Question 1 – (15 marks) – Start a new page **Marks**

a) Find all pairs of integers, x and y, which satisfy $(x+iy)^2 = 21+20i$ 3

b) On separate Argand diagrams sketch the locus of the point representing the complex number z if :

(i) $|z-1| = |z-3i|$ 1

5.b i)

(ii) $\arg(z-1) = \arg(z-3i)$ 1

c) On an Argand diagram shade the region specified by $1 \le \text{Im } z \le 3$ and $\frac{\pi}{4} \le \arg z \le \frac{2\pi}{3}$ 2

d) (i) Express $z = -\sqrt{3}+i$ in modulus-argument form. 2

(ii) Hence express z^5 in the form $x+iy$ where x and y are real numbers (in simplest form). 2

e) The complex number $z = x+iy$ is such that $\dfrac{z-8i}{z-6}$ is pure imaginary. Find the equation of the locus of the point P representing z and clearly show this locus on an Argand diagram. 4

Question 2 – (15 marks) – Start a new page **Marks**

a) Evaluate $\int_0^1 \frac{x}{4-x^2}\,dx$ 3

b) Find $\int \frac{1}{x^2+6x+18}\,dx$ 2

c) Find $\int \frac{10-6x}{(x+3)(x^2+5)}\,dx$ 4

d) Using the substitution $u = a - x$ show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ 2

Hence or otherwise evaluate

(i) $\int_0^1 x^2(1-x)^5\,dx$ 2

(ii) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$

Question 3 – (15 marks) – Start a new page **Marks**

a) Let the expansion of $(2+3x)^{12}$ be written in the form $\sum_{r=0}^{12} t_r x^r$.

(i) Write down expressions for t_r and t_{r+1}, and show that 2

$$\frac{t_{r+1}}{t_r} = \frac{36-3r}{2r+2}$$

(ii) Hence, find the greatest coefficient in the expansion of $(2+3x)^{12}$. You need not simplify your answer. 2

b) (i) Show that the coefficient of x^n in the expansion of $(1+x)^n(1+x)^n$ is given by

$$\sum_{r=0}^{n} ({}^nC_r)^2$$ 2

(ii) Hence, by equating the coefficients of x^n on both sides of the identity

$(1+x)^n\ (1+x)^n = (1+x)^{2n}$, prove that $\sum_{r=0}^{n} ({}^nC_r)^2 = \frac{(2n)!}{(n!)^2}$ 2

c) The velocity of a particle moving along the x-axis starting initially at $x = 1{\cdot}8$ is given by $V = e^{-2x}\sqrt{2x^2-6}$, $x \geq 1{\cdot}8$, where x is the displacement of the particle from the origin.

(i) Show that the acceleration of the particle in terms of its displacement can be expressed as:

$$a = -2e^{-4x}(2x^2 - x - 6)$$ 2

(ii) Hence, find the displacement of the particle at which the maximum speed occurs. 1

(iii) Show that the time T in seconds taken by the particle to move from $x = 2$ to $x = 3$ can be expressed as $T = \int_2^3 \frac{e^{2x}}{\sqrt{2x^2-6}}\,dx$ 2

(iv) Use Simpson's Rule with three function values to obtain an approximate value for T. 2

Question 4 – (15 marks) – Start a new page **Marks**

a) Sketch the graph of $y = e^{-x}$. Using this graph, and without the use of calculus, sketch the following:

(i) $y = -e^{-x}$ 1

(ii) $y = 1 - e^{-x}$ 1

(iii) $y = \dfrac{1}{1-e^{-x}}$ 2

(iv) $y = \left|\dfrac{1}{1-e^{-x}}\right|$ 2

b) Classify the following curves as ODD, EVEN or NEITHER and sketch each one on separate diagrams for the domain $-2\pi \le x \le 2\pi$

(i) $y = |\sin x|$ 1

(ii) $y = \sin|x|$

(iii) $|y| = \sin|x|$ 2

(iv) $y^2 = \sin x$ 2

c) Find the equation of the tangent to the curve:

$x^3 + y^3 - 8y + 7 = 0$ at the point (1, 2). 2

Question 5 – (15 marks) – Start a new page **Marks**

a) $P(x) = x^3 + 4x - 2$

If α, β, γ are the roots of $P(x) = 0$ find:

(i) $\alpha + \beta + \gamma$ 1

(ii) $\alpha^2 + \beta^2 + \gamma^2$ 1

and hence,

(iii) $\alpha^4 + \beta^4 + \gamma^4$ 2

b) (i) If a complex number $z = x + iy$ is a root of the cubic equation $az^3 + bz^2 + cz + d = 0$ where a, b, c, d are real numbers, prove that $\bar{z} = x - iy$ is also a root of the equation. (You may assume properties of conjugates of complex numbers). 2

(ii) Given that $1 + 2i$ is a root of the cubic equation $x^3 - 6x^2 + 13x - 20 = 0$ find all the roots of the equation. 2

c) $x^3 - 3x^2 + 2x - 7 = 0$ has roots α, β, γ. Find the polynomial equation which has roots.

(i) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ 1

(ii) $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ 2

d) The roots of the polynomial equation $x^3 - 6x^2 + mx + 6 = 0$ are in the arithmetic progression.

Find: (i) the value of m. 4

and (ii) all the roots of the equation.

