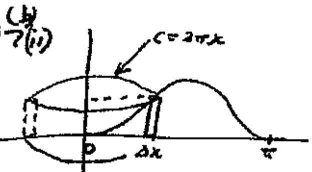
$= \pi\left[\frac{\sin 2x}{2}\right]_0^\pi$

$= 0$

$\therefore \int_0^\pi x\cos 2x\,dx = 0$ (2)

HSC Trial Maths Ext

7(b)(ii)



Opening out the cylindrical shell

$\therefore \Delta V = 2\pi x y\,\Delta x$

$= 2\pi x \cdot \sin^2 x\,\Delta x$

$\therefore V = \lim_{\Delta x \to 0} \sum 2\pi x \cdot \sin^2 x\,\Delta x$

$= \int_0^\pi 2\pi x \cdot \sin^2 x\,dx$

$= 2\pi \int x\left(\frac{1 - \cos 2x}{2}\right)dx$

$= 2\pi \int_0^\pi \frac{1}{2}x\,dx - \frac{2\pi}{2}\int_0^\pi x\cos 2x\,dx$

$= \pi\left[\frac{x^2}{2}\right]_0^\pi - 0$ by (i) above

$= \frac{\pi^3}{2}$ u^3 (4)

8(a) $x=H, v=0, t=T$ $\downarrow Mg$ $\downarrow Mkv$

$\uparrow$ $t=0, v=u, x=0$

$F = -Mg - Mkv$

$\therefore M.\frac{v\,dv}{dx} = -Mg - Mkv$

$\therefore \int_u^0 \frac{v\,dv}{g+kv} = \int_0^H -dx$

$\therefore \int \frac{g+kv-g}{g+kv}\,dv = \int_0^H -k\,dx$

$\therefore \int \left(1 - \frac{g}{k}\frac{k}{g+kv}\right)dv = \left[-kx\right]_0^H$

$\therefore \left[v - \frac{g}{k}\ln|g+kv|\right]_u^0 = -kH$

$\therefore -\frac{g}{k}\ln g - u + \frac{g}{k}\ln|g+ku| = -kH$ ①

Also $M.\frac{dv}{dt} = -Mg - Mkv$

$\therefore \int_u^0 \frac{dv}{g+kv} = \int_0^T -dt$

$\therefore \left[\ln(g+kv)\right]_u^0 = \left[-kt\right]_0^T$

$\therefore -\ln|g+ku| + \ln g = -kT$

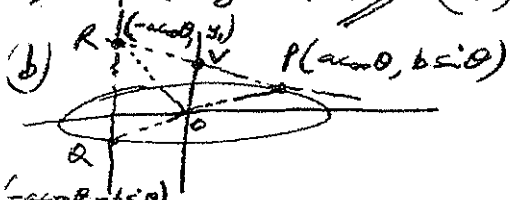
$\therefore \ln|g+ku| = kT + \ln g$ ②

Subs into ①

$\therefore -\frac{g}{k}\ln g - u + \frac{g}{k}(kT + \ln g) = -kH$

$\therefore u = gT + kH$ (4)

(b)



(i) $x = a\cos\theta$, $y = b\sin\theta$

$\therefore \frac{dx}{d\theta} = -a\sin\theta$, $\frac{dy}{d\theta} = b\cos\theta$

$\therefore \frac{dy}{dx} = \frac{-b\cos\theta}{a\sin\theta}$

$\therefore$ tangent $\frac{y - b\sin\theta}{x - a\cos\theta} = \frac{-b\cos\theta}{a\sin\theta}$

$\therefore ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$\therefore bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$

$\therefore \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ (2)

(b)(ii) area $\Delta PQR = \frac{1}{2}.PQ.PR.\sin\angle QPR$

& area $\Delta POV = \frac{1}{2}.PO.PV\sin\angle OPV$

$\therefore$ RTP $PQ.PR = 4PO.PV$

Now $PQ = 2.PO$ (diameter = twice rad)

& $PR = 2PV$ (ratio of intercepts on parallel lines)

$\therefore PQ.PR = 4PO.PV$ and area $\Delta PQR = 4\times\Delta POV$ (1)

(iii) To find V: $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ & $x = 0$

$\therefore y = \frac{b}{\sin\theta}$ $\therefore V\left(0, \frac{b}{\sin\theta}\right)$

$\therefore$ area $\Delta POV = \left|\frac{1}{2}\times\frac{b}{\sin\theta}\times a\cos\theta\right|$ using $A = \frac{1}{2}bh$

$= \frac{ab}{2|\tan\theta|}$

$\therefore$ area of $\Delta PQR = \frac{2ab}{|\tan\theta|}$ using (ii) above. (2)

(c)(i) $x = t$ and $y = t^{-1}$

$\therefore \frac{dx}{dt} = 1$ & $\frac{dy}{dt} = \frac{-1}{t^2}$

$\therefore \frac{dy}{dx} = -\frac{1}{t^2}$

$\therefore$ tangent $\frac{y - \frac{1}{t}}{x - t} = -\frac{1}{t^2}$

$\therefore t^2y - t = -x + t$

$\therefore x + t^2y = 2t$ (2)

(ii) grad of tangent $= \frac{-1}{t^2}$

$\therefore$ grad of $OM = t^2$

eqn $\frac{y-0}{x-0} = t^2$

$\therefore y = t^2x$ (1)

(iii) to find M: Solve $y = xt^2$

$y = -\frac{1}{t^2}x + \frac{2}{t}$

$\therefore xt^2 = -\frac{1}{t^2}x + \frac{2}{t}$

$\therefore xt^4 = -x + 2t$

$\therefore x = \frac{2t}{t^4+1}$

Subs back, $y = \frac{2t}{t^4+1}\times t^2$

$\therefore M\left(\frac{2t}{t^4+1}, \frac{2t^3}{t^4+1}\right)$

RTS $x^2 + y^2 = 2\sqrt{xy}$

Subs $LHS = x^2 + y^2$

$= \frac{4t^2}{\ldots} + \frac{4t^6}{\ldots} = \frac{4t^2}{\ldots}$

$RHS = 2\sqrt{xy}$

$= 2\sqrt{\frac{2t}{t^4+1}\times\frac{2t^3}{t^4+1}}$

$= \frac{4t^2}{t^4+1}$

$= LHS$ (3)