St. George Girls' High School

1998

TRIAL HIGHER SCHOOL CERTIFICATE

Year 12

MATHEMATICS

4 Unit

TIME ALLOWED: 3 HOURS
(Plus 5 minutes' reading time)

INSTRUCTIONS TO CANDIDATES:

1. All questions may be attempted
2. All necessary working must be shown
3. Marks may be deducted for careless or poorly presented work
4. Begin each question on a NEW PAGE
5. A list of standard integrals is included at the end of this paper
6. The mark allocated for each question is listed at the side of the question

Students are advised that this a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

QUESTION 2

a) i) Verify that $\alpha_1 = -1-3i$ is a root of the equation 1

$$z^2 + iz + 5(1-i) = 0$$

ii) By considering the coefficient of z in the above equation, or otherwise, find the second root α_2 1

iii) Find the modulus and argument of β where $\dfrac{1}{\beta} = \dfrac{1}{\alpha_1} + \dfrac{1}{\alpha_2}$ 2

b) Given that $z_1 = 2\left(\cos 30^0 + i\sin 30^0\right)$ and $z_2 = 3\left(\cos 40^0 + i\sin 40^0\right)$ show the vector $z_1 z_2$ on two different Argand diagrams using the different methods:

i) similar triangles (showing full reasons) 2

AND ii) basic relationships involving the modulus and argument of the product of complex numbers (no need to prove these relationships) 2

c) Using these same values for z_1 and z_2, show on an Argand diagram the locus of z if 2

$$\arg(z - z_1) - \arg(z - z_2) = 180^0$$

d) i) If $W = \dfrac{z+2i}{z-4}$, where $z = x + iy$, express the real and imaginary parts of W in terms of x and y. 2

ii) P is the point which represents z in the Argand diagram.

α) If W is purely imaginary, prove that the locus of P is a circle. 2

β) If W is purely real, find the locus of P. 1

QUESTION 3

a) i) Prove $\dfrac{d^2x}{dt^2} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$, where x denotes displacement, and v denotes velocity. 2

ii) The acceleration of a particle moving in a straight line is given by: $\ddot{x} = -2e^{-x}$, where x is the displacement from 0. The initial velocity of the particle is 2 m/s. The particle starts at $x = 0$.

α) Prove that $v^2 = 4e^{-x}$ 2

β) Describe the subsequent motion of the particle, making reference to its speed and direction. 2

b) The polynomial $P(x) = 2x^5 + 7x^4 + 26x^3 + 66x^2 + 72x + 27$ over the complex field has one known zero of $3i$. It is also known that it has a double real zero. Find all of its zeros. 4

c) i) Solve the equation $\tan^{-1}3x - \tan^{-1}2x = \tan^{-1}\dfrac{1}{5}$ 2

ii) Give the general solution of the equation : $\sin 3x + \sin x = \cos x$ 3

QUESTION 4

a) i) Prove $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ 2

ii) Hence, show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x}\,dx = \frac{\pi}{4}$ 3

for all real values of m.

b) P is any point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

and Q is the point on the auxiliary circle $x^2 + y^2 = a^2$ which has the same abscissa (x coordinate) as P. O is the origin. A line through P parallel to OQ cuts the x-axis at R and the y-axis at S.

i) Draw a diagram depicting this information. 1

ii) Prove that $PS = a$ units and $PR = b$ units. 3

c) PQ is a chord of the rectangular hyperbola $xy = c^2$.

i) Show that PQ has the equation $x + pq\,y = c\,(p+q)$, where P and Q have parameters p and q respectively. 2

ii) If PQ has a constant length of k^2, show that:

$$c^2\left[(p+q)^2 - 4pq\right]\left(p^2q^2 + 1\right) = K^4 p^2 q^2$$ 2

iii) Find the locus of R, the midpoint of PQ, in cartesian form. 2

QUESTION 5

a) i) Show that for any complex numbers z_1 and z_2 : 2

$$|z_1+z_2|^2+|z_1-z_2|^2 = 2\left(|z_1|^2+|z_2|^2\right)$$

ii) In an Argand diagram, P and Q are the points representing the complex numbers z_1 and z_2 respectively. By considering the parallelogram $OPRQ$, where O is the origin, interpret the above result (in part (i)) geometrically. 2

b) Find these integrals:

i) $\displaystyle\int \cos\left(\log_e x\right) dx$ 3

ii) $\displaystyle\int \frac{dx}{3\sin x + 4\cos x}$ 4

iii) $\displaystyle\int \frac{2-3x}{9x^2-16}\, dx$ 3

c) Solve the inequality 4

$$|x+3| + |x-2| \geq 6$$

by firstly solving an appropriate equation and then by using an appropriate sketch.

QUESTION 6

a) Q is a fixed point on the circumference of a circle, centre O and radius 1 metre. A point, P, moves at a uniform speed around the circumference so as to describe it (ie. go around it completely) in one second. 4

When the angle POQ is $\frac{\pi}{3}^c$, find the rate of change of the length of the chord PQ.

(let arc $PQ = x$, $\angle POQ = \theta$, *chord* $PQ = y$)

b) If x, y and z are any three positive numbers, it can be shown that: 5

$$(x+y)(x+z)(y+z) \geq 8xyz$$

DO NOT PROVE THIS RESULT, but use it to prove that:

If a, b and c are any three positive numbers such that each is less than the sum of the other two then:

$$(a+b-c)(b+c-a)(c+a-b) \leq abc$$

c) Sketch i) $y = \dfrac{x^2 - x - 6}{x - 1}$ 3

ii) $y^2 = \dfrac{x^2 - x - 6}{x - 1}$ 3

QUESTION 7

a) i) If $I_n = \int \tan^n x \; dx$, prove that:

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$ 2

ii) Hence, find $\int \tan^5 x \; dx$ 2

b) $P(4 \sec\theta, \; 2 \tan\theta)$ is a variable point on a hyperbola.

i) Write down the equation of the hyperbola 1

ii) Write down the coordinates of the foci (S and S^1) 1

iii) Write down the equationsof the directrices 1

iv) Write down the equations of the asymptotes 1

v) Sketch the curve 1

vi) Show that the equation of the tangent at P is:
$x \sec\theta - 2y \tan\theta = 4$ 3

vii) Express the equation of this tangent in terms of its gradient, m
(ie. eliminate θ from the equation in part (vi)) 2

viii) Write down the equations of the two tangents to the above hyperbola that have a gradient equal to 2. 1

QUESTION 8

a) i) For the complex number $z = \cos\theta + i\sin\theta$ 1

show that $z^n + z^{-n} = 2\cos n\theta$

(You may state de Moivre's theorem without proof).

