Student Number:____________________________

St Catherine's School

1856 160 YEARS 2016

2016 **HIGHER SCHOOL CERTIFICATE**
TRIAL EXAMINATION

Extension 2 Mathematics

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total Marks – 100

Section I Pages 3 – 6

10 marks
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 7 – 14

90 marks
- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section.

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Section I

10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following is the correct expansion of $\int \sin^3 x \, dx$?

(A) $\frac{1}{3}\cos^3 x - \cos x + c$

(B) $\frac{1}{3}\cos^3 x + \cos x + c$

(C) $\frac{1}{3}\sin^3 x - \sin x + c$

(D) $\frac{1}{3}\sin^3 x + \sin x + c$

2 If α, β and γ are the roots of $4x^3 - 6x^2 + 11x - 5 = 0$ then the polynomial equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ is

(A) $12x^3 + 9x^2 - 16x + 2 = 0$

(B) $3x^3 - 7x^2 + 18x - 11 = 0$

(C) $5x^3 - 11x^2 + 6x - 4 = 0$

(D) $2x^3 - 3x^2 - 22x + 10 = 0$

3 Six people are divided into three groups of two. The number of different ways this can be done is

(A) 90

(B) 45

(C) 30

(D) 15

4 The directrices of the hyperbola $\frac{y^2}{9} - \frac{x^2}{16} = 1$ are

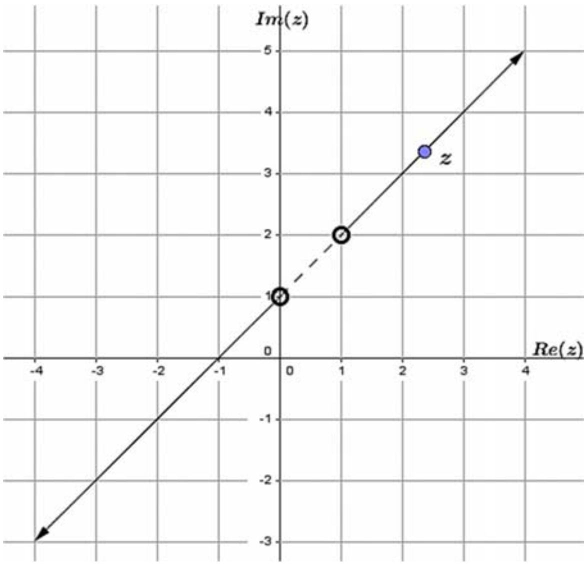
(A) $x = \pm\frac{9}{5}$

(B) $y = \pm\frac{9}{5}$

(C) $y = \pm 5$

(D) $x = \pm 5$

5 Which of the following defines the locus of the complex number z sketched in the diagram below



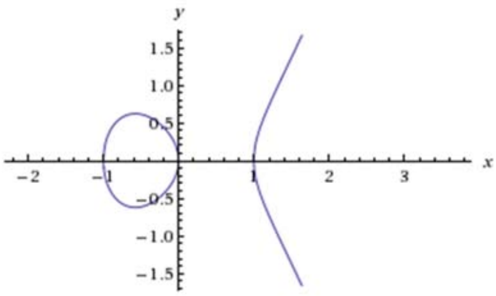
(A) $arg\left(\frac{z-i}{z-1-2i}\right) = \pi$

(B) $arg(z + i) = arg(z - 1 - 2i)$

(C) $arg(z - i) = arg(z - 1 - 2i)$

(D) $arg\left(\frac{z+i}{z-1-2i}\right) = \pi$

6 The diagram below shows the graph of $y^2 = f(x)$



Which expression best represents the function $f(x)$?

(A) $x^2(x-1)$

(B) $x^2(1-x)$

(C) $x(x^2-1)$

(D) $x(1-x^2)$

7 The complex number z lies on the curve $|z-(1+i)| = 1$.

What is the maximum value of $|z|$?

(A) $2+\sqrt{2}$

(B) 2

(C) $\sqrt{2}-1$

(D) $\sqrt{2}+1$

8 What is the gradient of the tangent to the curve $-8x^2+y^2+2y=0$ at the point $(1, 2)$?

(A) 2

(B) $\dfrac{8}{3}$

(C) -1

(D) $\dfrac{4}{5}$

9 Without evaluating the integrals, which of the following will give an answer of zero?

(A) $$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^3\theta+1}{\sin^2\theta}d\theta$$

(B) $$\int_{-1}^{1}\left(x^2-1\right)\left(1-x^2\right)^3 dx$$

(C) $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\cos x\ dx$$

(D) $$\int_{-3}^{3}\left|x^2-9\right| dx$$

10 Given that $\int \sec^n x\ dx = \frac{1}{n-1}\tan x\sec^{n-2} x + \frac{n-2}{n-1}\int \sec^{n-2} x\ dx$, then

$$\int_{0}^{\frac{\pi}{4}} \sec^4 x\ dx =$$

(A) $\frac{4}{3}$

(B) 1

(C) $\frac{5}{6}$

(D) $\frac{6+4\sqrt{2}}{9}$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) If $z = 1 - i\sqrt{3}$ and $w = 1 + i$,

i) Express z and w in modulus argument form, **2**

ii) find in modulus–argument form the complex number $\frac{z^2}{w^3}$. **2**

(b) If $(1+i)^n = x + iy$, show that $x^2 + y^2 = 2^n$ **3**

(c) i) The polynomial $P(x) = x^4 - 2x^3 - 3x^2 + ax + b$ has a double root at $x = 2$. Show that $a = 4$ and $b = 4$. **2**

ii) Factorise P(x) fully. **2**

(d) i) Use the substitution $t = \tan\frac{x}{2}$ to evaluate **3**

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$$

ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}\,dx$ **1**

Question 12 (15 marks)

(a) Find the locus of Z: $|z+1|<|z|$ **2**

(b) i) Consider the expansion of $(cos\theta + i\ sin\ \theta)^5$.
By writing each of $\sin 5\theta and\ cos5\theta$ in terms of $cos\theta\ and\ \sin\theta$, Show that **2**

$$tan5\theta = \frac{tan^5\theta - 10tan^3\theta + 5tan\theta}{1 - 10tan^2\theta + 5tan^4\theta}$$

ii) Find the values of θ, for which $tan\theta$ is a solution of the equation **2**
$x^4 - 10x^2 + 5 = 0$

iii) By solving the equation $x^4 - 10x^2 + 5 = 0$, find the exact values of **3**

$\tan\frac{\pi}{5}\ and \tan\frac{2\pi}{5}$

(c) i) By writing $\frac{(2x-1)(x+1)}{x-1}$ in the form $mx + b + \frac{a}{x-1}$, find the equation of the oblique **2**
asymptote of $y = \frac{(2x-1)(x+1)}{x-1}$.