Question 6 – (15 marks) – Start a new page

Marks

(i) Draw a careful sketch of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$, showing the vertices, the foci, the directrices and the asymptotes. Write on your diagram the equations of both directrices and of both asymptotes. (Show full working involved in finding the above points and lines). 6

(ii) Let $P=(4\sec\theta, 3\tan\theta)$ be any point on this hyperbola. Find the equations of:

(α) the tangent at P. 3

(β) the normal at P.

(iii) The tangent and normal at P meet the y-axis at T and N respectively.

Show that T is $(0,-3\cot\theta)$ and N is $(0, \frac{25}{3}\tan\theta)$. 2

(iv) Show that the circle with diameter NT passes through both foci. 4

(It will be sufficient to show that it passes through one focus, and that it will similarly pass through the other).

Question 7 – (15 marks) – Start a new page **Marks**

a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x+\sin x}\,dx$ using the substitution $t=\tan\frac{x}{2}$. 3

b) Let $I_n=\int_1^e x\,(\ln x)^n\,dx,\ \ n=0,1,2,3\,..$

(i) Using integration by parts, show that $I_n=\frac{e^2}{2}-\frac{n}{2}I_{n-1}\ \ (n=0,1,2,3\,..)$ 3

and (ii) hence, evaluate $\int_1^e x(\ln x)^3\,dx$. 2

c) (i) Sketch $y=x\ln x$, showing any turning points. 2

(ii) Deduce that $x\ln x=1$ has one root, and this root lies between $\sqrt{e}$ and e. 2

(iii) Show that if Newton's method is used to solve $x\ln x=1$, with the first approximation to the root being a_1, then the next approximation in the sequence is $a_2=\frac{1+a_1}{1+\ln a_1}$. 1

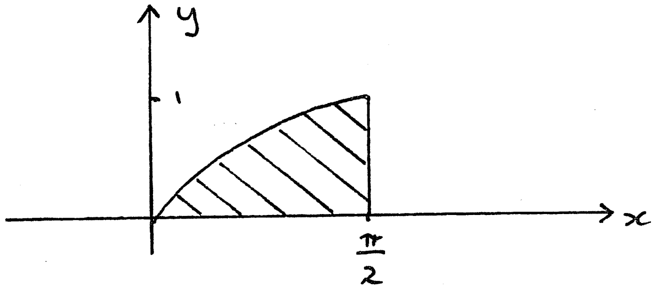
(iv) Hence, approximate the root of $x\ln x=1$ by using an integer a_1, where $\sqrt{e}<a_1<e$, as the first approximation, and by using the above iterative process twice. Give this answer to 2 decimal places. 2

Discuss whether this answer is necessarily the value of the root of $x\ln x=1$ to 2 decimal places.

Question 8 – (15 marks) – Start a new page

Marks

a) The diagram shows the region between $y = \sin x$ and the x-axis for $0 \le x \le \frac{\pi}{2}$.



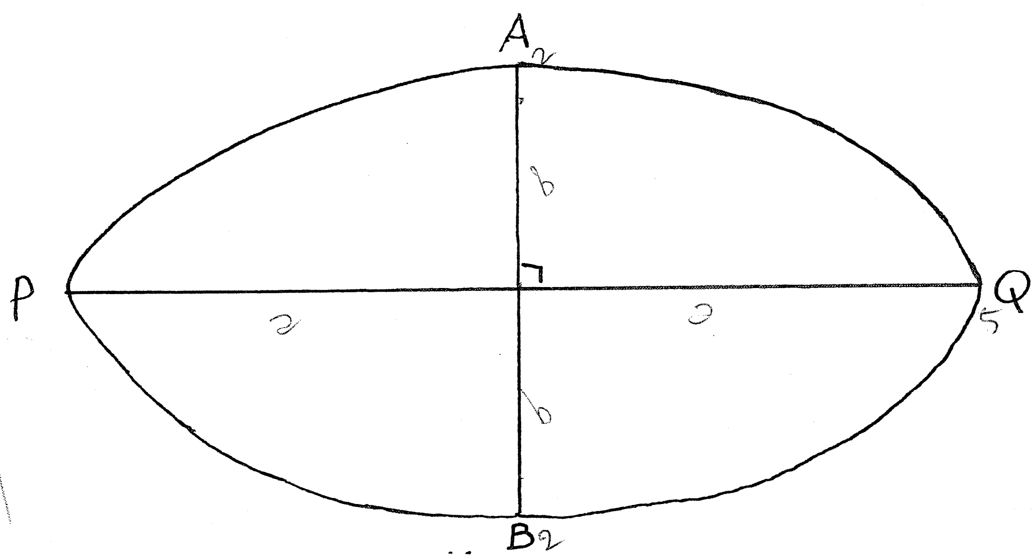
This region is rotated about the y-axis to form a solid.

(i) Use the method of cylindrical shells to find the volume of the solid. 4

(ii) Prove that: $\frac{d}{dy}\left[y(\sin^{-1} y)^2 + 2(\sin^{-1} y)\sqrt{1-y^2} - 2y\right] = (\sin^{-1} y)^2$ 2

(iii) By considering slices of thickness Δy perpendicular to the y-axis, and using the result of (ii), show the volume of the solid has the same value as the method used in part (i). 4

b)



The diagram shows an ellipse with major axis PQ and minor axis AB. If $PQ = 2a$ and $AB = 2b$, then the area of the ellipse is given by $A = \pi ab$.

