ST CATHERINE'S SCHOOL

YEAR 12 - 4 UNIT (ADDITIONAL) MATHEMATICS

TIME ALLOWED: 3 HOURS *(plus 5 mins reading time)*

DATE: AUGUST, 1997

Student Number: 55

INSTRUCTIONS:

- All questions are to be attempted.
- All questions are of equal value.
- All necessary working should be shown in every question, as part of your solution.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators and geometrical instruments are required.
- Standard Integrals are printed on the last page.
- Each question should be started in a *separate* Writing Booklet, clearly marked with the question number and your student number on the cover.
- You may ask for extra Writing Booklets if you need them.
- Tie your Booklets in 2 bundles (no staples are to be used):

  Section A: Questions 1, 2, 3and 4.
  Section B: Questions 5, 6, 7 and 8.

- Hand in Section A, Section B and this examination paper separately

TEACHERS USE ONLY TOTAL MARKS
A
B
TOTAL

SECTION A

Question 1 ***(Use a separate Writing Booklet)*** **Marks**

a) Find the following integrals: 4

i) $\int \sin 2x \;\cos x \; dx$

ii) $\int \sin 2x \;\cos 2x \; dx$

b) Show that $\int_4^5 \frac{2t^2 dt}{(t-1)\,(t-2)\,(t-3)} = 19\ln 2 - 9\ln 3$ 3

c) Using integration by parts, or otherwise, find $\int e^x \;\sin 2x \; dx$ 3

d) Show that $\int_0^m \frac{(m-x)^2}{m^2 + x^2}\, dx = m(1-\ln 2)$ 2

Question 2 *(Use a separate Writing Booklet)* **Marks**

a) If $z_1 = 1+3i$, $z_2 = 1-i$, **4**

i) Find in the form $a + ib$, where a and b are real, the numbers $z_1\ z_2$ and $\frac{z_1}{z_2}$.

ii) On an Argand Diagram the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ represent the complex numbers $z_1\ z_2$ and $\frac{z_1}{z_2}$ respectively (where z_1 and z_2 are given above).

Show this on an Argand Diagram, giving the co-ordinates of A and B.

iii) From your diagram, deduce that $\frac{z_1}{z_2} - z_1\ z_2$ is real.

b) Given that $z = \sqrt{3} - i$, **4**

i) Express z in modulus-argument form.

ii) Hence, evaluate the following in the form $x + iy$:

(α) z^5

(β) $(\bar{z})^5$

(γ) $\dfrac{z^5}{(\bar{z})^5}$

c) On an Argand diagram, sketch the locus of z if: **4**

i) $|z+3| < |z-1-4i|$

ii) $4 \arg \dfrac{z-1}{z+3} = \pi$

Question 3 *(Use a separate Writing Booklet)* | **Marks**

a) i) Show that $2 + i$ is a root of $2z^3 - 5z^2 - 2z + 15 = 0$ — 3

ii) Find the other roots.

b) If the roots of the equation $x^3 - px^2 + qx - r = 0$ are α, β and γ: — 4

i) Find the equation with roots $\alpha + \beta$, $\beta + \gamma$, $\gamma + \alpha$.

ii) Hence, or otherwise, find the value of $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$

c) i) Show that if the equation $P(x) = 0$ has a root of multiplicity m, then the equation $P^1(x)$ has a root of multiplicity *(m-1)*. — 5

ii) Solve the equation $x^4 - x^3 - 9x^2 - 11x - 4 = 0$, given that it has a root of multiplicity 3.

Question 4 ***(Use a separate Writing Booklet)***

a) i) By expanding $(\cos\theta + i\ \sin\theta)^5$ in two different ways, obtain an expression for $\cos 5\theta$ in terms of powers of $\cos\theta$. — 6

ii) Hence solve the equation $16x^4 - 20x^2 + 5 = 0$ giving solutions in the form $x = \cos\alpha$.

) If $I_n = \int \frac{x^n}{\sqrt{x^2 - a^2}}\ dx$ — 6

i) Show that $nI_n - (n-1)a^2 I_{n-2} = x^{n-1}\sqrt{x^2 - a^2}$

ii) Hence evaluate $\int_2^4 \frac{x^4}{\sqrt{x^2 - 4}}\ dx$

SECTION B

Question 5 ***(Use a separate Writing Booklet)*** **Marks**

a) For the curve $y = \dfrac{x^2}{x^3 + 4}$ **7**

i) Find any horizontal or vertical asymptotes.

ii) Find any maximum or minimum turning points.

iii) Sketch the curve.