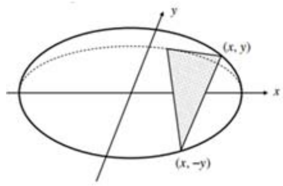
ii) Show that the turning points are (0,1) and (2,9). **2**

iii) Hence sketch the graph of $y = \frac{(2x-1)(x+1)}{x-1}$, clearly indicating the intercepts, the **2**
asymptotes and the turning points.

Question 13 (15 marks)

(a) The base of a solid is in the shape of an ellipse with equation $\frac{x^2}{9}+\frac{y^2}{4}=1$.

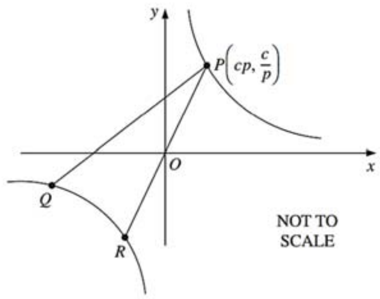
Sections parallel to the y-axis are equilateral triangles, with one side sitting in the base of the solid, as shown in the diagram.



i) Show that the volume of the solid is given by $V=\frac{8\sqrt{3}}{9}\int_0^3 (9-x^2)dx$

ii) Hence find the volume of the solid.

(b)



$P: (cp, \frac{c}{p})$, where $p \neq \pm 1$ is a point on the hyperbola $xy = c^2$.

The normal to the Hyperbola at P meets it again at the point Q.

The line through P and the origin meets the second branch of the hyperbola at R.

You are given that the equation of the normal at P is

$py - c = p^3(x - cp)$ **Do not prove this.**

i) Show that if the point Q *is* $(cq, \frac{c}{q})$, then $q = -\frac{1}{p^3}$

ii) Show that the coordinates of R is $(-cp, -\frac{c}{p})$.

iii) Show that angle QRP is a right angle.

Question 13 continues on the next page

(c) Find $\int \frac{\ln x}{x^2}\,dx$ **2**

(d) i) Find the values of a, b and c such that **2**

$$\frac{3x^2+4x+11}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4}$$

ii) Hence, or otherwise, find **2**

$$\int \frac{3x^2+4x+11}{(x+1)(x^2+4)}\,dx$$

END OF QUESTION 13

Question 14 (15 marks)

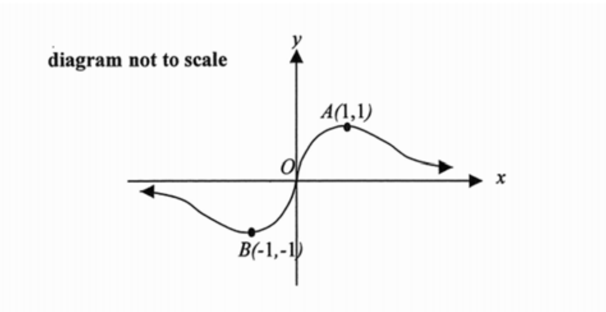
(a) A particle is fired vertically with initial velocity of $\boldsymbol{u}$ metres per second, and is subject both to gravity, g, and air resistance, which is proportional to the square of the speed v.

i) Show that the equation of motion is given by $\ddot{x} = -g - kv^2$, where k is a constant. **1**

ii) By taking $\ddot{x} = v\dfrac{dv}{dx}$ and integrating, show that the greatest height H reached by the particle is given by $H = \dfrac{1}{2k}\ ln\dfrac{g+ku^2}{g}$ **2**

iii) The particle returns to the point of projection. By considering a suitable equation of motion, show that the velocity w, with which it returns to the point of projection is given by $w^2 = \frac{g}{k}\,(1 - e^{-2kH})$ **3**

(b) In the diagram below the graph of $y = \frac{2x}{1+x^2}$ is sketched showing the turning points A: (1,1) and B: (-1,-1)



i) Find the real values of k, for which $\dfrac{2x}{1+x^2} = kx$ has one solution. **3**

ii) Sketch the graph of $y = \ln\left(\frac{2x}{1+x^2}\right)$ **2**

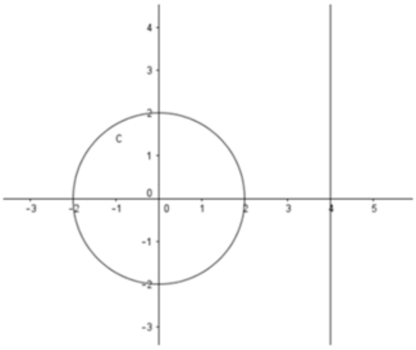
(c) For the hyperbola $xy = 9$

i) Show that the coordinates of the foci are $\left(3\sqrt{2}, 3\sqrt{2}\right)$ and $\left(-3\sqrt{2}, -3\sqrt{2}\right)$. **2**

ii) Find the equation of the directrices of this hyperbola. **2**

Question 15 (15 marks)

(a) The circle $x^2 + y^2 = 4$ is rotated about the line $x = 4$.



i) Using the method of cylindrical shells to show that the volume is given by **1**

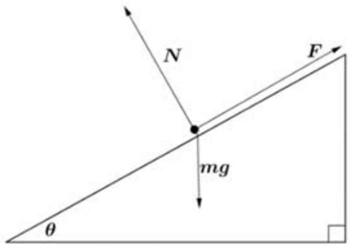
$$V = 4\pi \int_{-2}^{2} (4 - x)\sqrt{4 - x^2}\ dx$$

ii) Hence find the volume of the solid formed. **4**

(b) i) A car of mass m Kg is travelling around a circular track of radius R metres, which is inclined at an angle θ to the horizontal. **2**

The car has no tendency to side slip. Show that the recommended speed of travel, u metres per second is given by $u^2 = Rg\ tan\theta$

ii)



2

For a car travelling with a speed of v metres per second, $v \neq u$, show that the sideways frictional force is given by $F = mgsin\theta - m\frac{v^2}{R}\cos\theta$

iii) If the car is travelling with a speed *one third* the recommended speed, show that the frictional force is given by $F = \dfrac{8mgu^2}{9\sqrt{u^4+g^2R^2}}$ **2**

Question 15 continues on the next page

(c) i) The depth of water in a harbour is 7 metres at low tide and 13 metres at high tide. On a given day, the low tide is at 3AM and high tide is at 9AM. **2**

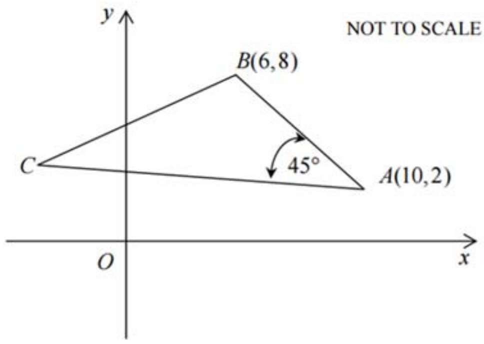
If the motion of the tide follows Simple Harmonic Motion, show that it can be represented by $x = -3\,Cos\frac{\pi}{6}t$, by suitable choice of axes. Explain your choice of axes clearly.