ii) Using the above result, express $\cos^7 x$ in the form: 4

$A\cos 7x + B\cos 5x + C\cos 3x + D\cos x$, and hence find:

$$\int \cos^7 x \, dx$$

iii) Find $\int \cos^7 x \, dx$ using a different method. (You may leave your final answer in a different form to that obtained in part (ii)). 2

b) Find the range of values of K such that $x^3 - x^2 - x + K = 0$ has: 5

i) one real solution
ii) two real solutions
iii) three real solutions

c) Prove that if: $\dfrac{z_2 - z_3}{z_3 - z_1} = \dfrac{z_3 - z_1}{z_1 - z_2}$, then the points that represent the complex numbers z_1, z_2, z_3 form an equilateral triangle. 3

4 unit Trial Solutions 1958

Q1 a) $\displaystyle\int \frac{dx}{x^2-4x-6} = \int \frac{dx}{x^2-4x+4-10}$

$$= \int \frac{dx}{(x-2)^2-(\sqrt{10})^2}.$$

let $u = x-2 \quad \therefore dx = du$

$$I = \int \frac{du}{u^2-(\sqrt{10})^2}$$

$$= \frac{1}{2\sqrt{10}} \ln\left(\frac{u-\sqrt{10}}{u+\sqrt{10}}\right) + C$$

$$= \frac{1}{2\sqrt{10}} \ln\left(\frac{x-2-\sqrt{10}}{x-2+\sqrt{10}}\right) + C$$

b) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^4 x \sin x \, dx$

let $u = \cos x$ $\qquad$ when $x=0$, $u=1$

$\therefore \dfrac{du}{dx} = -\sin x$ $\qquad$ $x=\frac{\pi}{2}$, $u=0$

$\therefore \sin x \, dx = -du$

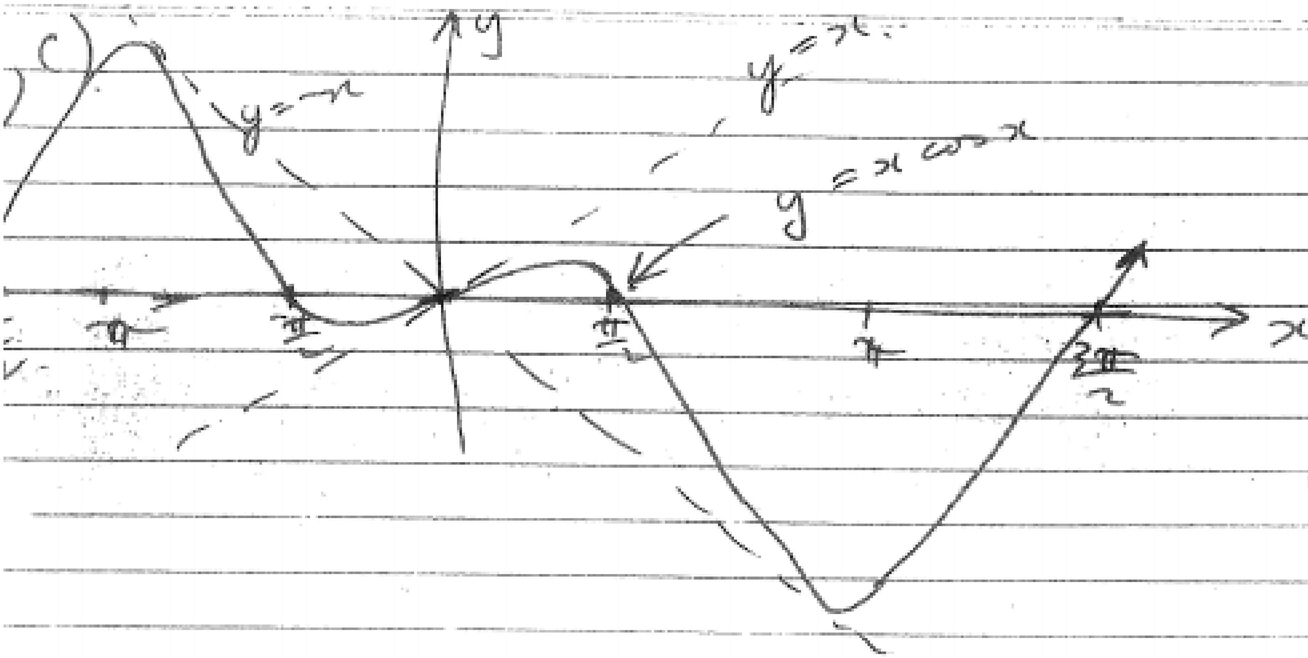
$$\therefore I = -\int_1^0 u^4 \, du$$

$$= \left[\frac{u^5}{5}\right]_0^1$$

$$= \frac{1}{5}$$

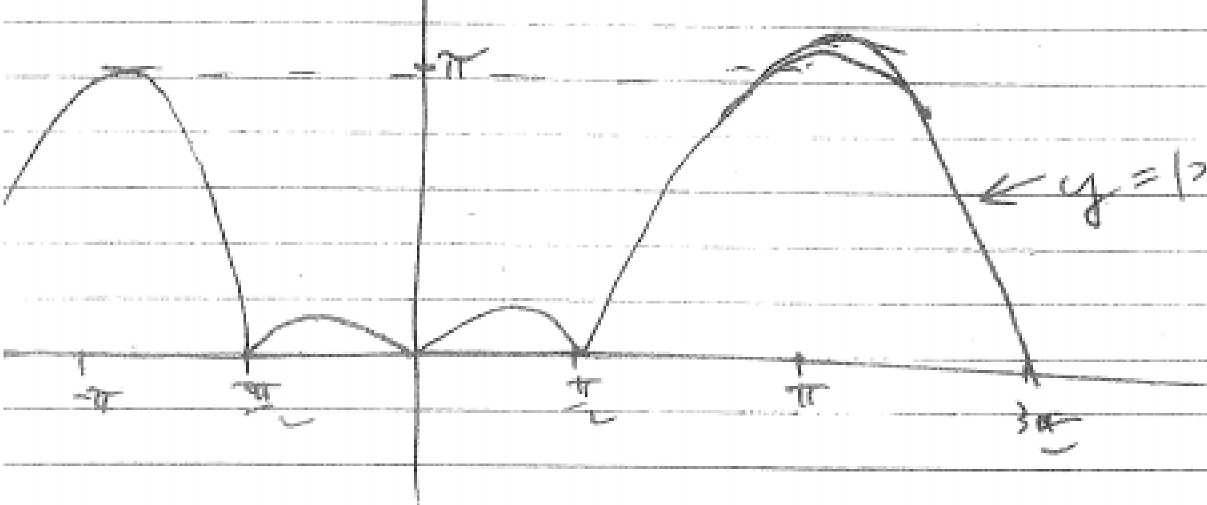
c) $y = x^2+4$

c)