An ellipse with a major axis of length 10 units and a minor axis of length 4 units forms the base of a right cone of height 10 units. Find the volume of the cone by integration. 5

Ext 2 2002 TRIAL (ST. GEORGE)

Solutions

Q1

a) $(x+iy)^2 = 21+20i$

$\therefore x^2+2xyi-y^2 = 21+20i$

$\therefore x^2-y^2 = 21$ ①

$2xy = 20$ ②

ie $xy = 10$

$\therefore y = \frac{10}{x}$

Sub $y = \frac{10}{x}$ into ① $\Rightarrow x^2 - \frac{100}{x^2} = 21$

$\therefore x^4 - 100 = 21x^2$

$x^4 - 21x^2 - 100 = 0$

let $u = x^2 \Rightarrow u^2 - 21u - 100 = 0$

(3) $\therefore (u-25)(u+4) = 0$

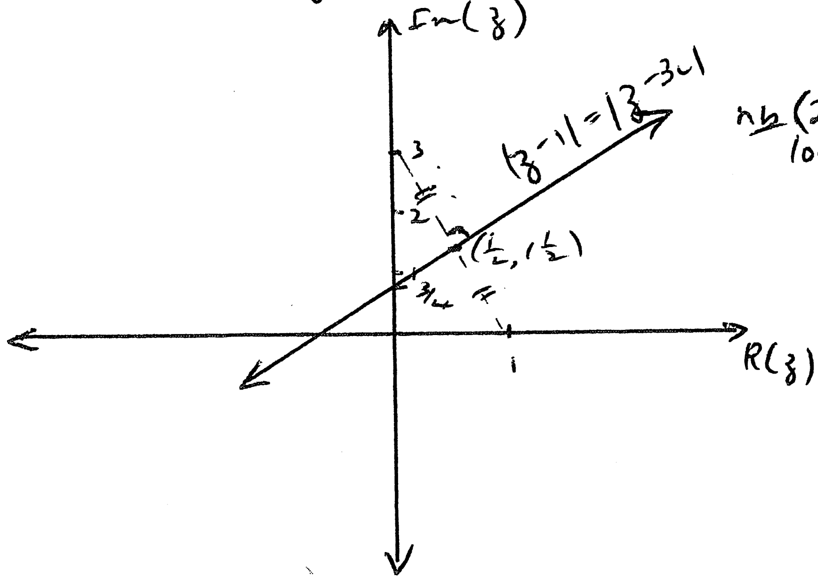
$u = 25, -4$

$\therefore x^2 = 25, \cancel{-4}$ (n/a)

$\therefore x = \pm 5$

If $x = 5$, $y = 2$ & if $x = -5$, $y = -2$.

b) (i)



nb (2,2) on locus. (1)

Algebraically: $|z-1| = |z-3i|$ let $z = x+iy$

$\therefore |x+iy-1| = |x+iy-3i|$

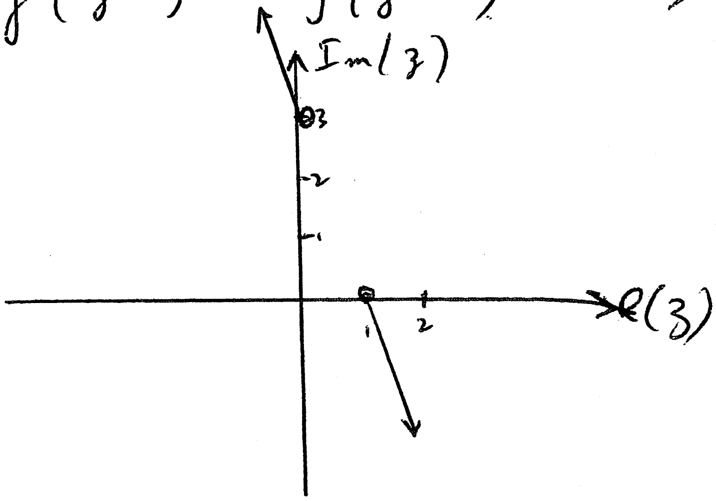
ie $\sqrt{(x-1)^2+y^2} = \sqrt{x^2+(y-3)^2}$

$\Rightarrow \cancel{x^2} - 2x + 1 + \cancel{y^2} = \cancel{x^2} + \cancel{y^2} - 6y + 9$

$\therefore 2x - 6y + 8 = 0$

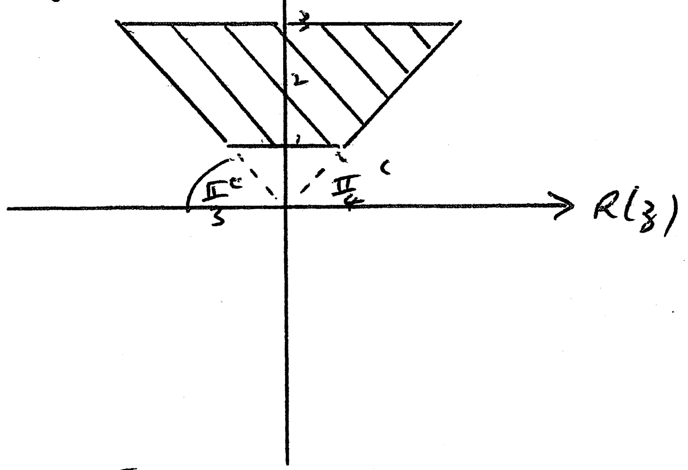
ie $x - 3y + 4 = 0$

(ii). $\arg(z-1) = \arg(z-3i)$ nb $\left.\begin{matrix} z \neq 1 \\ z \neq 3i \end{matrix}\right\}$