ii) A ship requires 11.5 metres of water to leave the harbour. Find the earliest time the ship can leave the harbour on that day. **2**

END OF QUESTION 15

Question 16 (15 marks)

(a)



In the figure above the length of AC is twice the length of AB.

i) Explain why $\overrightarrow{AB}$ represents the complex number $-4 + 6i$. **1**

ii) Explain why $\overrightarrow{AC}$ represents the complex number $-10\sqrt{2} + 2\sqrt{2}\, i$. **2**

iii) Find the complex number C represents. **1**

(b) i) Show that $\int x^2\sqrt{1-x^3}\,dx = -\frac{2}{9}\sqrt{\left(1-x^3\right)^3} + c$ **1**

ii) Let $I_n = \int_0^1 x^n\sqrt{1-x^3}\,dx$ for $n \geq 2$. **3**

By writing $x^n\sqrt{1-x^3} = x^{n-2} \times x^2\sqrt{1-x^3}$, or otherwise, show that

$$I_n = \frac{2n-4}{2n+5} I_{n-3} \text{ for } n \geq 5 .$$

iii) Hence find I_8 **2**

(c) The polynomial $f(x) = x^3 + cx + d$ has three distinct real roots and hence two turning points at $x = u \text{ and } x = v$.

i) Show that $u \text{ and } v$ are the roots of the equation $x^2 = -\frac{c}{3}$. **1**

ii) Explain why $f(u).f(v) < 0$ **1**

iii) Hence or otherwise show that $27d^2 + 4c^3 < 0$. **3**

END OF PAPER

Student Number: Solution.

2016 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics Extension 2

Multiple Choice Answer Sheet

Completely fill the response circle representing the most correct answer

	A	B	C	D
1.	●	○	○	○
2.	○	○	●	○
3.	○	○	○	●
4.	○	●	○	○
5.	○	○	●	○
6.	○	○	●	○
7.	○	○	○	●
8.	○	●	○	○
9.	○	○	●	○
10.	●	○	○	○

Extension 2 Trial Examination 2016

Q.1. $\int \sin^3 x \, dx = \int (1-\cos^2 x)\sin x \, dx$

$= \int \sin x - \int \cos^2 x \sin x \, dx$

$= -\cos x + \dfrac{\cos^3 x}{3} + C$ $\boxed{A}$

Q.2 $4\left(\frac{1}{x}\right)^3 - 6\left(\frac{1}{x}\right)^2 + 11\left(\frac{1}{x}\right) - 5 = 0$

$4 - 6x + 11x^2 - 5x^3 = 0$ $\boxed{C}$

Q.3 $\dfrac{{}^6C_2 \times {}^4C_2 \times {}^2C_2}{3!} = 15$ $\boxed{D}$

Q.4 $b^2 = a^2(e^2 - 1)$

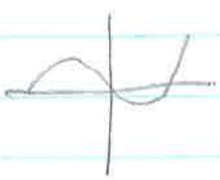
$e = \frac{5}{3}$

$y = \pm a/e \, ; \quad \pm \dfrac{3}{5/3}$

$= \pm \frac{9}{5}$ $\boxed{B}$

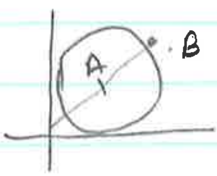
Q.5 $\operatorname{Arg}(z - (1+2i)) = \operatorname{Arg}(z - i)$ $\boxed{C}$

Q.6.



$y = x(x^2 - 1)$ $\boxed{C}$

Q.7



$OA = \sqrt{1^2 + 1^2}$

$AB = 1$

OB is max; $\sqrt{2} + 1$ $\boxed{D}$

Q. 8

$$-16x + 2yy' + 2y = 0$$

$$-16 + 4y' + 2y' = 0$$

$$6y'' = 16$$

$$y'' = \frac{8}{3}$$

$\boxed{B}$

Q. 9.

$\boxed{E}$ $\sin^7 x \cos x$ is an odd fn.

Q.10

$$\int_0^{\pi/4} \sec^4 x = \left(\frac{1}{3} \tan x \sec^2 x\right)_0^{\pi/4} + \frac{2}{3}\int_0^{\pi/4} \sec^2 x \, dx$$

$$= \frac{2}{3} + \frac{2}{3}(\tan x)_0^{\pi/4}$$

$$= \frac{4}{3}$$

$\boxed{A}$

Q.11

$z = 1 - i\sqrt{3}$

(i) $z = 2\,\text{cis}(-\pi/3)$ $\boxed{1M}$

$w = 1 + i$

$= \sqrt{2}\,\text{cis}\frac{\pi}{4}$ $\boxed{1M}$

(ii) $\left|\frac{z^2}{w^3}\right| = \frac{|z|^2}{|w|^3} = \frac{4}{2\sqrt{2}} = \sqrt{2}$ $\frac{1}{2}$

$\text{Arg}\,\frac{z^2}{w^3} = 2\,\text{Arg}\,z - 3\,\text{Arg}\,w$ $\frac{1}{2}$

$= 2\left(-\frac{\pi}{3}\right) - 3\left(\pi/4\right)$

$= -\frac{17\pi}{12}$

$= \frac{7\pi}{12}.$ 1.

$\therefore \frac{z^2}{w^3} = \sqrt{2}\,\text{cis}\,\frac{7\pi}{12}$

b). $(1+i)^n = x + iy$ $1 + i = \sqrt{2}\,\text{cis}\frac{\pi}{4}$ 1M.

$\left(\sqrt{2}\,\text{cis}\frac{\pi}{4}\right)^n = x + iy$

$x = \sqrt{2}^n \cos\frac{n\pi}{4}$ $y = \sqrt{2}^n \sin\frac{n\pi}{4}$ 1M.

$x^2 + y^2 = (\sqrt{2}^n)^2\left(\cos^2\frac{n\pi}{4} + \sin^2\frac{n\pi}{4}\right)$ 1M.

$= 2^n$

Q. 11 c) $P(x) = x^4 - 2x^3 - 3x^2 + ax + b$

$P'(x) = 4x^3 - 6x^2 - 6x + a$

$P(2) = 0$ $\quad P'(2) = 0$ 1M.

$16 - 16 - 12 + 8 + b = 0$ $\quad b = 4$ 1M

$32 - 24 - 12 + a = 0$ $\quad a = 4$ 1M.

(ii) $(x-2)^2$ is a factor of $P(x)$; Ordinarily.

