St. George Girls' High School

1999

TRIAL HIGHER CERTIFICATE EXAMINATION

# MATHEMATICS

4 Unit

TIME ALLOWED: 3 HOURS
(Plus 5 minutes' reading time)

INSTRUCTIONS TO CANDIDATES:

1. All questions may be attempted
2. All necessary working must be shown
3. Marks may be deducted for careless or poorly presented work
4. Begin each question on a NEW PAGE
5. A list of standard integrals is included at the end of this paper
6. The mark allocated for each question is listed at the side of the question

**Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.**

QUESTION 1 (15 marks)

a) Evaluate $\displaystyle\int_{e}^{e^2} \frac{dx}{x(\ln x)^3}$ 3

b) i) Show that $\displaystyle\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}$ 4

ii) Find $\displaystyle\int x^2 \tan^{-1} x \, dx$

c) Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sin x+\cos x}$ 3

d) $\displaystyle I_n = \int_0^{\pi/2} \cos^n x \, dx \quad n \ge 0$ 5

i) Show that $\displaystyle I_n = \frac{n-1}{n} I_{n-2}$

ii) Hence evaluate $\displaystyle\int_0^{\pi/2} \cos^4 x \, dx$

QUESTION 2 (15 marks)

a) The points $P$ and $Q$ represent the complex numbers $3 + 4i$ and $1 - 2i$ respectively. 3

i) Plot the points $P$ and $Q$ on an Argand diagram and plot the point $R$ such that $POQR$ is a parallelogram.

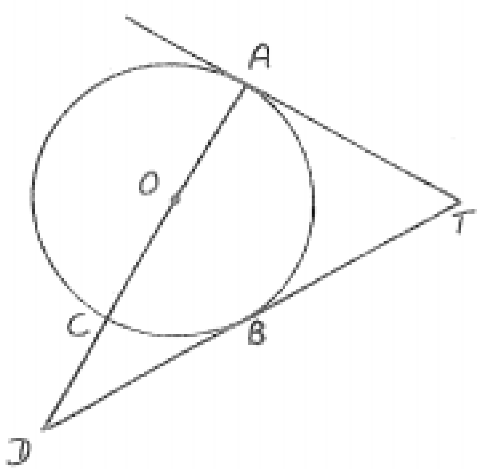
ii) What complex number does $R$ represent?

b) On separate diagrams sketch the locus specified by: 4

i) $\arg(z-2) = \arg(z+i)$

ii) $|z-2-3i| = 2$

c) 8




From an external point $T$, two tangents $TA$ and $TB$ are drawn to touch a circle with centre $O$ at $A$ and $B$ respectively. $A\hat{T}B$ is acute. The diameter $AC$ produced meets $TB$ produced at $D$.

i) Prove that $C\hat{B}D = \frac{1}{2} A\hat{T}B$

ii) Prove that $\Delta ABC$ is similar to $\Delta TBO$

iii) Deduce that $BC.OT = 2.(OA)^2$

QUESTION 3 (15 marks)

The hyperbola $H$ has Cartesian equation $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

a) Find: i) its eccentricity 4
ii) the coordinates of its foci, $S_1$ and $S_2$
iii) the equations of its directrices
iv) the equations of its asymptotes

b) Sketch $H$, clearly showing any intercepts with the coordinate axes and the details found in (a). 2

c) $P$ is the point $(4\ \sec\theta,\ 3\ \tan\theta)$ 3

i) Show that $P$ lies on $H$.

ii) Show that the tangent to $H$ at $P$ has equation

$$\frac{x\sec\theta}{4} - \frac{y\tan\theta}{3} = 1$$

d) The tangent at $P$ cuts the asymptotes at the points $C$ and $D$. 6

i) Prove that $CP = DP$

ii) Show that $OC.OD = 25$

iii) Hence show that the area of $\Delta OCD$ is independent of the position of $P$ on $H$.

QUESTION 4 (15 marks)

a) i) On the same set of axes sketch the graphs of 9

$$y = |x+1| \quad and \quad y = |x^2 - 1|$$

ii) Hence, or otherwise, sketch the graph of:

$$f(x) = |x+1| + |x^2 - 1|$$

iii) For which values of $x$ is $f(x) > 2$

iv) Discuss the differentiability of $f(x)$ at $x = 1$

b) i) Show that the area of an isosceles right angled triangle with hypotenuse $h$ is $\frac{h^2}{4}$ 6

ii) The base of a solid is the region enclosed by the curve $y = 9 - x^2$ and the $x$-axis. Each cross-section perpendicular to the $y$-axis is an isosceles right angled triangle with hypotenuse lying in the base. Find the volume of the solid.