iv) Use the graph to show that there are three solutions to the equation $x^3 - 4x^2 + 4 = 0$

b) On a new set of axes sketch $y = \left|\dfrac{x^2}{x^3 + 4}\right|$ **2**

c) Find the domain of $y = \pm\sqrt{\dfrac{x^2}{x^3 + 4}}$ **1**

d) Without further use of calculus, sketch $y^2 = \dfrac{x^2}{x^3 + 4}$ **2**

Question 6 *(Use a separate Writing Booklet)* — Marks

a) Find $\int \cos^2 y \, dy$ — 1

b) A hyperbola has asymptotes $y = x$ and $y = -x$. It passes through the point $(3, 2)$. — 4

i) Find the equation of the hyperbola.

ii) Determine its eccentricity and foci.

c) Find the equation of the tangents to the ellipse $\frac{x^2}{27} + \frac{y^2}{9} = 1$ — 3

which are perpendicular to the line $x + y = 10$.

d) For the hyperbola $xy = 16$, P is the variable point $\left(4p, \frac{4}{p}\right)$. — 4

i) Find the equations of the tangent and normal to the hyperbola at the point P.

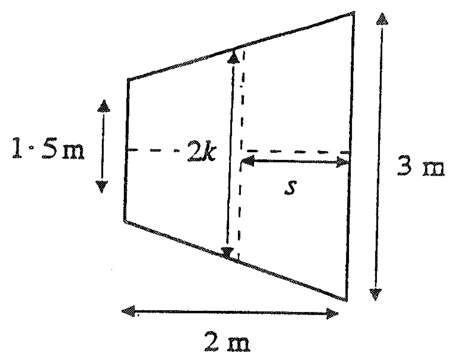
ii) If the tangent intersects the X axis at A and the normal intersects the Y axis at B, find the area of ΔPAB.

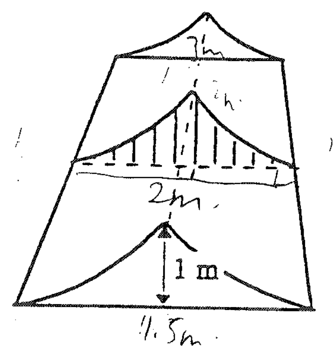
Question 7 (*Use a separate Writing Booklet*) **Marks**

a) The region bounded by $y = x^2 + 2$, the x axis, the y axis and the line $x = 2$ is rotated about the line $x = 2$. **4**

Use cylindrical shells to find the volume generated.

b) *Trapezium base of the tent* *Tent showing typical cross-section* **8**





The base of a tent is a trapezium with parallel sides of length 1.5 metres at the back of the tent and 3 metres at the front of the tent. The base has an axis of symmetry perpendicular to the parallel sides and 2 metres long. The roof of the tent is formed by draping material over a horizontal ridge pole of length 2 metres directly above the axis of symmetry of the base and at a height of 1 metre, as shown in the diagram above.

A vertical cross-section taken perpendicular to the axis of symmetry of the base has the shape of the region shaded above and has area $\frac{1}{3}k$ square units, where $2k$ metres is the width of the cross-section where it meets the trapezium base.

i) Show that if at a distance s metres from the front of the tent (measured along the axis of symmetry of the trapezium) the width of the trapezium base is $2k$ metres, as shown in the diagram, then $k = \frac{3}{2}\left(1 - \frac{1}{4}s\right)$.

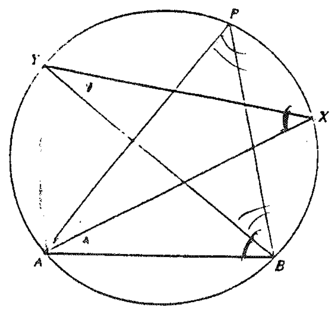
ii) Deduce that the area of typical cross-section as shaded above, taken at a distance s metres from the front of the tent, is $\frac{1}{2}\left(1 - \frac{1}{4}s\right)$ square units.

iii) If the tent has vertical flaps front and back, calculate the volume of the interior of the tent.

Question 8 *(Use a separate Writing Booklet)* **Marks**

a) AB is a fixed chord of a circle. P is any point on the major arc. **4**

The bisectors of $\angle PAB$ and $\angle PBA$ meet the circle at X and Y respectively.



i) Copy the diagram into your Writing Booklet, showing the information given.

ii) Prove that XY is constant.

b) A particle is projected from origin O with speed V m/s and angle θ to the horizontal. **4**

i) Show that the cartesian equation of its path is given by

$$y = x\tan\theta - \frac{g\sec^2\theta}{2V^2}x^2.$$

ii) Prove that there are two paths possible through a point (a, h) on the path if

$$\left(V^2 - gh\right)^2 > g^2\left(a^2 + h^2\right).$$

c) Using mathematical induction, show that for each positive integer n there are unique positive integers p_n and q_n such that $\left(1+\sqrt{2}\right)^n = p_n + q_n\sqrt{2}$. **4**

END OF EXAMINATION

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

St. Catherine's School

Yr. 12 4U. Trial HSC Solutions — Aug. 1997

Sect. A.