$x^4 - 2x^3 - 3x^2 + 4x + 4 = (x^2 - 4x + 4)(x^2 + 2x + 1)$ 1M

$= (x-2)^2 (x+1)^2$ 1M.

d) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\, dx$ $\quad t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\, dx$

$\displaystyle = \int_0^2 \frac{(1+t^2)}{(1+t)^2} \frac{2\,dt}{1+t^2}$ $1\frac{1}{2}$ $\quad \frac{dt}{dx} = \frac{1}{2}(1+t^2)$

$dx = \frac{2\,dt}{1+t^2}$

$\displaystyle = 2\int_0^2 (1+t)^{-2}\, dt$ 1M.

$1 + \sin x$

$= 1 + \frac{2t}{1+t^2}$

$\displaystyle = 2\left(-\frac{1}{1+t}\right)_0^2$ 1 $\quad = \frac{(1+t)^2}{1+t^2}$

$= 1$

$x = 0;\ t = 0$

$x = \frac{\pi}{2}\ \ t = 2$

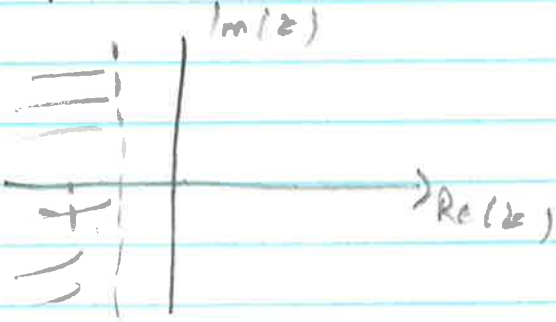
$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}\,dx = \int_0^{\pi/2} \frac{\sin x + 1 - 1}{1+\sin x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} dx - \int_0^{\pi/2} \frac{1}{1+\sin x}$$

$$= \frac{\pi}{2} - 1. \qquad \boxed{1M}$$

Question 12

a) $|z+1| < |z|$



$x < -\frac{1}{2}$

b) $(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5\cos^4\theta\,(i\sin\theta) + 10\cos^3\theta\,(i\sin\theta)^2$
$+ 10\cos^2\theta\,(i\sin\theta)^3 + 5\cos\theta\,(i\sin\theta)^4$
$+ (i\sin\theta)^5$

$$\left.\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ \sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\end{aligned}\right\} 1M.$$

$$\tan 5\theta = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$$

Divide by $\cos^5\theta$

$$= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}. \qquad 1M.$$

11) $x^4 - 10x^2 + 5 = 0$.

Let $x = \tan\theta$; if $\tan 5\theta = 0$.

$$5\tan\theta - 10\tan^3\theta + \tan^5\theta = 0$$

$$\tan\theta(\tan^4\theta - 10\tan^2\theta + 5) = 0 \quad \boxed{1M}$$

$\tan 5\theta = 0$; $5\theta = 0, \pi, 2\pi, 3\pi, 4\pi$

$$\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$$

$\theta = 0$ is a solution to $\tan\theta = 0$

$\therefore \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$ are the roots to

$\tan^4\theta - 10\tan^2\theta + 5 = 0$ $\boxed{1M}$

Equivalently,

$\tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}$ are the roots

of $x^4 - 10x^2 + 5 = 0$.

(iii) $x^4 - 10x^2 + 5 = 0$

$$x^2 = \frac{10 \pm 4\sqrt{5}}{2} = 5 \pm 2\sqrt{5}\ ; \quad x = \pm\sqrt{5 \pm 2\sqrt{5}} \quad \boxed{1M}$$

$\tan\frac{\pi}{5} > 0$; $\tan\frac{2\pi}{5} > 0$ and $\tan\frac{\pi}{5} < \tan\frac{2\pi}{5}$ $\boxed{1M}$

$$\therefore \tan\frac{\pi}{5} = \sqrt{5 - 2\sqrt{5}} \quad \boxed{1M}$$

$$\tan\frac{2\pi}{5} = \sqrt{5 + 2\sqrt{5}}$$

Q.12 c) $\dfrac{(2x-1)(x+1)}{x-1}$;

$$\begin{array}{r} 2x+3 \\ x-1\,\overline{)\,2x^2+x-1} \\ \underline{2x^2-2x} \\ 3x-1 \\ \underline{3x-3} \\ 2 \end{array}$$

$y = 2x+3 + \dfrac{2}{x-1}$ 1M.

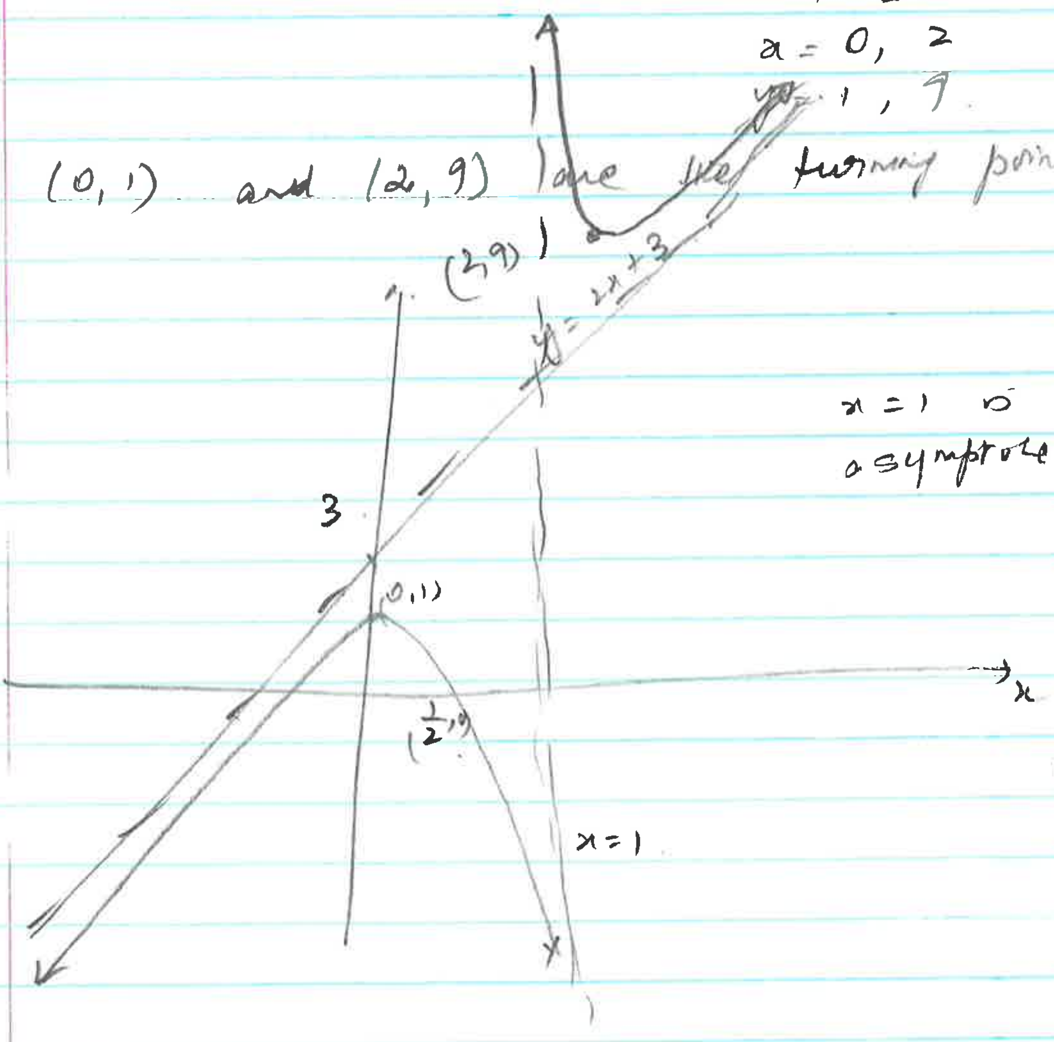
Equation of the oblique asymptote.
$y = 2x+3$. 1M.