(1)

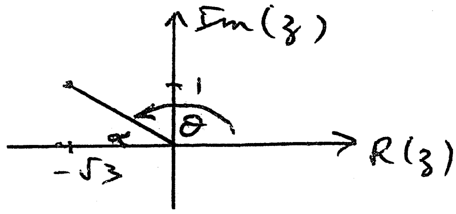
c) $1 \leq \operatorname{Im} z \leq 3$
$\wedge\ \frac{\pi}{4} \leq \arg z \leq \frac{2\pi}{3}$



(2)

a) (i) $z = -\sqrt{3} + i$

$r = |z| = \sqrt{(-\sqrt{3})^2 + 1^2}$
$= 2.$



$\tan \alpha = \frac{1}{\sqrt{3}}$. $\therefore\ \theta = \frac{5\pi}{6}^c$

$\therefore\ \alpha = \frac{\pi}{6}$

$\therefore\ z = 2\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$ (2)

(ii) $z^5 = 2^5 \operatorname{cis}\left(\frac{5\pi}{6} \times 5\right)$
$= 32 \operatorname{cis}\left(\frac{25\pi}{6}\right)$

$= 32 \operatorname{cis}\left(4\pi + \frac{\pi}{6}\right)$
$= 32 \operatorname{cis} \frac{\pi}{6}$. (2)
$= 32\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$
$= 32\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = 16\sqrt{3} + 16i$

e) $\dfrac{z-8i}{z-6}$ is pure imaginary.

$\therefore \dfrac{x+iy-8i}{x+iy-6}$ has zero real component.

$$\frac{z-8i}{z-6} = \frac{x+i(y-8)}{(x-6)+iy} \times \frac{x-6-iy}{x-6-iy}$$

$$= \frac{x(x-6)-ixy+i(y-8)(x-6)+y(y-8)}{(x-6)^2+y^2}$$

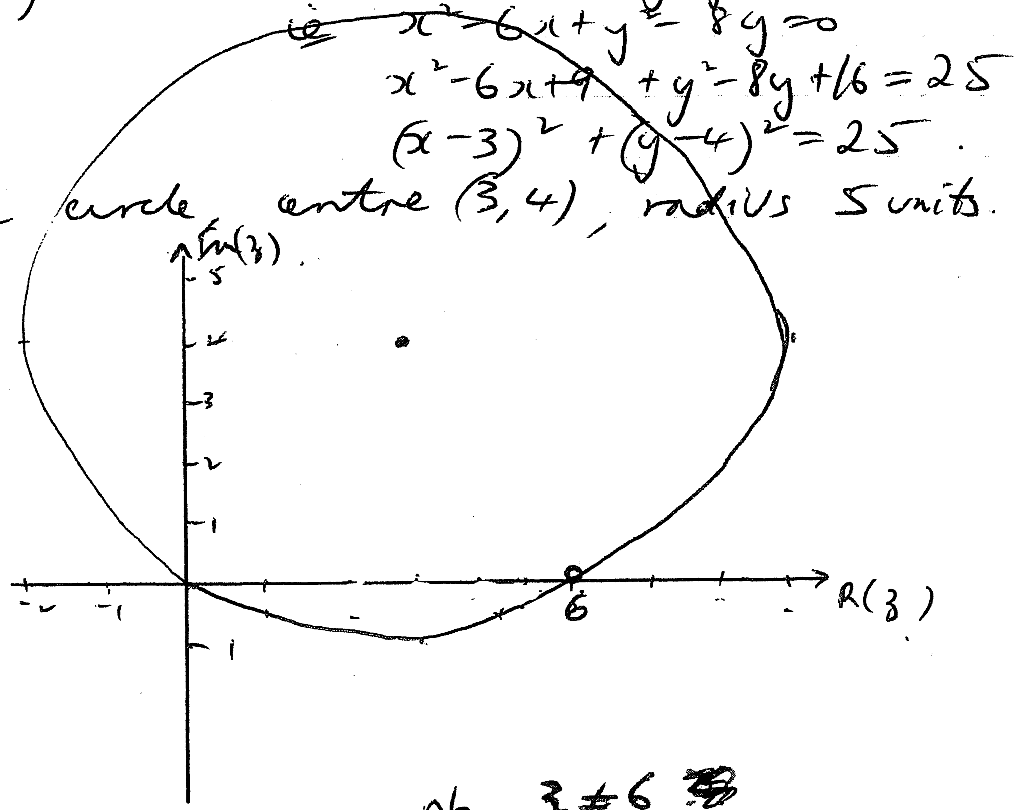
if $R\left(\dfrac{z-8i}{z-6}\right)=0$, then $x(x-6)+y(y-8)=0$.

ie $x^2-6x+y^2-8y=0$

$x^2-6x+9+y^2-8y+16=25$

$(x-3)^2+(y-4)^2=25$.

(4) ie circle centre $(3,4)$, radius 5 units.



nb $z \neq 6$

Question 2

a) $\displaystyle\int_0^1 \frac{x}{4-x^2}\,dx$.

let $u = 4-x^2$ when $x=0$, $u=4$

$\therefore \frac{du}{dx} = -2x$. $x=1$, $u=3$.