a) (i) $\int \sin 2x . \cos x \, dx = \int 2 \sin x \cos x \cos x \, dx$

$= \int 2 \sin x . \cos^2 x \, dx$

let $u = \cos x$

$du = -\sin x \, dx$

$= 2 \int -u^2 . du$

$= -\frac{2}{3} u^3 + c \quad = -\frac{2}{3} \cos^3 x + c$ (2)

(ii) $\int \sin 2x \cos 2x \, dx$

$= \int \frac{1}{2} \sin 4x \, dx \quad = -\frac{1}{8} \cos 4x + c$

— Too complex — look for easy way out!

b) $\dfrac{2t^2}{(t-1)(t-2)(t-3)} = \dfrac{A}{t-1} + \dfrac{B}{t-2} + \dfrac{C}{t-3}$

$2t^2 = A(t-2)(t-3) + B(t-1)(t-3) + C(t-1)(t-2)$

$t = 1$, $\quad 2 = A.-1.-2$

$2 = 2A \quad$ ie $A = 1$

$t = 2$, $\quad 8 = B.1.-1$

$8 = -B \quad$ ie $B = -8$

$t = 3$, $\quad 18 = C.2.1$

$18 = 2C \quad$ ie $C = 9$

A lot [illegible] these methods & made it very difficult for themselves.

$I = \int_4^5 \left(\frac{1}{t-1} - \frac{8}{t-2} + \frac{9}{t-3} \right) dt$

$= \left[\ln(t-1) - 8 \ln(t-2) + 9 \ln(t-3) \right]_4^5$

$= \left[\ln \frac{(t-1)(t-3)^9}{(t-2)^8} \right]_4^5$

$= \ln \frac{4 \times 2^9}{3^8} - \ln \frac{3 \times 1}{2^8}$

$= \ln \frac{2^{11}}{3^8} \times \frac{2^8}{3} \quad = 19 \ln 2 - 9 \ln 3$

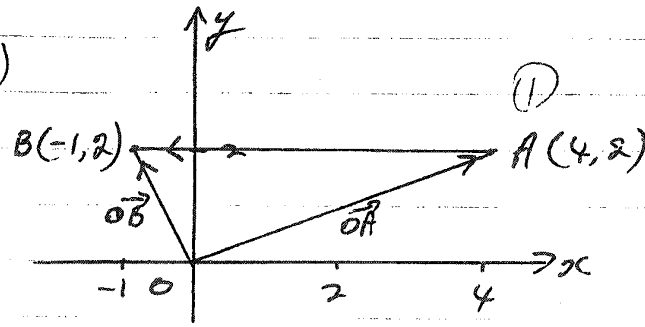
(a)(i) $z_1 z_2 = (1+3i)(1-i) = 1 - i + 3i + 3$

$= \underline{4 + 2i}$ ①

$\frac{z_1}{z_2} = \frac{1+3i}{1-i} \times \frac{1+i}{1+i} = \frac{1+4i-3}{2}$

$= \frac{-2+4i}{2} = \underline{-1+2i}$ ①

(ii) ①



(iii) $\frac{z_1}{z_2} - z_1 z_2 = \vec{AB}$

$= -1 - 4$

$= -5$

which is real $\therefore$

Also, as from diagram (see No: 36 or 13)

(b) (i) $z = \sqrt{3} - i$

$= 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \underline{2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]}$

(ii)(α) $z^5 = 2^5\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right]$

$= 32\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right] = \underline{-16(\sqrt{3}+i)}$

(β) $\bar{z} = 2\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$

$(\bar{z})^5 = 2^5\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right] = \underline{16(-\sqrt{3}+i)}$

(γ) $\frac{z^5}{(\bar{z})^5} = \frac{16\left(\cos -\frac{5\pi}{6} + i\sin -\frac{5\pi}{6}\right)}{16\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$

$\boxed{\text{OR } \frac{-16(\sqrt{3}+i)}{16(-\sqrt{3}+i)}}$

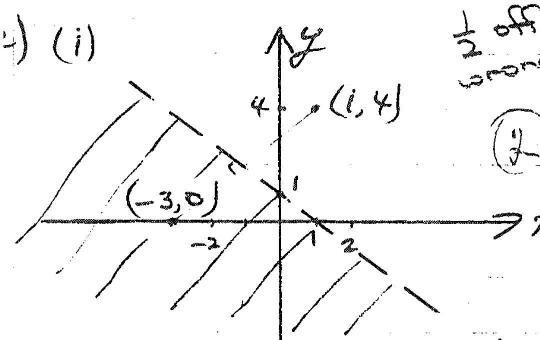
$= \cos\left(-\frac{10\pi}{6}\right) + i\sin\left(-\frac{10\pi}{6}\right)$

$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \underline{\frac{1}{2} + i\frac{\sqrt{3}}{2}}$ ①

(c) (i)