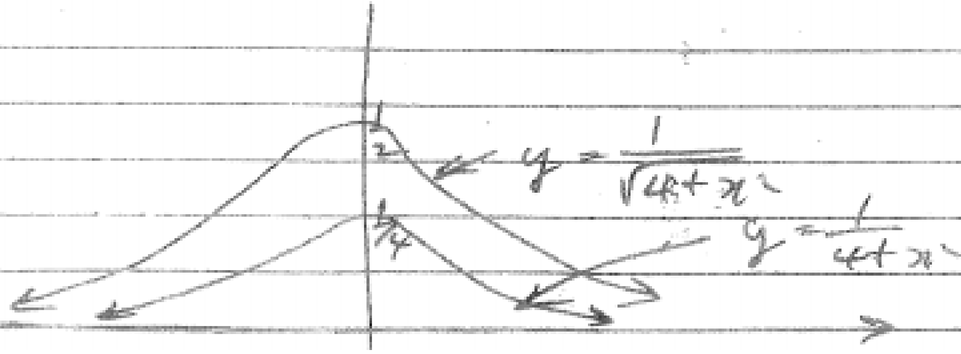
y
$y = x$
$y = -x$
$y = x\cos x$
x
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

(ii)



π
$y = |x\cos x|$
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

) (i)



$\frac{1}{2}$
$\frac{1}{4}$
$y = \frac{1}{\sqrt{4+x^2}}$
$y = \frac{1}{4+x^2}$

$\therefore \beta = \dfrac{-5(1-i)}{i} \times \dfrac{i}{i}$

$= 5+5i$

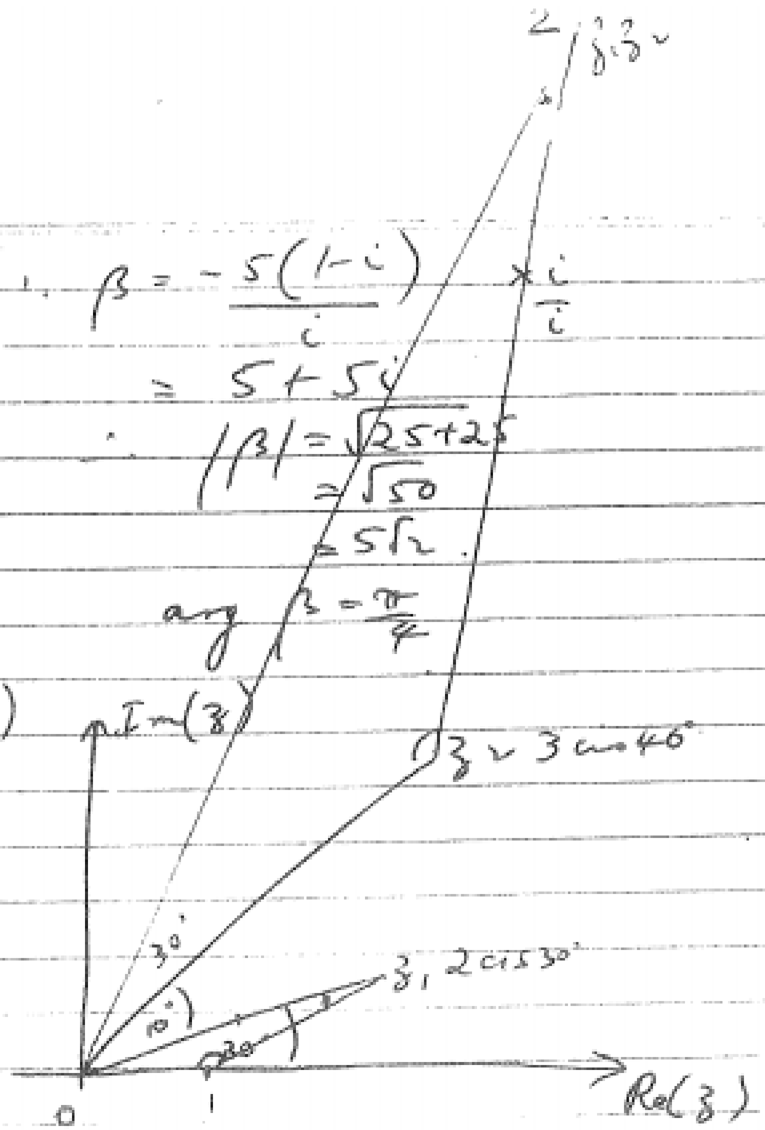
$\therefore |\beta| = \sqrt{25+25}$

$= \sqrt{50}$

$= 5\sqrt{2}$

$\arg \beta = \frac{\pi}{4}$

b) (i)



$\dfrac{\vec{OZ}}{\vec{Oz_1}} = \dfrac{\vec{Oz_2}}{1}$ ratio of sides in similar triangles

$\therefore \vec{OZ} = \vec{Oz_1} \times \vec{Oz_2}$

$= |z_1||z_2|$

i.e. $\vec{OZ} = |z_1 z_2|$

$\arg Z = \arg z_1 + \arg z_2$

$= \arg(z_1 z_2)$

$\therefore Z = z_1 z_2$

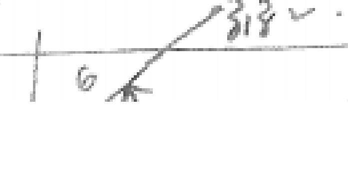
(ii) $z_1 z_2 = 2 \text{ cis } 30° \times 3 \text{ cis } 40°$

$= 6 \text{ cis } (30° + 40°)$ using $|z_1 z_2| = |z_1||$

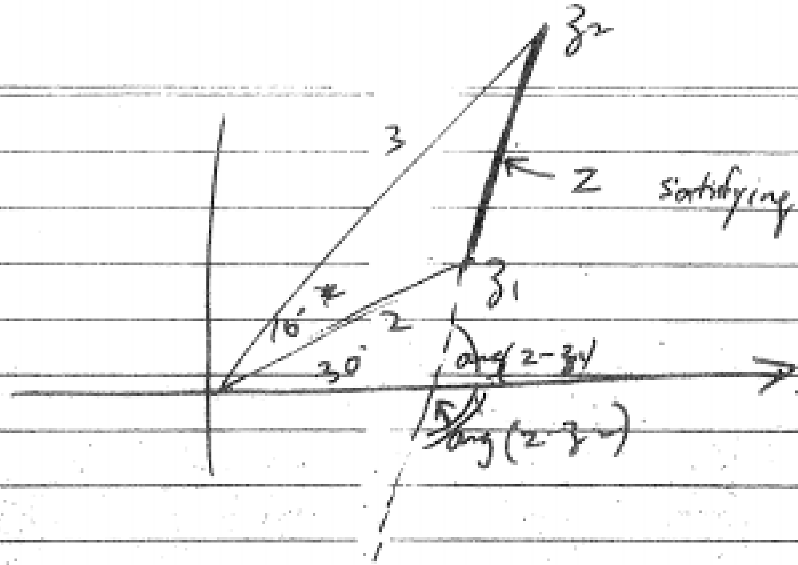
$= 6 \text{ cis } 70°$

and $\arg z_1 z_2$

$= \arg z_1 + \arg$



i)