$\therefore x\,dx = -\frac{1}{2}\,du$.

$$\therefore I = \int_4^3 \frac{-\frac{1}{2}\,du}{u}.$$

$$= -\frac{1}{2}\left[\ln u\right]_4^3$$

$$= -\frac{1}{2}(\ln 3 - \ln 4)$$

$$= -\frac{1}{2}\ln 3/4$$

$$= \ln (3/4)^{-\frac{1}{2}}$$

$$= \ln \sqrt{\tfrac{4}{3}}.$$

$$= \ln \frac{2}{\sqrt{3}}.$$

$$= \ln \frac{2\sqrt{3}}{3}.$$

3

b) $\displaystyle\int \frac{dx}{x^2+6x+18}$

$$= \int \frac{dx}{x^2+6x+9+9}$$

$$= \int \frac{dx}{(x+3)^2+3^2}$$

2

let $u = x+3 \Rightarrow dx = du$.

$$\therefore I = \int \frac{du}{u^2+9}$$

$$= \frac{1}{3}\tan^{-1}\frac{u}{3} + c.$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x+3}{3}\right) + c$$

c) $\int \frac{10-6x}{(x+3)(x^2+5)} \, dx.$

Using partial fractions, $\frac{10-6x}{(x+3)(x^2+5)} \equiv \frac{a}{x+3} + \frac{bx+c}{x^2+5}$.

ie $10-6x \equiv a(x^2+5) + (bx+c)(x+3)$

$\equiv ax^2+5a + bx^2+3bx+cx+3c$

$\equiv (a+b)x^2 + (3b+c)x + 5a+3c$

Equating coefficients gives

$a+b=0$. ① $\Rightarrow a=-b$ ④

$3b+c=-6$ ② $\Rightarrow -3a+c=-6$ ⑤.

$5a+3c=10$ ③

From ⑤, $c = 3a-6$.

Sub into ③ $\Rightarrow 5a+3(3a-6) = 10$

(4) $\therefore 14a-18=10$

$a=2$; $b=-2$.

and $c=0$.

ie $\int \frac{10-6x \, dx}{(x+3)(x^2+5)} = \int \frac{2}{x+3} - \frac{2x}{x^2+5} \, dx.$

$= 2\ln(x+3) - \ln(x^2+5) + C$

or $\ln \frac{(x+3)^2}{x^2+5} + C$.

d) $I = \int_0^a f(x) \, dx$

let $u = a-x$ ie $\frac{du}{dx} = -1$ or $dx = -du$.

when $x=0$, $u=a$

" $x=a$, $u=0$

$\therefore I = \int_a^0 f(a-u)(-du)$

(2) $= -\int_a^0 f(a-u) \, du.$

$= \int_0^a f(a-u) \, du = \int_0^a f(a-x) \, dx.$

Q2 a) (i) $\int_0^1 x^2(1-x)^5\,dx = \int_0^1 (1-x)^2(1-(1-x))^5\,dx.$

$$= \int_0^1 (1-x)^2 \, x^5 \, dx.$$

$$= \int_0^1 x^5(1-2x+x^2)\,dx.$$

$$= \int_0^1 x^5 - 2x^6 + x^7 \, dx$$

$$= \left[\frac{x^6}{6} - \frac{2}{7}x^7 + \frac{x^8}{8}\right]_0^1$$

$$= \frac{1}{6} - \frac{2}{7} + \frac{1}{8} - 0.$$

$$= \frac{1}{168}.$$

(ii) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin(\frac{\pi}{2}-x)}{\sin(\frac{\pi}{2}-x)+\cos(\frac{\pi}{2}-x)}\,dx$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x}\,dx$$

Now, $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} + \frac{\cos x}{\sin x + \cos x}\,dx$

$$= \int_0^{\frac{\pi}{2}} 1 \, dx$$

$$= [x]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

$\therefore$ each integral must be $\frac{\pi}{4}$. (as they are equal).

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{\pi}{4}.$$

Question 3

a) (i) $(2+3x)^{12} = \sum_{r=0}^{12} t_r x^r$.

$t_r = {}^{12}C_r \, 2^{12-r} \, 3^r$.

$t_{r+1} = {}^{12}C_{r+1} \, 2^{11-r} \, 3^{r+1}$

$\therefore \dfrac{t_{r+1}}{t_r} = \dfrac{{}^{12}C_{r+1} \, 2^{11-r} \, 3^{r+1}}{{}^{12}C_r \, 2^{12-r} \, 3^r}$

$= \dfrac{12!}{(r+1)!(11-r)!} \times \dfrac{r!(12-r)!}{12!} \times \dfrac{3}{2}$.

$= \dfrac{12-r}{r+1} \times \dfrac{3}{2}$

2. $= \dfrac{36-3r}{2r+2}$

(ii) For increasing coefficients, $\dfrac{t_{r+1}}{t_r} > 1$.

ie $\dfrac{36-3r}{2r+2} > 1$

$36-3r > 2r+2$

$r < 6\frac{4}{5}$.

ie when $r = 1 \ldots 6$, $t_{r+1} > t_r$

ie $t_7 > t_6 > \ldots > t_1$

If $r = 7, 8 \ldots$, $t_{r+1} < t_r$

ie $t_7 > t_8 > t_9 \ldots$

2. ie greatest coefficient is t_7

where $t_7 = {}^{12}C_7 \, 2^5 . 3^7$.

b) $(1+x)^n \times (1+x)^n = ({}^nC_0 + {}^nC_1 x + \ldots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n)$

- the terms involving x^n are:

