# St Catherine's School

Year: 12
Subject: 4 Unit Mathematics
Time Allowed: 3 hours
Date: August 1999

Exam number:_______________

**Directions to candidates:**

- All questions are to be attempted.
- All questions are of equal value.
- All necessary **working** must be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Each question attempted should be started on a **new page**.
- Approved calculators are required.
- This page is a cover sheet for **Section A**. Write a cover page for **Section B** and include your student number.
- Hand in your work in **2 bundles**:
  Section A Questions 1, 2, 3 and 4
  Section B Question 5, 6, 7 and 8.

| TEACHER'S USE ONLY Total Marks |
|---|
| A |
| B |
| |
| |
| TOTAL |

**Question 1.**

a) Integrate (i) $\int \frac{x}{\sqrt{2x+3}}dx$

(ii) $\int \sin^{-1}x\, dx$ (7m)

b) Evaluate $\int_0^1 \frac{x^2}{x+1}dx$ (3m)

c) if $I_n = \int \sec^n x\, dx$ use integration by parts to show that

$I_n = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}I_{n-2}$. and hence evaluate $\int_0^{\frac{\pi}{4}} \sec^4 x\, dx$ (5m)

**Question 2. (Start a new page)**

a) Find two complex numbers which satisfy the equation
$3z\bar{z} + 2(z-\bar{z}) = 39 + 12i$ (3m)

b) (i) Given that $(\cos\theta + i\sin\theta)^n = \cos n\theta + \sin n\theta$, where n is a positive integer, show that the result is also true for n,where n is a negative integer.

(ii) Write $(1+i)^{-6}$ in the form a+ib. (5m)

c) (i) Factorise $z^6 - 1$ in the field of reals.

(ii) Factorise $z^6 - 1$ in the field of Complex numbers.

(iii) Find the four roots of $z^4 + z^2 + 1 = 0$. (7m)

**Question 3. ( Start a new page)**

a) Consider the polynomial equation f(x) $= x^n + nkx + (n-1) = 0$.
If $\alpha$ is a double root of this equation,

(i) show that $\alpha = \frac{-1}{k}$.

(ii) Find the possible values of k
when n is odd and when n is even. (6m)

c) If $\int_1^x \frac{dx}{x^{1+\frac{1}{n}}} < \int_1^x \frac{dx}{x} < \int_1^x \frac{dx}{x^{1-\frac{1}{n}}}$ and n is a positive integer, show that

$n(1-x^{-\frac{1}{n}}) < \log_e x < n(x^{\frac{1}{n}}-1)$ and by choosing a suitable value of n,

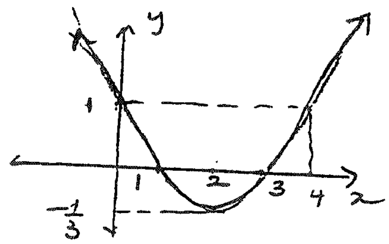
show that $5.6 < \log_e 10^6 < 54$

(6m)

**Section B.**

**Question 5.(Start a new page)**

a)




The sketch above shows the parabola y=f(x) where f(x) is the quadratic
$f(x) = \frac{1}{3}(x-1)(x-3)$.

Without using calculus draw careful sketches of the following curves highlighting features like intercepts,asymptotes and turning points

(i) $y = \frac{1}{f(x)}$

(ii) $y = f(x)^2$

(iii) $y = \tan^{-1} f(x)$

(iv) $y = f(\log_e x)$

(10m)

b) (i) On the same number plane sketch the graphs of
$y = |x| - 3$ *and* $y = 5 + 4x - x^2$

(ii) Hence or otherwise solve $\frac{|x|-3}{5+4x-x^2} > 0$ (5m)

b) A sequence of numbers $U_n$ is such that $U_1 = 3$ *and* $U_2 = 21$ and $U_n = 7U_{n-1} - 10\,U_{n-2}$ for $n \geq 3$. Use the method of Mathematical Induction to show that $U_n = 5^n - 2^n$, $n \geq 1$ (5m)

c) Sketch the locus of Z where Z moves such that:

i) $|Z-2| < |Z-4i|$

(ii) $\text{Arg}(1-Z) = \dfrac{\pi}{4}$ (4m)