satisfying $\arg(z-z_1)-\arg(z-z_2)$
$=180$

a) $w = \frac{z+2i}{z-4}$

$= \frac{x+(2+y)i}{(x-4)+iy} \times \frac{x-4-iy}{x-4-iy}$

$= \frac{x(x-4) - ixy + (x-4)(2+y)i + y(y+2)}{(x-4)^2+y^2}$

$= \frac{x^2-4x - \cancel{xy}i + 2xi + \cancel{xy}i - 8i - 4yi + y^2 + 2y}{(x-4)^2+y^2}$

$= \frac{x^2-4x+y^2+2y + (2x-4y-8)i}{(x-4)^2+y^2}$

$\therefore Re(w) = \frac{x^2-4x+y^2+2y}{(x-4)^2+y^2}$

$Im(w) = \quad 2x-4y-8$

(i) If w is pure imaginary, $Re(w) = 0$

ie $x^2 - 4x + y^2 + 2y = 0$

$x^2 - 4x + 4 + y^2 + 2y + 1 = 5$

$(x-2)^2 + (y+1)^2 = 5$

ie circle, centre $(2, -1)$

radius $= \sqrt{5}$.

(ii) If w is purely real, $Im(w) = 0$.

ie $2x - 4y - 8 = 0$

ie straight line $y = \frac{1}{2}x - 2$

Q3

a) (i) $\frac{d^2x}{dt^2} = \frac{dv}{dt}$ and $\frac{d}{dx}\frac{1}{2}v^2 = \frac{d}{dv}\frac{1}{2}v^2 \times \frac{dv}{dx}$

$= \frac{dv}{dx}\frac{dx}{dt}$ $= v\frac{dv}{dx}$

$= v\frac{dv}{dx}$

$\therefore \frac{d}{dx}\frac{1}{2}v^2 = \ddot{x}$

(ii) (α) $\ddot{x} = -2e^{-x}$ at $t = 0, v = 2, x = 0$

$v^2 = 2\int a\,dx$

$= 2\int -2e^{-x}\,dx$

$= 4e^{-x} + C$

when $x = 0$, $v = 2 \Rightarrow C = 0$

$v^2 = 4e^{-x}$

(β)

— particle continues in a positive direction with velocity approaching zero. Velocity never reaches zero as $4e^{-x} \neq 0$.

b) $2x^5+7x^4+26x^3+66x^2+72x+27 = P(x)$

If $3i$ is a zero, then $(x-3i)$ is a factor. Also $-3i$ is a zero (conjugate root theorem) $\therefore$ $x+3i$ is a factor

$\therefore (x+3i)(x-3i) = x^2+9$ is a factor.

$$
\begin{array}{r l}
 & 2x^3 + 7x^2 + 8x + 3 \\
x^2+9 & \overline{)\,2x^5 + 7x^4 + 26x^3 + 66x^2 + 72x + 27} \\
 & \underline{2x^5 \qquad\quad + 18x^3} \\
 & 7x^4 + 8x^3 + 66x^2 \\
 & \underline{7x^4 \qquad\quad + 63x^2} \\
 & 8x^3 + 3x^2 + 72x \\
 & \underline{8x^3 \qquad\quad + 72x} \\
 & 3x^2 + 27 \\
 & \underline{3x^2 + 27}
\end{array}
$$

$\therefore P(x) = (x^2+9)(2x^3+7x^2+8x+3)$

Let $Q(x) = 2x^3+7x^2+8x+3$ — this has repeated real zero — use multiple roots theorem

i.e. find $Q'(x) = 6x^2+14x+8$

$= 2(3x+4)(x+1)$

test $x = -\frac{4}{3}, -1$ in $Q(x)$

$Q(-1) = 0$, $Q(-\frac{4}{3}) \neq 0$

$(x+1)^2$ is a factor of $Q(x)$

$$
\begin{array}{r}
2x+3 \\
x^2+2x+1 \overline{\smash{)}\,2x^3+7x^2+8x+3} \\
\underline{2x^3+4x^2+2x} \\
3x^2+6x+3 \\
\underline{3x^2+6x+3}
\end{array}
$$

$\therefore P(x) = (x^2+9)(x+1)^2(2x+3)$

ie zeros are $\pm 3i, -1, -1, -3/2$

(c) (i) $\tan^{-1} 3x - \tan^{-1} 2x = \tan^{-1} \frac{1}{5}$.

take "tan" of both sides

LHS $= \tan(\tan^{-1} 3x - \tan^{-1} 2x)$

$= \dfrac{\tan(\tan^{-1} 3x) - \tan(\tan^{-1} 2x)}{1 + \tan(\tan^{-1} 3x)\tan(\tan^{-1} 2x)}$

$= \dfrac{3x - 2x}{1 + 6x^2}$

RHS $= \tan(\tan^{-1} \frac{1}{5})$

$= \frac{1}{5}$

$\therefore \dfrac{x}{1+6x^2} = \dfrac{1}{5}$

$6x^2 + 1 = 5x$

$6x^2 - 5x + 1 = 0$

$(3x-1)(2x-1) = 0$

$x = \frac{1}{3}, \frac{1}{2}$

(C) (ii) $\sin 3x + \sin x = \cos x$.

~~LHS =~~

Use $\sin(\overset{x}{A+B}) + \sin(\overset{y}{A-B}) = 2\sin \overset{\frac{x+y}{2}}{A} \cos \overset{\frac{x-y}{2}}{B}$

with $A = 2x$, $B = x$.

$LHS = 2\sin 2x \cos x$

$= 4\sin x \cos^2 x$.

$\therefore 4\sin x \cos^2 x = \cos x$.

$\cos x(4\sin x \cos x - 1) = 0$.

$\cos x = 0$, $\quad \sin x \cos x = \frac{1}{4}$.

$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$. $\quad \sin 2x = \frac{1}{2}$

$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$.

~~$\therefore x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}$~~

$2x = n\pi + (-1)^n \frac{\pi}{6}$

$x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}$

$\therefore x = \frac{(2n+1)\pi}{2}, \quad \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}$.