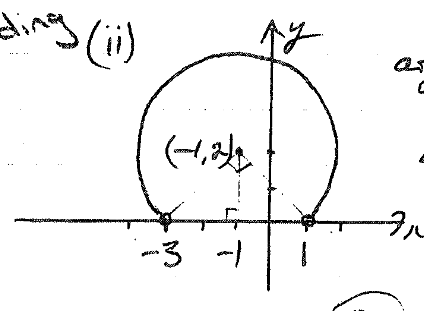
$\frac{1}{2}$ off for wrong shading

②



$|z-(-3)| < |z-(1+4i)|$

(ii)



②

$\arg(z-1) - \arg(z+3) = \frac{\pi}{4}$

$\angle$ at circum. $= \frac{\pi}{4}$

$\angle$ at centre $= \frac{\pi}{2}$

Centre $(-1,2)$

$2r^2 = 4^2$

$r^2 = 8$

(c) $I = \int e^x \sin 2x \, dx$ $\quad$ let $u = \sin 2x$ $\quad \frac{dv}{dx} = e^x$

$= uv - \int v \, du$ $\quad \frac{du}{dx} = 2\cos 2x$ $\quad v = e^x$

$= e^x \sin 2x - \int e^x . 2\cos 2x \, dx$ $\quad$ let $u_1 = \cos 2x$ $\quad \frac{dv_1}{dx} = e^x$

$= e^x \sin 2x - 2(e^x \cos 2x - \int e^x - 2\sin 2x \, dx)$ $\quad \frac{du_1}{dx} = -2\sin 2x$ $\quad v_1 = e^x$

$= e^x \sin 2x - 2e^x \cos 2x - 4\int e^x \sin 2x \, dx$

$I = e^x \sin 2x - 2e^x \cos 2x - 4I$

$5I = e^x \sin 2x - 2e^x \cos 2x$

$I = \frac{e^x \sin 2x - 2e^x \cos 2x}{5} = \frac{e^x}{5}(\sin 2x - 2\cos 2x) + c$

3

d) $\int_0^m \frac{(m-x)^2}{m^2+x^2} dx = \int_0^m \frac{m^2 - 2mx + x^2}{m^2+x^2} dx$

$= \int_0^m \left(1 - \frac{2mx}{m^2+x^2}\right) dx$

$= \left[x - m\ln(m^2+x^2)\right]_0^m$

$= m - m\ln(2m^2) - (0 - m\ln m^2)$

$= m - m\ln(2m^2) + m\ln m^2$

$= m\left(1 + \ln\left(\frac{m^2}{2m^2}\right)\right)$

$= m(1 + \ln \frac{1}{2}) = m(1 - \ln 2)$ # 2 (a)(i) $P(z) = 2z^3 - 5z^2 - 2z + 15$

$P(2+i) = 2(2+i)^3 - 5(2+i)^2 - 2(2+i) + 15$

$= 2(8+12i-6-i) - 5(4+4i-1) - 2(2+i) + 15$

$= 4 + 22i - 15 - 20i - 4 - 2i + 15 = 0$

$\therefore (2+i)$ is a root of $P(z) = 0$ # (1)

(ii) Since coeff. of $P(z)$ are integers, $2-i$ is also a root of $P(z)=0$

$(z-2-i)(z-2+i) = z^2 - 4z + 5$ See No 56

By division, $P(z) = (z^2-4z+5)(2z+3)$

Roots are $\underline{2+i,\ 2-i,\ -\frac{3}{2}}$ (2)

(b) (i) $x^3 - px^2 + qx - r = 0 \quad \therefore \alpha + \beta + \gamma = p$

Let $y = p - x \quad \therefore x = p - y$

Eqn with roots $\alpha+\beta,\ \beta+\gamma,\ \gamma+\alpha$ is

$(p-y)^3 - p(p-y)^2 + q(p-y) - r = 0$

$p^3 - 3p^2y + 3py^2 - y^3 - p(p^2 - 2py + y^2) + pq - qy - r = 0$

$-p^2y + 2py^2 - y^3 + pq - qy - r = 0$

$\underline{y^3 - 2py^2 + (p^2+q)y - pq + r = 0}$ (3)

(OR) $x^3 - 2px^2 + (p^2+q)x - pq + r = 0$

(ii) Product of roots $= (\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) = \underline{pq - r}$ (1)

(c)(i) Let $P(x) = (x-a)^m . Q(x)$

then $P'(x) = m(x-a)^{m-1}. Q(x) + (x-a)^m . Q'(x)$

$= (x-a)^{m-1} . R(x)$

which has multiplicity $(m-1)$ # (2)

(ii) Let $P(x) = x^4 - x^3 - 9x^2 - 11x - 4 = 0$

$P'(x) = 4x^3 - 3x^2 - 18x - 11 = 0$

$P''(x) = 12x^2 - 6x - 18 = 0$

ie $6(x+1)(2x-3) = 0$

Check for $P(-1)$ & $P(\frac{3}{2})$. $P(-1) = 0$, $P(\frac{3}{2}) \neq 0$

$P(4) = 0$ by inspection.