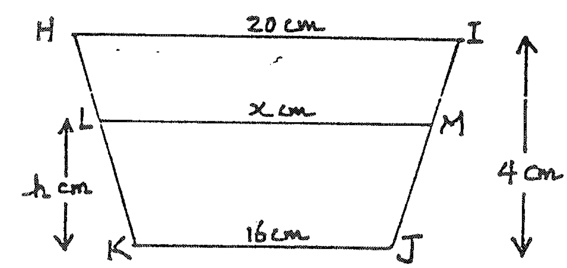
**Question 4. (Start a new page)**

a) If a,b >0 and a ≠b, show that

i) $a^2 + b^2 > 2ab$

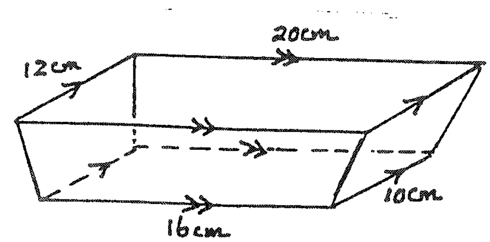
i) $a^4 + b^4 > a^3b + ab^3$ (3m)

b) (i)( Figure not drawn to scale)



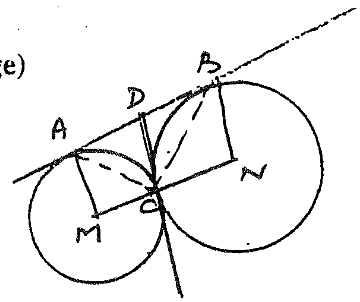

An isosceles trapezium HIJK has parallel sides KJ = 16cm.and HI = 20 cm.. The distance between these sides is 4cm. L lies on HK and M lies on IJ. The shortest distance from K to LM is h cm and LM has length x cm. Show that x=16+h.

(ii) (Figure not drawn to scale)




The above diagram is of a cake tin with a rectangular base with sides of 16cm and 10cm. Its top is also rectangular with dimensions 20cm and 12cm. The tin has a depth of 4cm and each of its four sides is a trapezium. Find its volume by integration. (6m)

**Question 6.(Start a new page)**




In the diagramMCN is a straight line. Circles are drawn with centre M, radius MC and centre N, radius NC.
AB is a common tangent to the two circles with points of contact at A and B respectively.
CD is the common tangent at C and meets AB at D.

(i) Copy the diagram.

(ii) Explain why AMCD and BNCD are cyclic quadrilaterals.

((iii) Show that $\Delta ACD \parallel \Delta CBN$

(iv) Show that $MD \parallel CB$. (7m)

b) If z is a complex number such that $|z-6|+|z+6|=60$.

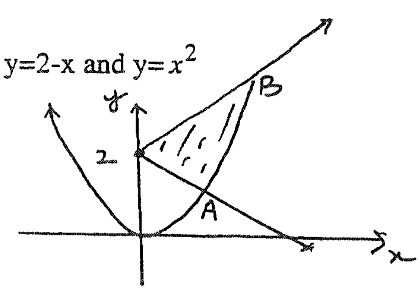
Describe the locus of z and determine its Cartesian Equation. (5m)

c) Find the locus of Z given that $z^2-\bar{z}^2=i$. (3m)

**Question 7. (Start a new page)**

a) The figure below shows y=2+x, y=2-x and $y=x^2$

area considered is shaded :




(i) Find the coordinates of A and B.

(ii) The area bounded by $y=x^2$ and y=2+x, is rotated about y axis.
Use the method of cylindrical shells to determine the volume generated.

(iii) The area bounded by y=2+x, y=2-x and $y=x^2$ is rotated about the y axis.
Use the method of cylindrical shells to determine the volume generated. (7m)

b) For the rectangular hyperbola $xy=1$,

(i) State the eccentricity of the hyperbola.

(ii) find the coordinates of the foci.

(4m)

c) If $\alpha$ and $-\alpha$ are the roots of the equation $x^4+px^3+qx+r=0$. prove that $q^2+p^2r=0$. ( Hint. Relate the roots to the coefficients)

(4m)

**Question 8.( Start a new page)**

a) P is a point $(cp, \frac{c}{p})$ on the hyperbola $xy=c^2$,

(i) The tangent at P intersects the x and y axes at K and L respectively.

Show that PK=PL. (You may assume that the equation of the tangent at P is $x+p^2y=2cp$)

(ii)Let the normal to the hyperbola at P meet the axes of symmetry of the hyperbola at M and N.,show that PK=PM=PN.

(You may assume that the equation of the normal at P is $p^3x-py=c(p^4-1)$)

(iii) Sketch the hyperbola illustrating the results proved above.

(iv) Explain why K,L,M and N are the vertices of a square.

(7m)

b) (i) State the domain and range of the following curves and draw careful sketchesof each of them highlighting main features.

(i) $y=\cos^{-1}x^2$

(ii) $y=x\cos^{-1}x^2$

(ii)Give reasons why $\int_0^1 \cos^{-1}x < \int_0^1 \cos^{-1}x^2$

(8m)

END OF EXAM.