${}^nC_0 \, {}^nC_n x^n + {}^nC_1 \, {}^nC_{n-1} x^n + \ldots + {}^nC_n \, {}^nC_0 x^n$

Now, ${}^nC_r = {}^nC_{n-r}$ $\therefore$ terms involving x^n are:

$({}^nC_0)^2x^n + ({}^nC_1)^2x^n + \dots + ({}^nC_n)^2x^n$.

ie coefficient of x^n is

$$\sum_{r=0}^{n} ({}^nC_r)^2.$$

(2)

(ii) The coefficient of x^n in the expansion of $(1+x)^{2n}$ is ${}^{2n}C_n = \frac{(2n)!}{(n!)(n!)}$

ie $$\sum_{r=0}^{n} ({}^nC_r)^2 = \frac{(2n)!}{(n!)^2}$$

(2)

c (i) $v = e^{-2x}\sqrt{2x^2-6}$, $x \geq 1.8$.

Now, $a = \frac{d}{dx}\frac{1}{2}v^2$.

$$= \frac{d}{dx}\cdot\frac{1}{2}\left(e^{-4x}(2x^2-6)\right)$$

$$= \frac{d}{dx}(x^2-3)e^{-4x}$$

(2)

$$= (x^2-3)\cdot -4e^{-4x} + e^{-4x}(2x)$$

$$= e^{-4x}(2x - 4x^2 + 12)$$

$$= -2e^{-4x}(2x^2 - x - 6)$$

(ii). Maximum speed occurs when $a = 0$.

ie $-2e^{-4x}(2x^2 - x - 6) = 0$

$\Rightarrow 2x^2 - x - 6 = 0$ since $e^{-4x} > 0\ \forall x$

ie $(2x+3)(x-2) = 0$

$\Rightarrow x = 2, -3/2$

$\therefore x = 2$ since $x \geq 1.8$. (1)

(iii) $v = \frac{dx}{dt}$ $\therefore \frac{dt}{dx} = \frac{1}{v}$

$$= \frac{e^{2x}}{\sqrt{2x^2-6}}$$

$$\therefore \ T = \int_2^3 \frac{e^{2x}}{\sqrt{2x^2-6}}\,dx \qquad (2)$$

(iv)

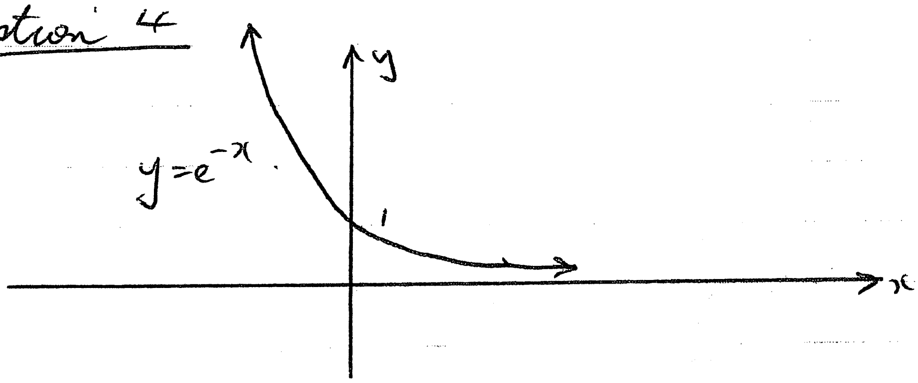
$$T \approx \frac{3-2}{6}\left[\frac{e^4}{\sqrt{2}} + 4.\frac{e^5}{\sqrt{2\times\left(\frac{5}{2}\right)^2-6}} + \frac{e^6}{\sqrt{12}}\right]$$

$$\approx \frac{1}{6}\left(\frac{e^4}{\sqrt{2}} + \frac{4e^5}{\sqrt{6.5}} + \frac{e^6}{\sqrt{12}}\right)$$

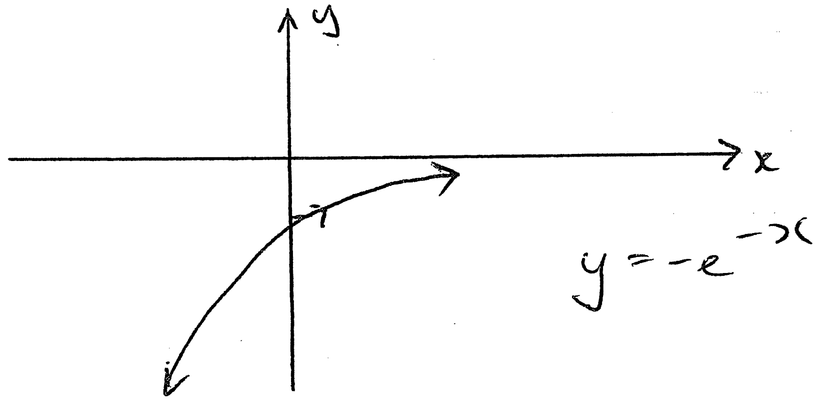
$$\doteqdot 64.65 \qquad (2)$$

Question 4

a)