Roots are $-1, -1, -1, 4$ (3)

Very well done!

(a) (i) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ ✓

Also $(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5\cos^4\theta . i\sin\theta + 10\cos^3\theta . i^2\sin^2\theta$

$+ 10\cos^2\theta . i^3\sin^3\theta + 5\cos\theta . i^4\sin^4\theta + i^5\sin^5\theta$

$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$= 11\cos^5\theta - 10\cos^3\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ 3

(ii) Let $x = \cos\theta$. $\cos 5\theta = 16x^5 - 20x^3 + 5x$

$= x(16x^4 - 20x^2 + 5)$

When $\cos 5\theta = 0$ then $x(16x^4 - 20x^2 + 5) = 0$

$\therefore 5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$

ie $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$

When $x = \cos\frac{\pi}{2}$, solution to $x = 0$

$\therefore$ Roots of $16x^4 - 20x^2 + 5 = 0$ are

$x = \cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}$. 3

(b) (i) $I_n = \int \frac{x^n}{\sqrt{x^2-a^2}}\,dx$

$= \int x^{n-1} . \frac{x}{\sqrt{x^2-a^2}}\,dx = \int x^{n-1} \frac{d}{dx}(\sqrt{x^2-a^2})\,dx$

$= x^{n-1}\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2} . (n-1)x^{n-2} . dx$

$= x^{n-1}\sqrt{x^2-a^2} - (n-1)\left\{\int \frac{x^n}{\sqrt{x^2+a^2}} - \int \frac{a^2 x^{n-1}}{\sqrt{x^2-a^2}}\right\}$

$I_n = x^{n-1}\sqrt{x^2-a^2} - (n-1)I_n + a^2(n-1)I_{n-2}$

$n . I_n - (n-1)a^2 . I_{n-2} = x^{n-1} . \sqrt{x^2-a^2}$. # 3

(b) (ii) $\displaystyle\int_2^4 \frac{x^4}{\sqrt{x^2-4}}\,dx = \; ?$ $\quad n=4, \; a=2$

$=4)$: $4I_4 - 3.4.I_2 = \left[x^3\sqrt{x^2-4}\right]_2^4 = 64\sqrt{12} = 128\sqrt{3}$

$=2)$: $2I_2 - 4I_0 = \left[x^1\sqrt{x^2-4}\right]_2^4 = 4\sqrt{12} = 8\sqrt{3}$

$$I_0 = \int_2^4 \frac{1}{\sqrt{x^2-4}}\,dx = \left[\ln\left(x+\sqrt{x^2-4}\right)\right]_2^4$$

$$= \ln\left(\frac{4+\sqrt{12}}{2}\right)$$

$$= \ln\left(2+\sqrt{3}\right)$$

$\therefore 2I_2 = 4\ln(2+\sqrt{3}) + 8\sqrt{3}$

$$4\,I_4 = 6.2I_2 + 128\sqrt{3}$$
$$= 24\ln(2+\sqrt{3}) + 48\sqrt{3} + 128\sqrt{3}$$
$$24\ln(2+\sqrt{3}) + 176\sqrt{3}$$

$\therefore\; \underline{I_4 = 6\ln(2+\sqrt{3}) + 44\sqrt{3}}$ ☺

5.(a) $y = \dfrac{x^2}{x^3+4}$

i) Vertical asymptote $x^3+4=0$ is $\underline{x = -\sqrt[3]{4}}$ $(\doteq -1.6)$

Horizontal " $x \to \infty$, $y = \dfrac{\frac{x^2}{x^3}}{\frac{x^3}{x^3}+\frac{4}{x^3}} = 0$ ie. $\underline{y=0}$