QUESTION 5 (15 marks)

a) $P(x) = x^3 + bx + 1$, $b$ is a real number 10

i) If the roots of $P(x) = 0$ are $\alpha, \beta, \gamma$ find the polynomial equation with roots of:

($\alpha$) $3\alpha, 3\beta, 3\gamma$

($\beta$) $\alpha^2, \beta^2, \gamma^2$

($\gamma$) $\alpha + \beta - \gamma,\ \beta + \gamma - \alpha,\ \gamma + \alpha - \beta$

ii) For which values of $b$ will $P(x) = 0$ have two complex roots.

b) $z = \dfrac{\sqrt{3}}{2} + \frac{1}{2}i$ 5

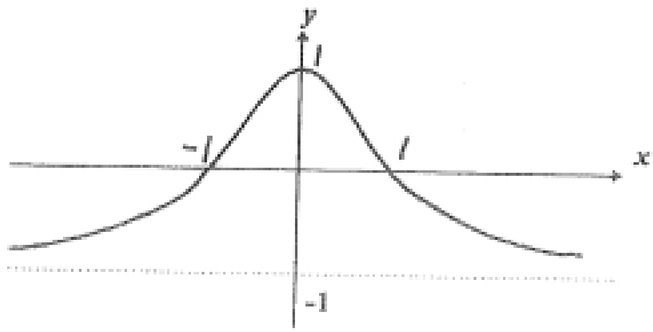
i) Express $z$ in modulus-argument form

ii) Find the smallest positive integer $n$ such that:

$1 + z + z^2 + \ \dots\ + z^{n-1} = 0$

QUESTION 6

a) The graph of $f(x) = \dfrac{1-x^2}{1+x^2}$ is shown below.




On separate diagrams, sketch the graphs of the following functions, clearly showing all intercepts and asymptotes.

i) $y = \dfrac{1}{f(x)}$ 2

ii) $y = [f(x)]^2$ 2

iii) $y^2 = f(x)$ 2

b) The acceleration $\ddot{x}\ cm/s^2$ of a particle moving along the $x$-axis is given by:

$\ddot{x} = f(x)$ (where $x$ is measured in cm)

where $f(x)$ is as shown above.

i) If the particle was initially at rest at the origin, find its velocity at $x = 1$.

ii) This particle next stops at $x = k$.
Show that $2 < k < 3$.

iii) Using an appropriate first approximation and one application of Newton's method, find a value for $k$.

QUESTION 7

a) An object of mass $100\ kg$ experiences air resistance of $\dfrac{v^2}{20}$ Newtons, where $v\ ms^{-1}$ is the velocity of the object. This object falls from rest from a height of $h$ metres above the ground. Let $x$ metres be the distance of the object from its starting point and the acceleration due to gravity be $9.8\ ms^{-2}$. 8

i) Show that $\ddot{x} = 9.8 - \dfrac{v^2}{2000}$

ii) Find the terminal velocity of the object

iii) If the object reaches a velocity of $60\%$ of its terminal velocity at the instant it hits the ground calculate the value of $h$ correct to one decimal place.

b) i) Show that $\displaystyle\int_0^2 xe^x\, dx = e^2 + 1$ 7

ii) Draw a neat sketch of the area bounded by the $y$ axis, the line $x = 2$ and the curves $y = e^x$ *and* $y = x^2$

iii) The area in (ii) is rotated about the line $x = 3$. Using cylindrical shells or otherwise show that the volume of the solid generated is given by:

$$V = 2\pi \int_0^2 (3 - x)\left(e^x - x^2\right) dx$$

iv) Hence find the exact value of $V$.

QUESTION 8 (15 marks)

a) i) Express $\sin z + \cos z$ in the form $R\sin(z + \alpha)$ where: $R > 0$ and $0 < \alpha < \frac{\pi}{2}$ 8

ii) Given that $y = e^x \sin x$ show that:

$$\frac{dy}{dx} = \sqrt{2}\, e^x \sin\left(x + \frac{\pi}{4}\right)$$

iii) Prove by mathematical induction that for $y = e^x \sin x$ and $n$ a positive integer

$$\frac{d^n y}{dx^n} = 2^{\frac{n}{2}} e^x \sin\left(x + \frac{n\pi}{4}\right)$$

where $\frac{d^n y}{dx^n}$ denotes the $n$th derivative of $y$ with respect to $x$

b) Given that $P(x) = x^4 + 2x^3 - 2x^2 + 8$ has a zero of multiplicity 2, find all solutions of $P(x) = 0$ over the field of complex numbers 4

c) Prove that: $\binom{n+2}{r} = \binom{n}{r-2} + 2\binom{n}{r-1} + \binom{n}{r}$ 3

The following list of standard integrals may be used:

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq 1; \; x \neq 0, \; \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2}).$$

Note: $\ln x = \log_e x, \quad x > 0$