(n is any the integer)

Q4

a) (i) $\int_0^a f(x)\,dx$ let $x = a - u$. $dx = -du$

$I = \int_a^0 f(a-u)(-du)$

when $x=0$, $u=a$

$x=a$, $u=0$

$= \int^a f(a-u)\,du$

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^m x \, dx}{\sin^m x + \cos^m x} = \int_0^{\frac{\pi}{2}} \frac{\sin^m\left(\frac{\pi}{2}-x\right)}{\sin^m\left(\frac{\pi}{2}-x\right)+\cos^m\left(\frac{\pi}{2}-x\right)}$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^m x}{\cos^m x + \sin^m x}$$

Now $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^m x \, dx}{\sin^m x + \cos^m x} + \int_0^{\frac{\pi}{2}} \frac{\cos^m x}{\cos^m x + \sin^m x} dx$

$$= \int_0^{\frac{\pi}{2}} 1 \, dx$$

$$= \left[x \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

$\therefore \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 \, dx$

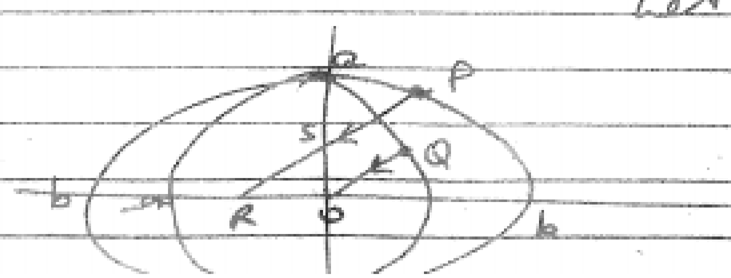
$$= \frac{\pi}{4}.$$

b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Let $P \equiv (a\cos\theta, b\sin\theta)$

(i)

$Q \equiv (a\cos\theta, a\sin\theta)$



OQ has gradient $m = \tan\theta$.
and equation $y = x\tan\theta$.

RP has equation

$$y - b\sin\theta = \tan\theta(x - a\cos\theta) \quad *$$

Let $x = 0$ for S

$$\therefore \quad y - b\sin\theta = \tan\theta(-a\cos\theta).$$

$$y = b\sin\theta - a\sin\theta$$

$$= (b-a)\sin\theta$$

$\therefore$ S is $(0, (b-a)\sin\theta)$.

$$PS = \sqrt{(a\cos\theta)^2 + (-a\sin\theta)^2}$$

$$= a\sqrt{\cos^2\theta + \sin^2\theta}$$

$$= a$$

For R, using $*$ & $y = 0$

$$\Rightarrow \quad -b\sin\theta = \tan\theta(x - a\cos\theta)$$

$$-b = \frac{x}{\cos\theta} - a.$$

$$x = (a-b)\cos\theta.$$

$\therefore$ R is $(\cos\theta(a-b), 0)$

$$\therefore \quad PR = \sqrt{(-b\cos\theta)^2 + (b\sin\theta)^2}.$$

$$= b.$$

(A)

c) $xy = c^2$

(i) Let P be $\left(cp, \frac{c}{p}\right)$ and Q be $\left(cq, \frac{c}{q}\right)$

$$\therefore m = \frac{\frac{c}{q} - \frac{c}{p}}{cq - cp}$$

$$= \frac{cp - cq}{cpq^2 - cp^2q}$$

$$= \frac{p-q}{pq(q-p)}$$

$$= \frac{-1}{pq}$$

$\therefore$ eqn of PQ is $y - \frac{c}{p} = \frac{-1}{pq}(x - cp)$

or $pqy - cq = -x + cp$

$x + pqy = c(p+q)$

(ii) $PQ^2 = (cp - cq)^2 + \left(\frac{c}{p} - \frac{c}{q}\right)^2$

$$= c^2(p-q)^2 + c^2\left(\frac{p-q}{pq}\right)^2$$

$$= c^2(p-q)^2\left(1 + \frac{1}{p^2q^2}\right)$$

$$= \frac{c^2(p-q)^2(p^2q^2+1)}{p^2q^2}$$

$$= (K^2)^2$$

$$c^2\left[(p+q)^2 - 4pq\right](p^2q^2+1) = k^4p^2q^2 \quad (1)$$

R is midpt of PQ

$$\therefore x_R = \frac{c(p+q)}{2} \Rightarrow p+q = \frac{2x}{c} \quad (2)$$

$$y_R = \frac{1}{2}\left(\frac{c}{p}+\frac{c}{q}\right) = \frac{c(p+q)}{2pq}$$

eliminate $p+q$ $\Rightarrow$

$$\frac{x}{y} = pq \quad (3)$$

Now, (1) becomes

$$c^2\left[\frac{4x^2}{c^2} - 4\frac{x}{y}\right]\left(\frac{x^2}{y^2}+1\right) = k^4\frac{x^2}{y^2}$$

$$\cancel{c^2}\left(\frac{4x^2y - 4xc^2}{y\cancel{c^2}}\right)\left(\frac{x^2+y^2}{y^2}\right) = \frac{k^4x^2}{y^2}$$

$$x\left(\frac{4xy - 4c^2}{y}\right)(x^2+y^2) = k^4x^2$$

ie $\quad 4(xy-c^2)(x^2+y^2) = k^4xy$

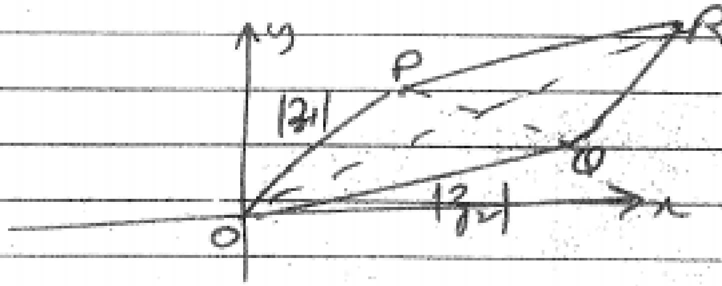
is locus of R

Question 5

(a) (i) $|z_1+z_2|^2 + |z_1-z_2|^2 = (z_1+z_2)(\overline{z_1+z_2}) + (z_1-z_2)(\overline{z_1-z_2})$

$= 2\left(|z_1|^2 + |z_2|^2\right)$

(ii)