1

) $\dfrac{dy}{dx} = \dfrac{(x^3+4)2x - x^2.3x^2}{(x^3+4)^2}$

$= \dfrac{2x^4+8x-3x^4}{(x^3+4)^2} = \dfrac{8x-x^4}{(x^3+4)^2}$

s.p. $8x - x^4 = 0$

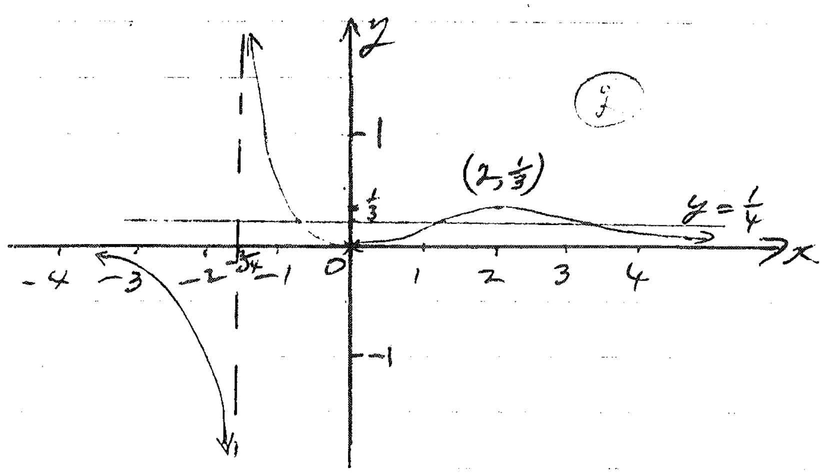
$x(8-x^3) = 0 \Rightarrow x = 0, 2$

when $x=0$, $y=0$ When $x=2$, $y=\frac{1}{3}$

$x=0^-$, $\frac{dy}{dx} = -ve$ $x=2^-$, $\frac{dy}{dx} = +ve$

$x=0^+$, $\frac{dy}{dx} = +ve$ $x=2^+$, $\frac{dy}{dx} = -ve$

$\therefore$ $\underline{(0,0) \text{ is Min. T.P}}$ $\therefore$ $\underline{(2,\frac{1}{3}) \text{ is Max. T.P.}}$ 3

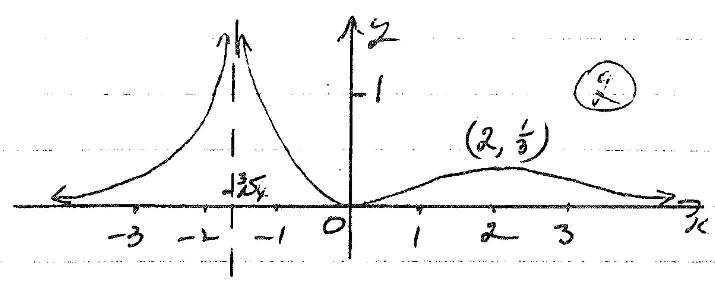


(iv) $4x^2 = x^3+4$

$\dfrac{x^2}{x^3+4} = \dfrac{1}{4}$

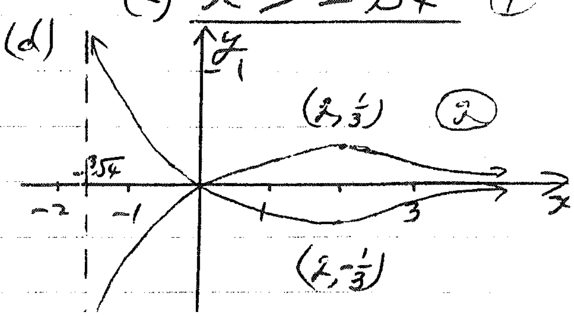
Draw $y = \frac{1}{4}$

It cuts the graph in $\underline{3 \text{ places}}$. 1



(c) $\underline{x > -\sqrt[3]{4}}$ 1

(d)



6. (a) $\int \cos^2 y \, dy = \frac{1}{2}\int (1 + \cos 2y)\, dy$

$= \frac{1}{2}\left(y + \frac{1}{2}\sin 2y\right) + c$

$= \underline{\frac{1}{2}y + \frac{1}{4}\sin 2y + c}$ (1)

(b) (i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Asymptotes $y = \pm \frac{b}{a}x$

$\frac{b}{a} = 1 \quad \therefore b^2 = a^2$

$\therefore \frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$

at $(3,2)$: $\frac{9}{a^2} - \frac{4}{a^2} = 1 \Rightarrow 9 - 4 = a^2 \quad \therefore a^2 = 5$

Eqn. $\underline{\frac{x^2}{5} - \frac{y^2}{5} = 1}$ (2)

OR

$x^2 - y^2 = 5$

(ii) $b^2 = a^2(e^2 - 1)$

$5 = 5(e^2 - 1) \quad \therefore e^2 - 1 = 1$

$e^2 = 2, \quad \underline{e = \sqrt{2}}$ (1)

Foci : $(\pm ae, 0)$, $(\pm\sqrt{5}.\sqrt{2}, 0)$ $\underline{(\pm\sqrt{10}, 0)}$ (1)

(c) $\frac{x^2}{27} + \frac{y^2}{9} = 1$ $\perp$ $x + y = 10 \Rightarrow$ grad $= 1$.

Subst. $y = x + c$ in eqn.