ii) $y' = 2 - 2(x-1)^{2}$ $\boxed{1M}$

Stationary points at $y'=0$ $(x-1)^2 = 1$
$x-1 = \pm 1$
$x = 0, 2$ $\boxed{1M}$
$y = 1, 9$.

$(0,1)$ and $(2,9)$ are the turning points.

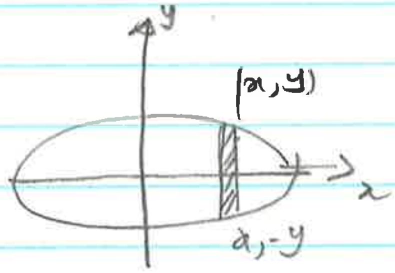
iii)



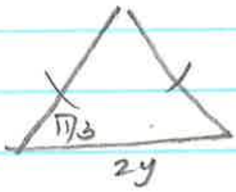
$x=1$ is an asymptote

$\boxed{2M}$

Question 13

a) 

A slice



Area of a slice $= \frac{1}{2} \cdot 2y \cdot 2y \cdot \sin\frac{\pi}{3}$

$= \sqrt{3}\, y^2$

$= \sqrt{3}\left(\frac{4}{9}(9-x^2)\right)$

$\frac{y^2}{4} = 1 - \frac{x^2}{9}$

$y^2 = \frac{4}{9}(9-x^2)$

$\Delta V = \sqrt{3}\, y^2\, \Delta x$

$\Delta V = \frac{4\sqrt{3}}{9}(9-x^2)\,\Delta x$ (1M)

$V = \frac{4\sqrt{3}}{9}\int_{-3}^{3}(9-x^2)\,dx = \frac{8\sqrt{3}}{9}\int_{0}^{3}(9-x^2)\,dx$ (M)

ii) $V = \frac{8\sqrt{3}}{9}\left(9x - \frac{x^3}{3}\right)_0^3$

$= \frac{8\sqrt{3}}{9} \times 18$

$= 16\sqrt{3}$ $\boxed{1M}$

(b) $xy = c^2$

$py - c = p^3(x - cp)$

passes through $\left(cq, \frac{c}{q}\right)$

8

$p \cdot \frac{c}{q} - c = p^3(cq - cp)$

$\frac{\cancel{c}(p-q)}{q} = cp^3(q-p)$ 1M.

$(p \neq q)$

$\frac{1}{q} = -p^3$

$q = -\frac{1}{p^3}$ 1M.

iii) Equation of OP : $y = \frac{c/p}{cp} x$

$y = \frac{1}{p^2} x$ 1M.

meets $xy = c^2$

$x \cdot \frac{x}{p^2} = c^2$

$x^2 = p^2 c^2$

$x = \pm pc$

$x = -cp$

$y = -\frac{cp}{p^2}$

$= -\frac{c}{p}$

$\therefore$ R : $(-cp, -c/p)$ 1M

iii) $m_{QR} = \frac{\frac{c}{q} + \frac{c}{p}}{cq + cp} = \frac{1}{pq}$

$m_{PR} = \frac{2c/p}{2cp} = \frac{1}{p^2}$

$m_{QR} \times m_{PR} = \frac{1}{pq \times p^2} = \frac{1}{p^3 q} = \frac{-q}{q} = -1$

$(q = -\frac{1}{p^3})$

$\therefore$ $\angle QRP$ is a right angle

9

c) $\int \frac{\ln x}{x^2} \, dx$

$u = \ln x \qquad v' = x^{-2}$

$u' = \frac{1}{x} \qquad v = -\frac{1}{x}$

$= -\frac{\ln x}{x} + \int \frac{1}{x} \cdot \frac{1}{x} \, dx$

$= \frac{-\ln x}{x} - \frac{1}{x} + c$

$\int uv' = uv - \int u'v$

d) $\frac{3x^2+4x+11}{(x+1)(x^2+4)} = \frac{a}{x+1} + \frac{bx+c}{x^2+4}$

$3x^2+4x+11 = a(x^2+4) + (bx+c)(x+1)$

$$\left.\begin{aligned} 3 &= a+b \\ 4 &= b+c \\ 11 &= 4a+c \end{aligned}\right\}$$

$c - a = 1$

$c + 4a = 11$

$5a = 10$

$\underline{a = 2}$

$\underline{b = 1}$

$\underline{c = 3}$

$\boxed{2M}$

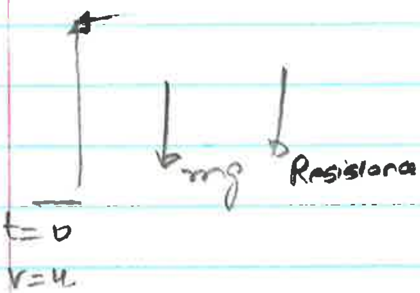
$\therefore \frac{3x^2+4x+11}{(x+1)(x^2+4)} = \frac{2}{x+1} + \frac{x+3}{x^2+4}$

$\int \frac{3x^2+4x+11}{(x+1)(x^2+4)} \, dx = \int \frac{2}{x+1} \, dx + \int \frac{x}{x^2+4} \, dx + \int \frac{3}{x^2+4} \, dx$

$= 2 \ln(x+1) + \frac{1}{2} \ln(x^2+4) + \frac{3}{2} \tan^{-1}\frac{x}{2} + c$

$\boxed{2M}$

Question 14



$R \propto v^2$

$R = mkv^2$

Equation of motion

$ma = -mg - mkv^2$

$\therefore a = -g - kv^2$ (i)