OPRQ is a parallelogram

$\vec{OR}$ represents $z_1 + z_2 \Rightarrow OR = |z_1 + z_2|$

$\vec{QP}$ represents $z_1 - z_2 \Rightarrow QP = |z_1 - z_2|$

* Hence, the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides

b)(i) $\int \cos(\ln x)\,\frac{dx}{dx}\,dx$

$= x\cos(\ln x) - \int x\left(-\sin(\ln x) \times \frac{1}{x}\right)dx$

$= x\cos(\ln x) + \int \sin(\ln x)\,\frac{dx}{dx}\,dx$

$= x\cos(\ln x) + x\sin(\ln x) - \int x\cdot\cos(\ln x)\,\frac{1}{x}\,dx$

$\therefore 2\int \cos(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x) + c$

$\therefore \int \cos(\ln x)\,dx = \frac{1}{2}\left(x\cos(\ln x) + x\sin(\ln x)\right) + c$

(ii) $\int \frac{dx}{\ }$ let $x = \tan$

$$\therefore \quad dx = \frac{2\,dt}{\sec^2 \frac{x}{2}} \qquad \cancel{\& \sin x = \frac{2x}{1+x^2}}$$

$$= \frac{2\,dt}{\tan^2 \frac{x}{2} + 1} \qquad \cancel{\cos x = \frac{1-x^2}{1+x^2}}$$

$$= \frac{2\,dt}{1+t^2}.$$

$$\& \sin x = \frac{2t}{1+t^2} \; ; \; \cos x = \frac{1-t^2}{1+t^2}$$

$$\therefore \quad I = \int \frac{2\,dt}{(1+t^2)\left[\frac{6t}{1+t^2} + \frac{4(1-t^2)}{1+t^2}\right]}$$

$$= 2\int \frac{dt}{6t + 4 - 4t^2}.$$

$$= -\int \frac{dt}{2t^2 - 3t - 2} \qquad \begin{matrix} 2t & & +1 \\ t & & -2 \end{matrix}$$

$$= -\int \frac{dt}{(2t+1)(t-2)}.$$

$$\text{let } \frac{1}{(2t+1)(t-2)} \equiv \frac{a}{2t+1} + \frac{b}{t-2}.$$

$$\therefore \quad a(t-2) + b(2t+1) \equiv 1$$

$$(a+2b)t - 2a + b = 1$$

$$\therefore \quad a + 2b = 0 \Rightarrow a = -2b$$

$$-2a + b = 1 \Rightarrow 5b = 1$$

$$b = 1/5, \; a = -2/5$$

$$\therefore \quad I = -\int \frac{-2/5}{2t+1}\,dt - \int \frac{\frac{1}{5}}{t-2}\,dt.$$

$$= +\frac{2}{5} \times \frac{1}{2} \ln(2t+1) - \frac{1}{5}\ln(t-2) + C$$

$$= +\frac{1}{5} \ln|2t+1| - \frac{1}{5} \ln|t-2| + c$$

$$= |2t+1|$$

(iii) $\displaystyle\int \frac{2-3x}{9x^2-16}\,dx$

$= -\frac{1}{6}\displaystyle\int \frac{18x\,dx}{9x^2-16} + 2\int \frac{dx}{9(x^2-\frac{16}{9})}$

$= -\frac{1}{6}\ln(9x^2-16) + \frac{2}{9}\displaystyle\int \frac{dx}{x^2-(\frac{4}{3})^2}$

$= -\frac{1}{6}\ln(9x^2-16) + \frac{2}{9} \times \frac{1}{8/3} \ln \frac{x-4/3}{x+4/3} + c$

$= -\frac{1}{6}\ln(9x^2-16) + \frac{6}{72}\ln\frac{3x-4}{3x+4} + c$

$= -\frac{1}{6}\ln(9x^2-16) + \frac{1}{12}\ln\frac{3x-4}{3x+4} + c$

e). $|x+3| + |x-2| > 6$

Solve $|x+3| + |x-2| = 6$

$-3 \qquad 2$

Consider $x < -3$

$-x-3 \;-x+2 = 6$

$-2x = 7$

$x = -3\frac{1}{2}$ ✓

$-3 < x < 2$

$x+3 \;-x+2 = 6$ ✗

$x > 2$

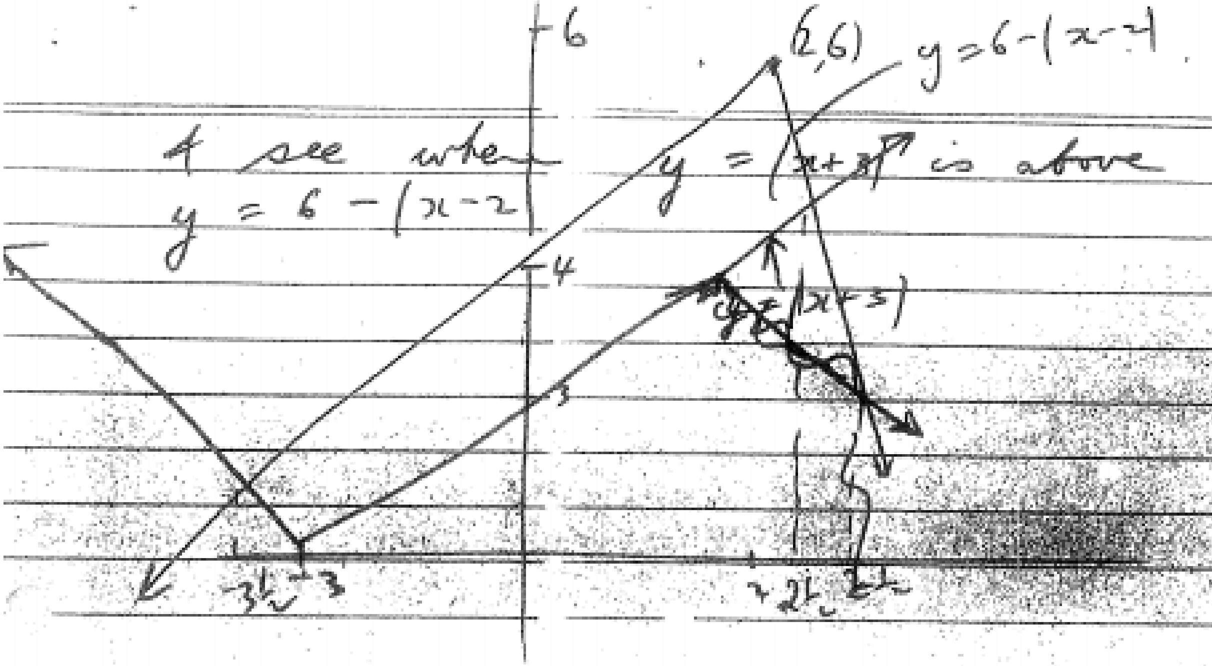
$x+3 + x-2 = 6$

$2x = 5$

$x = 2\frac{1}{2}$ ✓

Sketch $y = |x+3|$ & $y = 6 - |x-2|$

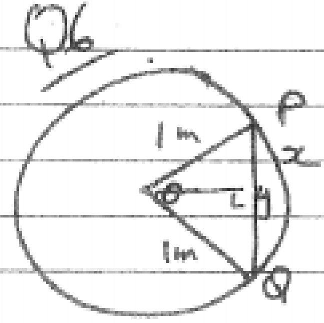
& see when $y = |x+3|$ is above $y = 6 - |x-2|$



$\therefore$ $y = |x+3|$ is above $y = 6 - |x-2|$ when $x < -3\frac{1}{2}$, $x > 2\frac{1}{2}$.