$\frac{x^2}{27} + \frac{x^2 + 2xc + c^2}{9} = 1$

$x^2 + 3x^2 + 6xc + 3c^2 = 27$

$4x^2 + 6xc + 3c^2 - 27 = 0$ $\Delta = 0$ if tangent

$\therefore 36c^2 - 16(3c^2 - 27) = 0$

$9c^2 - 4(3c^2 - 27) = 0$

$9c^2 - 12c^2 - 108 = 0 \Rightarrow 3c^2 = 108$

$c^2 = 36$

$c = \pm 6$

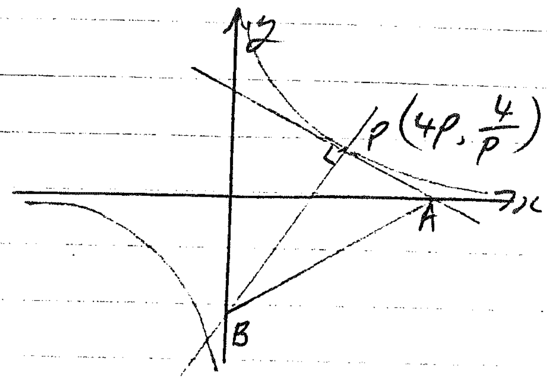
Tangents are $y = x + 6$

$\underline{y = x - 6}$ (3)

e) (i) $y = \frac{16}{x}$

$\frac{dy}{dx} = -\frac{16}{x^2}$

$4p, \quad \frac{dy}{dx} = -\frac{1}{p^2}$



of tangent:

$-\frac{1}{p^2} = \frac{y - \frac{4}{p}}{x - 4p}$

$-x + 4p = p^2 y - 4p$

$\underline{x + p^2 y = 8p}$ (1)

Eqn. of normal:

$p^2 = \frac{y - \frac{4}{p}}{x - 4p}$

$p^2 x - 4p^3 = y - \frac{4}{p}$

$p^3 x - 4p^4 = py - 4$

$\underline{p^3 x - py = 4(p^4 - 1)}$ (1)

At A, $y = 0 \quad \therefore x = 8p$

At B, $x = 0 \quad \therefore y = \frac{4}{p}(1 - p^4)$

Dist. $AP = \sqrt{(4p)^2 + (\frac{4}{p})^2}$

$= \sqrt{16p^2 + \frac{16}{p^2}}$

$= 4\sqrt{\frac{p^4+1}{p^2}}$

Dist. $PB = \sqrt{(4p)^2 + (4p^3)^2}$

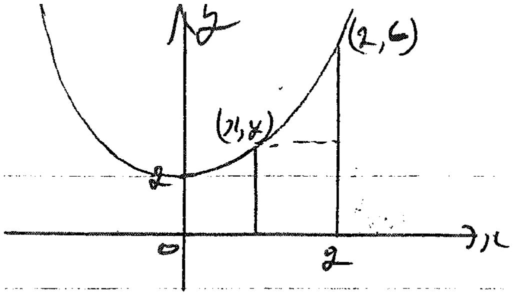
$= \sqrt{16p^2 + 16p^6}$

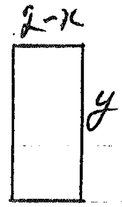
$= 4\sqrt{p^2 + p^6}$

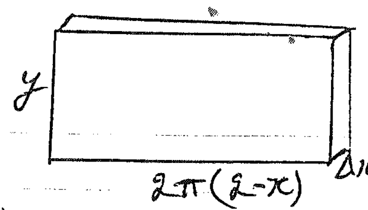
Area $\Delta PAB = \frac{1}{2} \cdot 4\sqrt{\frac{p^4+1}{p^2}} \cdot 4\sqrt{p^2 + p^6}$

$= \underline{8\sqrt{(p^4+1)(p^3+1)}}$ sq. units. (3)

(a)







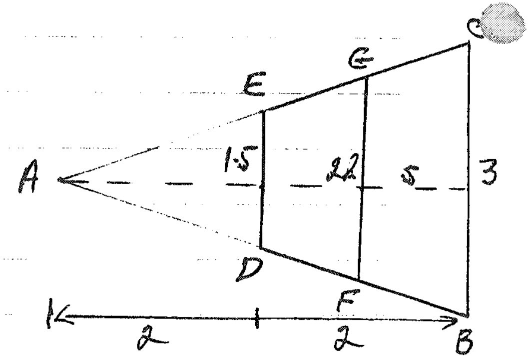
$\Delta A = (2-x)\cdot y$

$\Delta V = 2\pi(2-x)y\,\Delta x$

$$\text{Vol.} = 2\pi\int_0^2 (2-x)(x^2+2)\,dx$$
$$= 2\pi\int_0^2 (2x^2+4-x^3-2x)\,dx$$
$$= 2\pi\left[\tfrac{2}{3}x^3+4x-\tfrac{1}{4}x^4-x^2\right]_0^2$$
$$= 2\pi\left\{\left(\tfrac{16}{3}+8-4-4\right)-(0)\right\} = \underline{\tfrac{32}{3}\pi\ u^3}$$

(4)

5) (i) $\Delta ADE \,|||\, \Delta ABC$ in ratio $1:2$



$$\therefore \frac{2h}{3} = \frac{4-s}{4}$$
$$8h = 12-3s$$
$$h = \tfrac{3}{2}\left(1-\tfrac{1}{4}s\right) \quad \# \quad (3)$$