(ii) $v\frac{dv}{dx} = -g - kv^2$

$\int \frac{v\,dv}{g+kv^2} = -\int dx$

$\frac{1}{2k} \ln(g+kv^2) = -x + c$

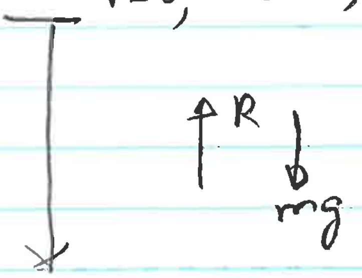
$x = 0;\ v = u \quad \therefore c = \frac{1}{2k} \ln(g+ku^2)$ 1M

$\therefore x = \frac{1}{2k}\left(\ln(g+ku^2) - \ln(g+kv^2)\right)$

$= \frac{1}{2k} \ln \frac{g+ku^2}{g+kv^2}$

$V = 0;\ x = H \quad \therefore H = \frac{1}{2k} \ln \frac{g+ku^2}{g}$ 1M

(iii) Downward motion follows a different equation of motion; Reset: $v=0;\ x=0;\ t=0$



$ma = mg - mkv^2$

$a = g - kv^2$ 1M

$V\frac{dv}{dx} = g - kv^2$ (iii).

$$\int \frac{v\,dv}{g - kv^2} = \int dx$$

$$-\frac{1}{2k} \ln(g - kv^2) = x + c$$

$x = 0;\quad v = 0 \qquad \therefore\ c = -\frac{1}{2k} \ln g$

$$\therefore\ x = \frac{1}{2k}\left(\ln g - \ln(g - kv^2)\right)$$

$$= \frac{1}{2k} \ln \frac{g}{g - kv^2} \qquad \boxed{1M}$$

$x = H;\quad v = w$

$$H = \frac{1}{2k} \ln \frac{g}{g - kw^2} \qquad \boxed{1M}$$

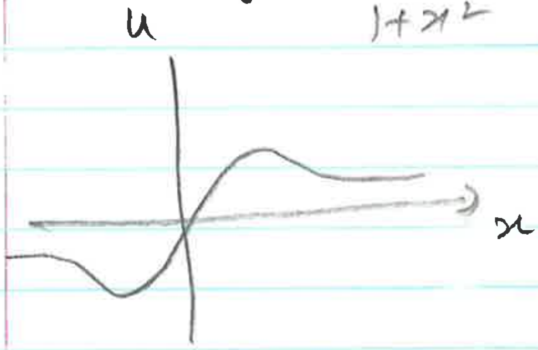
$$2kH = \ln \frac{g}{g - kw^2}$$

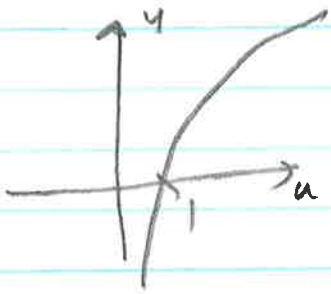
$$e^{-2kH} = \frac{g - kw^2}{g}$$

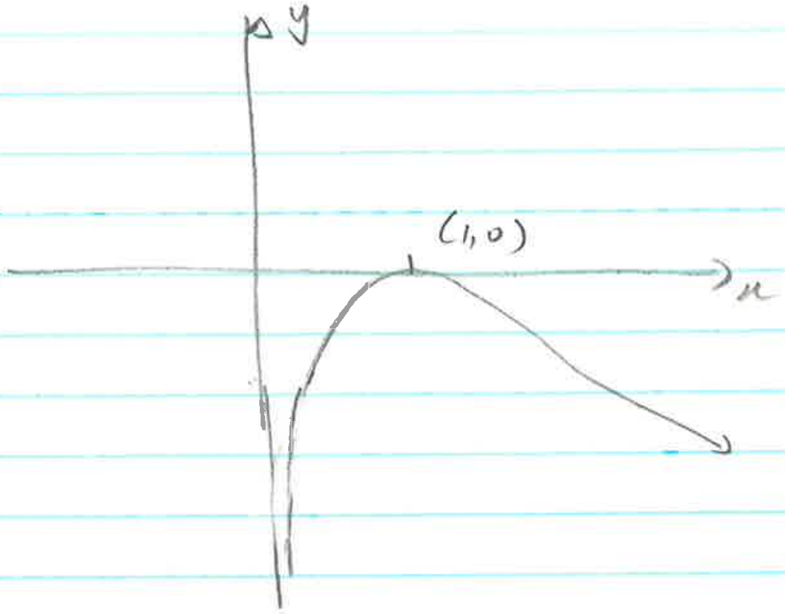
$$\frac{kw^2}{g} = 1 - e^{-2kH}$$

$$w^2 = \frac{g}{k}\left(1 - e^{-2kH}\right) \qquad \boxed{1M}$$

b) (ii) $\quad y = \dfrac{2x}{1+x^2}$







Notes: (not tested)

$y = \ln \frac{2x}{1+x^2}$ is defined when $\frac{2x}{1+x^2} > 0$; when $x > 0$

$\frac{2x}{1+x^2} = 1$; $x = 1$; $\ln 1 = 0$

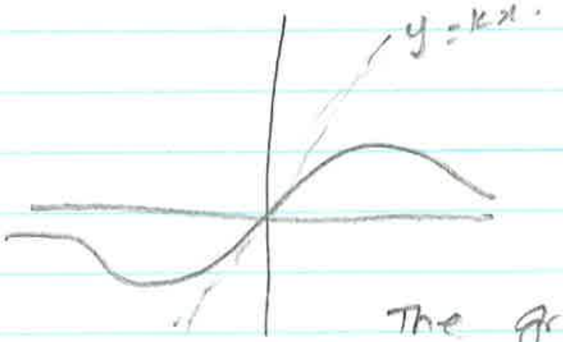
$0 < x < 1$ $\quad \frac{2x}{1+x^2} < 1$; $\quad \ln \frac{2x}{1+x^2} < 0$

$x > 1$ $\quad \frac{2x}{1+x^2} < 1$ $\quad \ln \frac{2x}{1+x^2} < 0$

$x \to 0$ $\quad \frac{2x}{1+x^2} \to 0$ $\quad \ln \frac{2x}{1+x^2} \to -\infty$

$x \to \infty$ $\quad \frac{2x}{1+x^2} \to 0$ $\quad \ln \frac{2x}{1+x^2} \to -\infty$

(1)



$y' = \frac{(x^2+1)(2) - 2x(2x)}{(1+x^2)^2}$

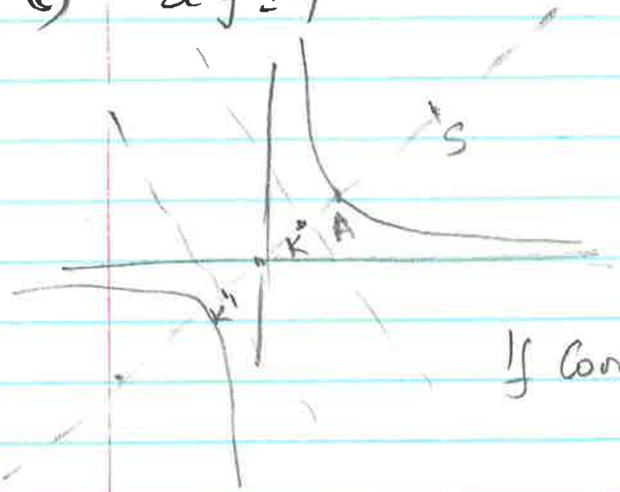
$y'_{\text{at } x=0} = 2$ $\quad \boxed{1M}$

The gradient of the tangent at (0,0) is 2.