Q6

a)



Let arc $PQ = x$ $\therefore$ $\dot{x} = \frac{2\pi}{1} = 2\pi$.

Also, $x = 1 \times \theta$
$= \theta$

$\therefore$ $\dot{x} = \dot{\theta} = 2\pi$.

Let chord $PQ = y$

$\therefore$ $\dot{y} = \dot{\theta} \times \frac{dy}{d\theta}$

But $\sin\frac{\theta}{2} = \frac{y}{2}$

$\therefore$ $\frac{dy}{d\theta} = \cos\frac{\theta}{2}$,

$\therefore$ $\dot{y} = 2\pi \times \cos\frac{\theta}{2}$

at $\theta = \frac{\pi}{3}$, $\dot{y} = 2\pi \times \frac{\sqrt{3}}{2}$
$= \sqrt{3}\pi\ ms^{-1}$

i.e. rate of change of chord PQ

b) $(x+y)(y+z)(z+x) \geq 8xyz$ (given)

let $x = a+b-c > 0$ (from question)

$y = b+c-a > 0$ (" ")

$z = c+a-b > 0$ (" ")

$\therefore x+y = 2b, \quad y+z = 2c, \quad z+x = 2a$

$\therefore 2b \times 2c \times 2a \geq 8(a+b-c)(b+c-a)(c+a-b)$

ie $(a+b-c)(b+c-a)(c+a-b) \leq abc$.

c)(i) $y = \dfrac{x^2-x-6}{x-1} = \dfrac{(x-3)(x-2)}{x-1}$

$= \dfrac{x(x-1)-6}{x-1}$

$= x - \dfrac{6}{x-1}$.

v. asymptote at $x=1$

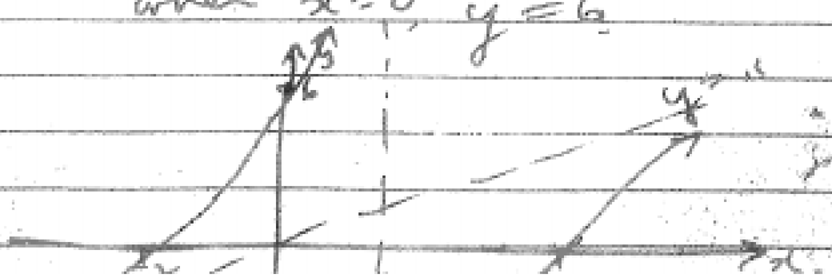
as $x \to \pm\infty$, $y \to x$.

If $x>1$, $y<x$.

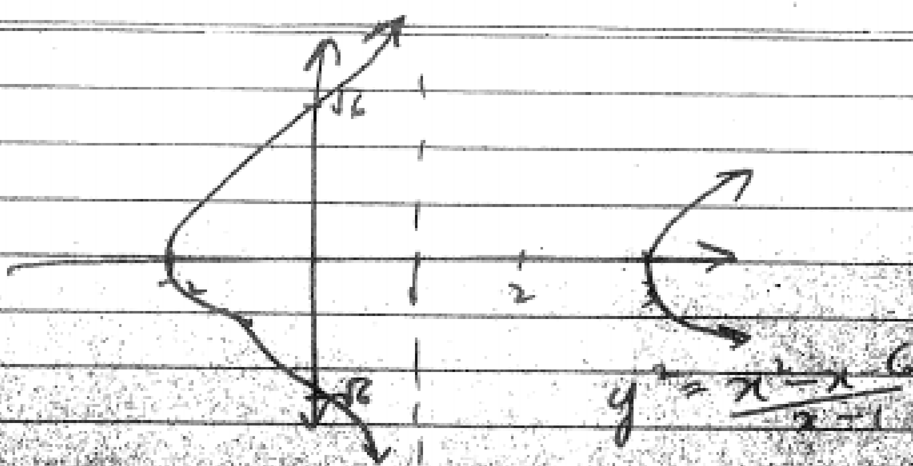
$x<1$ $y>x$

$y=0$ when $x=2, 3$.

when $x=0$, $y=6$



(ii)



(Rub out part below x axis & reflect it, adjust curvature of curve on x axis)

Q7

a) $I_n = \int \tan^n x \, dx$

$= \int \tan^{n-2} x (\sec^2 x - 1) \, dx$

$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} dx$

For $\int \tan^{n-2} x \sec^2 x \, dx$, let $u = \tan x$

$\therefore \frac{du}{dx} = \sec^2 x$

$\therefore I = \int u^{n-2} \, du$

$= \frac{1}{n-1} u^{n-1} = \frac{1}{n-1} \tan^{n-1} x$

$\therefore I_n = \frac{1}{n-1} \tan^{n-1} x - I_{n-2}$

$\therefore I_5 = \frac{1}{4} \tan^4 x - I_3$

$I_3 = \frac{1}{2} \tan^2 x - I_1$

$I_1 = \int$

$\therefore I_3 = \frac{1}{2}\tan^2 x + \ln \cos x$.

$\& \ I_5 = \frac{1}{4}\tan^4 x - \left(\frac{1}{2}\tan^2 x + \ln(\cos x)\right) + c$

$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln(\cos x) + c$

b) $P(4\sec\theta, 2\tan\theta)$

(i) $x = 4\sec\theta \qquad y = 2\tan\theta$

$\frac{x^2}{16} = \sec^2\theta \qquad \frac{y^2}{4} = \tan^2\theta$

$\sec^2\theta - \tan^2\theta = 1$

$\Rightarrow \frac{x^2}{16} - \frac{y^2}{4} = 1. \qquad a=4, b=2.$

(ii) $b^2 = a^2(e^2-1)$ for a hyperbola.