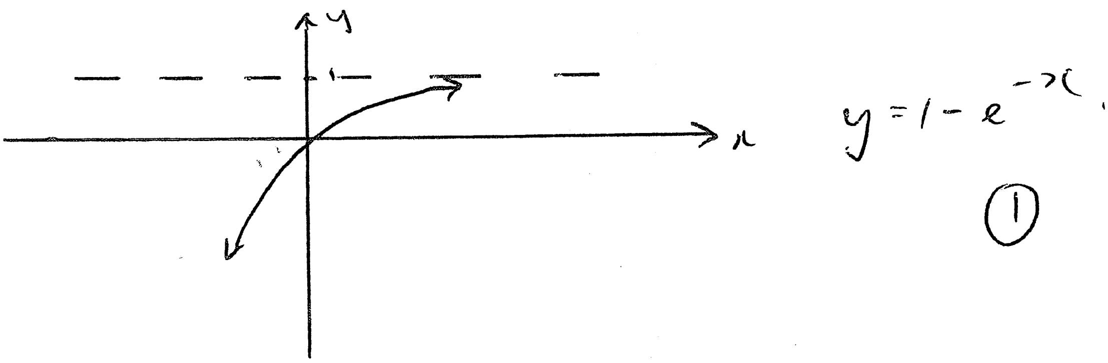


(i)



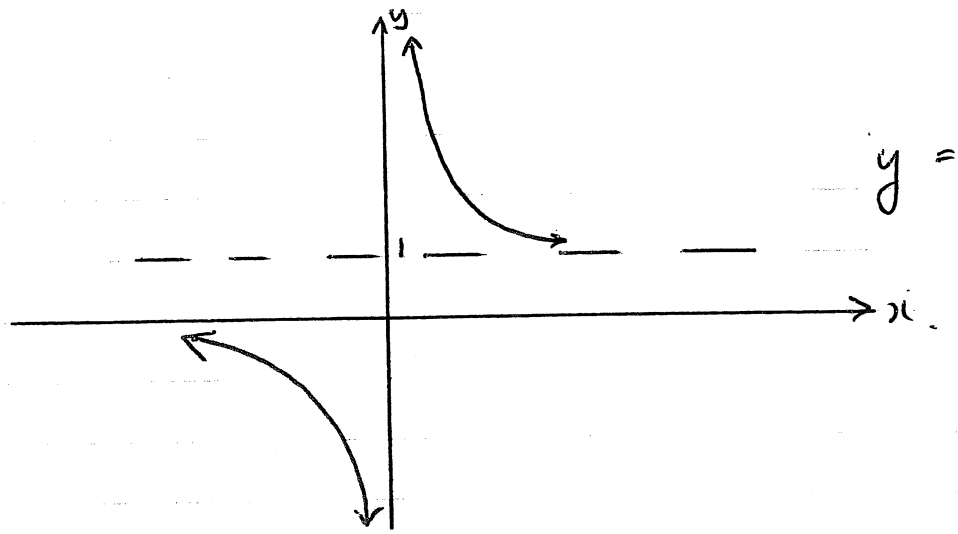
(1)

(ii)



(1)

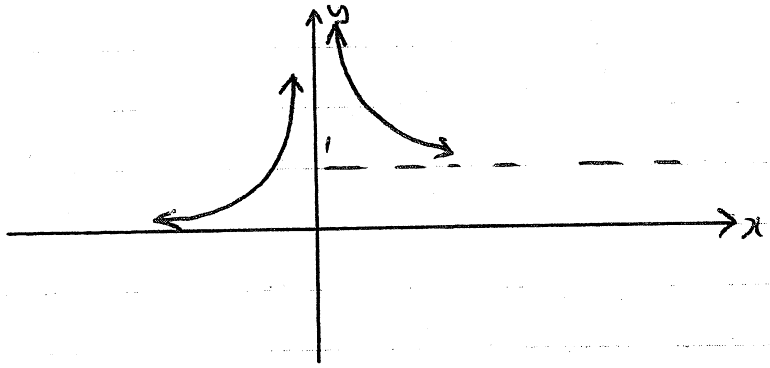
(iii)



$y = \dfrac{1}{1-e^{-x}}$

2

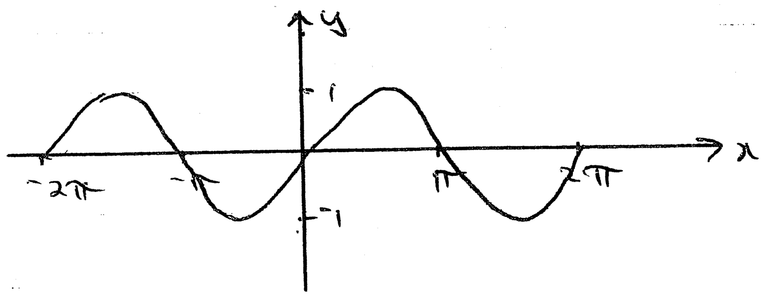
(iv)



$y = \left| \dfrac{1}{1-e^{-x}} \right|$

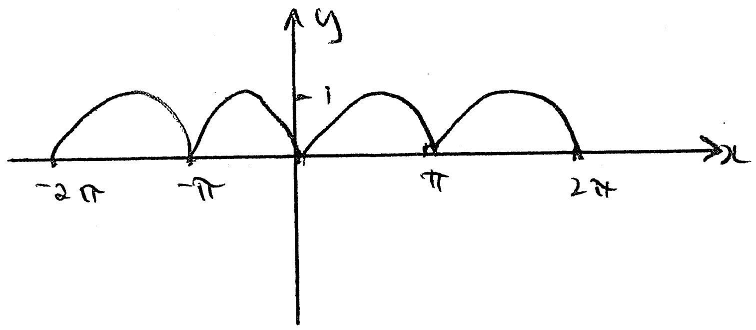
2

b)