$\frac{2x}{1+x^2} = kx$ has one solution;

$k \leq 0$; $\quad k > 2$
$\boxed{1M}$ $\quad \boxed{1M}$

(c) $xy = 9$ Note $e = \sqrt{2}$



Note $OS = OA \times \sqrt{2}$

$= \sqrt{3^2+3^2} \cdot \sqrt{2}$

$= 3\sqrt{2} \cdot \sqrt{2}$

$= 6$ (1M)

If Cords. of S: (s, s)

$s^2 + s^2 = 6^2$

$s = \pm 3\sqrt{2}$

foci: $(3\sqrt{2}, 3\sqrt{2})$ and $(-3\sqrt{2}, -3\sqrt{2})$ (1M)

(ii) K: where $y = x$ meets the directrix (in quad. 1)

$OK = \frac{OA}{e} = \frac{3\sqrt{2}}{\sqrt{2}} = 3$.

if $K: (k, k)$ $2k^2 = 9$

$k = \pm \frac{3}{\sqrt{2}}$

$K: \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ $K'\left(\frac{-3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right)$ (1M)

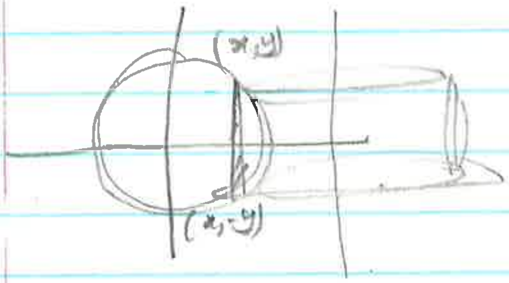
Eqn of directrix

$y - \frac{3}{\sqrt{2}} = -1\left(x - \frac{3}{\sqrt{2}}\right)$ (1M)

$x + y = \frac{6}{\sqrt{2}}$ and also $x + y = \frac{-6}{\sqrt{2}}$

Question 15

a)



$x^2+y^2=4$

$y=\pm\sqrt{4-x^2}$

$\Delta V = 2\pi(4-x)(2y)\Delta x$

$= 4\pi(4-x)\sqrt{4-x^2}\,\Delta x$

$$V = 4\pi\int_{-2}^{2}(4-x)\sqrt{4-x^2}\,dx \qquad \boxed{1M}$$

(ii) $V = 16\pi\int_{-2}^{2}\sqrt{4-x^2}\,dx - 4\pi\int_{-2}^{2}x\sqrt{4-x^2}\,dx$

$\int_{-2}^{2}x\sqrt{4-x^2}\,dx = 0$ $\quad x\sqrt{4-x^2}$ being an odd fn $\quad \boxed{1M}$

$\therefore V = 16\pi\int_{-2}^{2}\sqrt{4-x^2}\,dx$

$= 16\pi\int_{-2}^{2}\sqrt{4-x^2}\,dx$

$= 16\pi\left(\dfrac{\pi\cdot(2)^2}{2}\right)$

$= 32\pi^2$

0

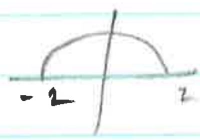
$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$x=0;\ \theta=0$

$x=2;\ \theta=\frac{\pi}{2}$

OR



OR

$V = 32\pi\int_0^2\sqrt{4-x^2}\,dx$ (even fn)

$= 32\pi\times\frac{4}{2}\int_0^{\pi/2}\cos^2\theta\,d\theta = 64\pi\int_0^{\pi/2}(1+\cos2\theta)\,d\theta$

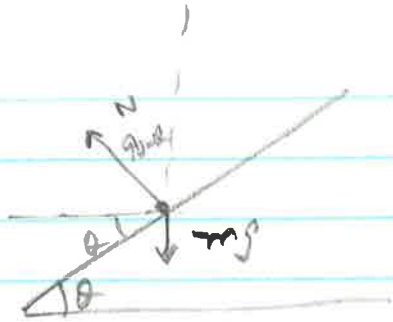
$= 64\pi\left(\theta+\frac{\sin2\theta}{2}\right)_0^{\pi/2}$

$= 32\pi^2$

15

15 b)

i)



Resolving forces N and mg horizontally and vertically.

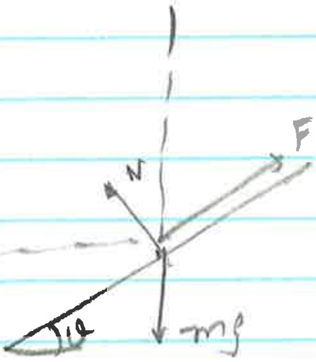
$F = 0$.

$N\cos\theta = mg$

$N\sin\theta = \dfrac{mv^2}{R}$

$\tan\theta = \dfrac{v^2}{Rg}$ $\qquad v^2 = Rg\tan\theta$.

ii)



Resolve F, N, mg horizontally and vertically,

$N\cos\theta + F\sin\theta = mg$ — ①

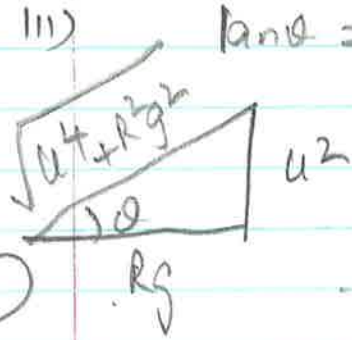
$N\sin\theta - F\cos\theta = \dfrac{mv^2}{R}$ — ② (1M)

eliminate N ① × $\sin\theta$ − ② × $\cos\theta$

$F\sin^2\theta + F\cos^2\theta = mg\sin\theta - \dfrac{mv^2}{R}\cos\theta$

$F = mg\sin\theta - \dfrac{mv^2\cos\theta}{R}$ (1M)

iii)



(1M)