$4 = 16(e^2-1)$

$e^2 = 5/4$

$e = \frac{\sqrt{5}}{2}$

foci are $(\pm ae, 0)$

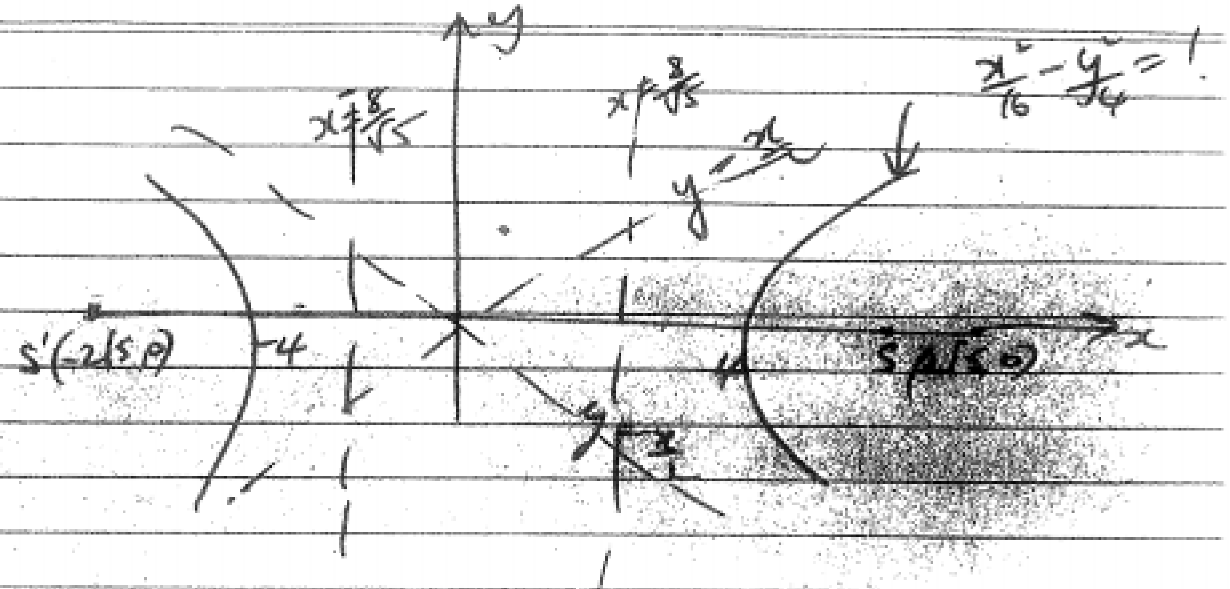
i.e. $(\pm 2\sqrt{5},)$

(iii) $x = \pm a/e$

$\Rightarrow x = \pm \cancel{\frac{4}{2\sqrt{5}}} \cdot \frac{4}{\sqrt{5}/2}$

$= \pm 8/\sqrt{5} \ldots = \pm \frac{8\sqrt{5}}{5}$

(iv) $y = \pm \frac{bx}{a} \Rightarrow y = \pm \frac{x}{2}$



(vi) $\frac{x^2}{16}-\frac{y^2}{4}=1$

$\therefore \frac{2x}{16}-\frac{2y}{4}\times y'=0$ or use parametric differentiation

$y'=\frac{x}{8}\times\frac{2}{y}$

$=\frac{x}{4y}$

at $(4\sec\theta, 2\tan\theta)$, $y'=\frac{4\sec\theta}{8\tan\theta}$

$=\frac{\sec\theta}{2\tan\theta}$.

$\therefore$ eqn of tangent is

$y-2\tan\theta=\frac{\sec\theta}{2\tan\theta}(x-4\sec\theta)$

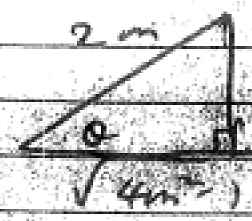
$\therefore 2y\tan\theta-4\tan^2\theta=x\sec\theta-4\sec^2\theta$

$\therefore x\sec\theta-2y\tan\theta=4\underbrace{(\sec^2\theta-\tan^2\theta)}_{=4}$

i) $\therefore y=x\frac{\sec\theta}{2\tan\theta}-2\cot\theta$.

$\therefore$ gradient of tangent $= m = \dfrac{\operatorname{cosec}\theta}{2}$

$\therefore \operatorname{cosec}\theta = 2m$



$\therefore \cot\theta = \pm\sqrt{4m^2-1}$

$\therefore$ eqn of tangent is

$$y = mx \pm 2\sqrt{4m^2-1}$$

(v(ii)) When $m=2$,

$$y = 2x \pm 2\sqrt{15}.$$

Q8

a) (i) $z = \cos\theta + i\sin\theta$

$\therefore z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta.$

deM theorem

$\therefore z^n + z^{-n} = 2\cos n\theta.$

(ii) let $z = \operatorname{cis} x°$.

(ii) $z + \frac{1}{z} = 2\cos x$ from (i).

$\left(z + \frac{1}{z}\right)^7 = 2^7 \cos^7$

ie $128\cos^7 x = z^7 + 7z^5 + 21z^3 + 35z + \frac{35}{z} + \frac{21}{z^3} + \frac{7}{z^5} + \frac{1}{z^7}$

$= z^7 + \frac{1}{z^7} + 7\left(z^5 + \frac{1}{z^5}\right) + 21\left(z^3 + \frac{1}{z^3}\right) + 35\left(z + \frac{1}{z}\right)$

$\therefore \cos^7 x = \frac{1}{64}\cos 7x + \frac{7}{64}\cos 5x + \frac{21}{64}\cos 3x + \frac{35}{64}\cos x$

$\therefore \int \cos^7 x \, dx = \frac{1}{448}\sin 7x + \frac{7}{320}\sin 5x$

$\quad + \frac{7}{64}\sin 3x + \frac{35}{64}\sin x + C$

(ii) $\int \cos^7 x \, dx = \int \cos^6 x \cdot \cos x \, dx$

$= \int (1-\sin^2 x)^3 \cos x \, dx.$

let $u = \sin x$. $\quad \therefore du = \cos x \, dx$

$\therefore I = \int (1-u^2)^3 \, du$

$= \int 1 - 3u^2 + 3u^4 - u^6 \, du.$

$= u - u^3 + \frac{3}{5}u^5 - \frac{u^7}{7}$

$= \sin x - \sin^3 x + \frac{3}{5}\sin^5 x$

$- \frac{1}{7}\sin^7 x + C$

b) $x^3 - x^2 - x + k = 0$

$y = x^3 - x^2 - x + k$

$y' = 3x^2 - 2x - 1$

$= (3x+1)(x-1)$

$\frac{dy}{dx} = 0$ at $x = -\frac{1}{3}, 1$

at $x = -\frac{1}{3}$, $y = -\frac{1}{27} - \frac{1}{9} + \frac{1}{3} + k = k + \frac{5}{27}$

at $x = 1$, $y = 1 - 1 - 1 + k = k - 1$

(i) to have one real solution both

[illegible]