(ii)
$$\Delta A = \tfrac{1}{3}h$$
$$= \tfrac{1}{3}\cdot\tfrac{3}{2}\left(1-\tfrac{1}{4}s\right)$$
$$\underline{\Delta A = \tfrac{1}{2}\left(1-\tfrac{1}{4}s\right)} \quad \# \quad (2)$$

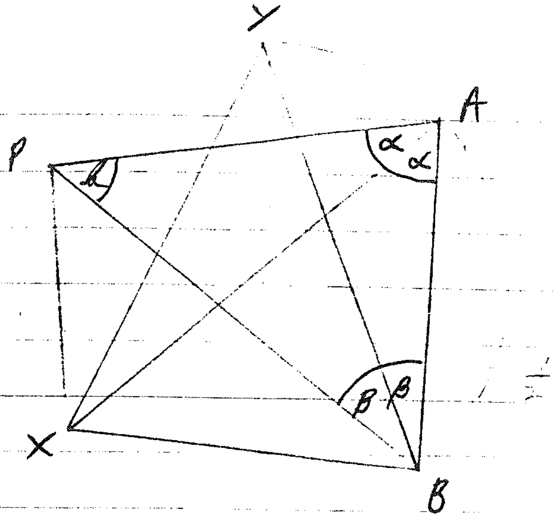
$\therefore h = -\frac{3}{8}s + \frac{3}{2}$

$\therefore h = \frac{3}{2}\left(1-\frac{1}{4}s\right)$

iii)
$$\Delta V = \tfrac{1}{2}\left(1-\tfrac{1}{4}s\right)\Delta s$$
$$V = \lim_{\Delta s\to 0}\sum_{s=0}^{2}\tfrac{1}{2}\left(1-\tfrac{1}{4}s\right)\Delta s$$
$$= \tfrac{1}{2}\int_0^2\left(1-\tfrac{1}{4}s\right)ds$$
$$= \tfrac{1}{2}\left[s-\tfrac{1}{8}s^2\right]_0^2$$
$$= \tfrac{1}{2}\left\{\left(2-\tfrac{4}{8}\right)-(0)\right\}$$
$$V. = \tfrac{1}{2}\cdot\tfrac{3}{2} = \underline{\tfrac{3}{4}\ m^3} \quad (3)$$

(a) Let $\angle PAB = 2\alpha$

$\angle PBA = 2\beta$



Since AB is fixed (constant length), $\angle APB = h$ will be constant for varying positions of P, since angles at cir. standing on the same chord are equal.

PX is a chord.

$\angle PAB = \angle PBX = \alpha$

In $\triangle PAB$, $h + 2\alpha + 2\beta = 180°$

$h + 2(\alpha+\beta) = 180°$

But $\angle XBY = \alpha+\beta$ is constant (angle at circum. subtended by XY)

A constant angle will be subtended by a chord that does not change in length. $\therefore$ XY is constant (4)

Projectiles

(i) $\begin{cases} x = V\cos\theta\, t & \text{ie. } t = \dfrac{x}{V\cos\theta} \\ y = V\sin\theta\, t - \dfrac{gt^2}{2} \end{cases}$

Sub.
$$y = \frac{V\sin\theta . x}{V\cos\theta} - \frac{g}{2}\cdot\frac{x^2}{V^2 . \cos^2\theta}$$

ie
$$y = x\tan\theta - \frac{g\sec^2\theta}{2V^2}x^2$$
(2)

(ii) At (a,h): $h = a\tan\theta - \dfrac{g(1+\tan^2\theta)a^2}{2V^2}$

$$\tan^2\theta\left(\frac{ga^2}{2V}\right) - a\tan\theta + h + \frac{ga^2}{2V^2} = 0$$

$\Delta > 0$ for 2 values: $a^2 > \dfrac{4ga^2}{2V^2}\left(h + \dfrac{ga^2}{2V^2}\right)$

$V^4 - 2ghV^2 > g^2a^2$

$(V^2 - gh)^2 > g^2(a^2+h^2)$ # (2)

D When $n=1$, LHS $= (1+\sqrt2)^1 = 1+\sqrt2$. RHS $= p_1 + q_1\sqrt2 =$ LHS when $p_1 = 1, q_1 = 1$. Thus unique integers p_1, q_1 can be found.

Assume $(1+\sqrt2)^k = p_k + q_k\sqrt2$.

$(1+\sqrt2)^{k+1} = (1+\sqrt2)^1(1+\sqrt2)^k$

$= (1+\sqrt2)(p_k + q_k\sqrt2)$

$= p_{k+1} + q_{k+1}\sqrt2$ (4)

Since p_k, q_k are unique, so are p_{k+1}, q_{k+1}.

③ Since it is true for n=1, then it is true