$\tan\theta = \dfrac{u^2}{Rg}$ $\qquad \sin\theta = \dfrac{u^2}{\sqrt{u^4+R^2g^2}}$ $\qquad \cos\theta = \dfrac{Rg}{\sqrt{u^4+R^2g^2}}$

$\therefore F = mg\dfrac{u^2}{\sqrt{u^4+R^2g^2}} - \dfrac{mv^2}{R}\cdot\dfrac{Rg}{\sqrt{u^4+R^2g^2}}$

$v = \dfrac{u}{3}$

$= \dfrac{mg\,u^2}{\sqrt{u^4+R^2g^2}} - m\dfrac{u^2}{9}\cdot\dfrac{g}{\sqrt{u^4+R^2g^2}}$

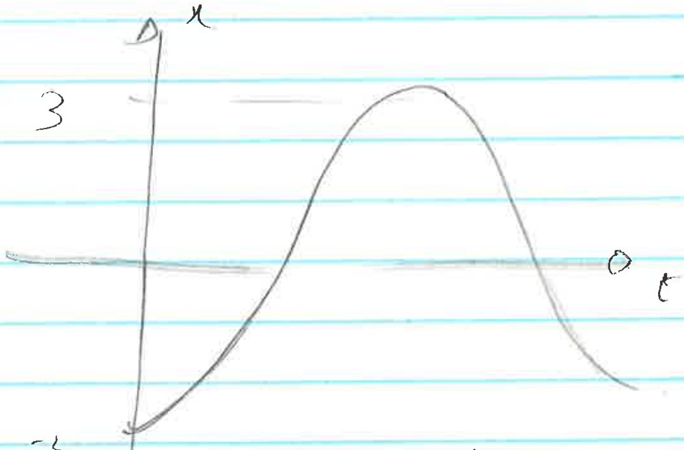
$= \dfrac{8mg\,u^2}{\sqrt{u^4+R^2g^2}}$ (1M)

(6)

c)

Set: $x = 0$ at 10m

$t = 0$ 3AM ; $x = 3$; 13m
$x = -3$; 7m.



amplitude is 3 Period : 12 hrs ;

$$\frac{2\pi}{n} = 12$$
$$n = \frac{\pi}{6}$$ (1M)

$\therefore \quad x = -3\cos\frac{\pi}{6}t$ (1M)

ii) $x = 1.5$ m at 11.5 meter

$$1.5 = -3\cos\frac{\pi}{6}t$$

$$\cos\frac{\pi}{6}t = -\frac{1}{2}$$
$$\frac{\pi}{6}t = \frac{2\pi}{3}$$ $t = 4$

The earliest time is 3AM + 4hr = 7AM.

Question 16

a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (6+8i) - (10+2i)$

$= -4 + 6i$ 1M.

$\overrightarrow{AC} = \overrightarrow{AB} \times 2\operatorname{cis}\frac{\pi}{4}$ 1M

$= (-4+6i)(\sqrt{2} + i\sqrt{2})$

$= (-4\sqrt{2} - 6\sqrt{2}) + i(6\sqrt{2} - 4\sqrt{2})$

$= -10\sqrt{2} + 2\sqrt{2}\,i$ 1M

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$= (10+2i) - 10\sqrt{2} + 2\sqrt{2}\,i$

$= (10 - 10\sqrt{2}) + i(2 + 2\sqrt{2})$

b) $\int x^3\sqrt{1-x^3}\,dx = -\frac{1}{3}\int (1-x^3)^{\frac{1}{2}}(-3x^2)\,dx.$

$= -\frac{1}{3}\cdot\frac{2}{3}(1-x^3)^{3/2} + c$

$= -\frac{2}{9}(1-x^3)^{3/2} + c.$ 1M

$I_n = \int_0^1 x^n\sqrt{1-x^3}\,dx$

$u = x^{n-2}$ $\quad v' = x^2\sqrt{1-x^3}$

$u' = (n-2)x^{n-3}$ $\quad v = -\frac{2}{9}(1-x^3)^{3/2}$

$\int uv' = uv - \int u'v$

$$I_n = -\frac{2}{9}\left(x^{n-2}(1-x^3)^{3/2}\right)_0^1 - \int_0^1 (n-2)x^{n-3}\left(\frac{-2}{9}\right)(1-x^3)^{3/2}\,dx$$ (1M)

$$= \frac{2}{9}(n-2)\int_0^1 x^{n-3}\sqrt{1-x^3}\,(1-x^3)\,dx$$

$$= \frac{2}{9}(n-2)\left[\int_0^1 x^{n-3}\sqrt{1-x^3}\,dx - \int_0^1 x^n\sqrt{1-x^3}\,dx\right]$$ (1M)

$$I_n = \frac{2}{9}(n-2)\left(I_{n-3} - I_n\right)$$

$$I_n\left(1+\frac{2}{9}(n-2)\right) = \frac{2n-4}{9} I_{n-3}$$

$$I_n \cdot \left(\frac{2n+5}{9}\right) = \frac{2n-4}{9} I_{n-3}$$

$$I_n = \frac{2n-4}{2n+5} I_{n-3}.$$ (1M)

$$I_8 = \frac{12}{21} I_5$$

$$= \frac{\cancel{12}^{4}}{21_7} \cdot \frac{\cancel{6}^{2}}{15} \cdot I_2$$ (1M)

$$= \frac{8}{35}\left[-\frac{2}{9}\sqrt{(1-x^3)^3}\right]_0^1$$

$$= \frac{8}{35} \times \frac{2}{9} = \frac{16}{315}.$$ (1M)

c) $f(x) = x^3 + cx + d$

$f'(x) = 3x^2 + c$

u & v are roots of $f'(x) = 0$; (since they are the stat. pts)

" " $x^2 = -\frac{c}{3}$

ii) 3 distinct real roots

$\Rightarrow$ 3 points of intersection with the x axis + 2 stationary points

possible when $(u, f(u))$ and $(v, f(v))$ are on opposite sides of the x axis

ie $f(u) \cdot f(v) < 0$

iii) $f(u) \cdot f(v) = (u^3 + cu + d)(v^3 + cv + d)$

$= u^3v^3 + cuv^3 + dv^3 + cvu^3 + c^2uv + cvd + du^3 + duv + d^2.$

note $u + v = 0$; $uv = \frac{c}{3}$

$\therefore f(u) \cdot f(v)$

$= \frac{c^3}{27} + cuv(u^2 + v^2) + d(v^3 + u^3) + cd(u+v) + \frac{c^3}{3} + d^2$

$= \frac{c^3}{27} + \frac{c^2}{3}\left(\frac{-2c}{3}\right) + 0 + 0 + \frac{c^3}{3} + d^2$

$= \frac{4c^3}{27} + d^2 < 0$ $\quad 4c^3 + 27d^2 < 0$