$y = \sin x$

(i)

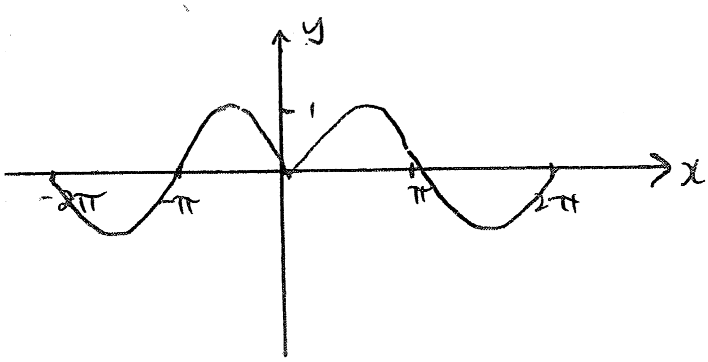


$y = |\sin x|$

even

1

(ii)

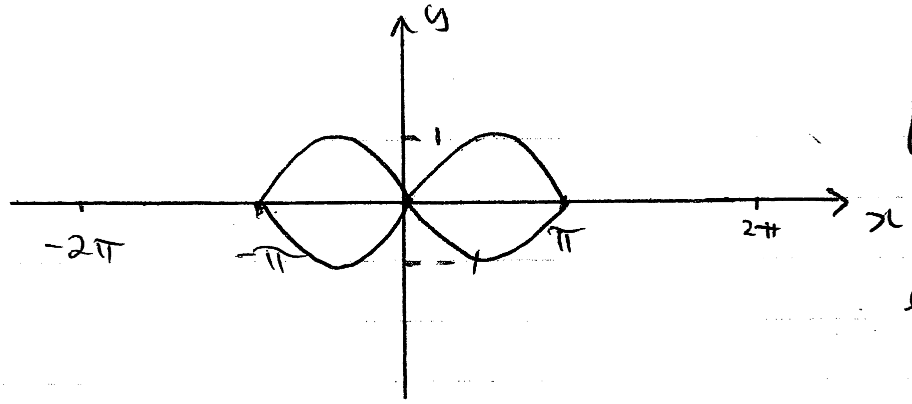


$y = \sin|x|$

even

2

(iii)

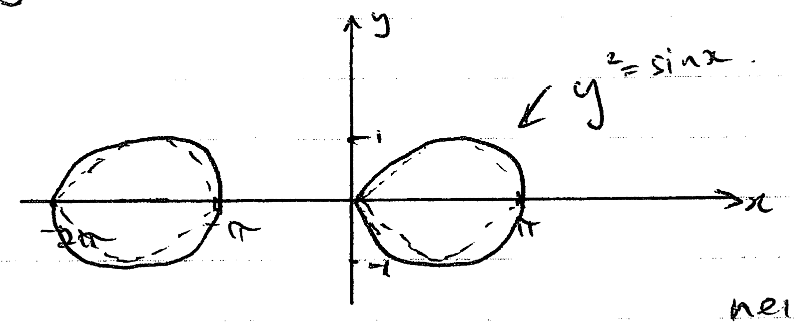


$|y| = \sin|x|$ ②

even.

(iv)

$y^2 = \sin x$ ie $y = \pm\sqrt{\sin x}$.



neither ②

c) $x^3 + y^3 - 8y + 7 = 0$.

$\therefore\ 3x^2 + 3y^2 \frac{dy}{dx} - \frac{8dy}{dx} = 0$

$\therefore\ \frac{dy}{dx}(3y^2 - 8) = -3x^2$

$\frac{dy}{dx} = \frac{-3x^2}{3y^2-8}$.

$\therefore$ at $(1,2)$, $y' = \frac{-3}{4}$.

$\therefore\ y - y_1 = m(x - x_1)$ becomes ②

$y - 2 = \frac{-3}{4}(x-1)$

or $y = -\frac{3}{4}x + 2\frac{3}{4}$.

(or $4y - 8 = -3x + 3$.
$\therefore\ 3x + 4y - 11 = 0$)

Question 5

a) $P(x) = x^3 + 4x - 2$.

(i) $\alpha+\beta+\gamma = \cancel{4}\ 0$ (1)

(ii) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= 0^2 - 2(4)$

$= -8$. (1)

(iii) $\alpha^4+\beta^4+\gamma^4 = ?$

α, β, γ satisfy $x^3+4x-2=0$

$\therefore$ satisfy $x^4+4x^2-2x=0$

ie $\alpha^4 + 4\alpha^2 - 2\alpha = 0$

$\beta^4 + 4\beta^2 - 2\beta = 0$.

$\gamma^4 + 4\gamma^2 - 2\gamma = 0$.

ie $\alpha^4+\beta^4+\gamma^4 = 2(\alpha+\beta+\gamma) - 4(\alpha^2+\beta^2+\gamma^2)$.

$= 2\times 0 - 4(-8)$.

$= 32$. (2)

b) (i) $az^3 + bz^2 + cz + d = 0$. (1)

$a, b, c, d \in \mathbb{R}$.

Taking conjugates of both sides gives

$\overline{az^3 + bz^2 + cz + d} = \bar{0}$

ie $\overline{az^3} + \overline{bz^2} + \overline{cz} + \bar{d} = \bar{0}$. Since conjugate of sum equals sum of conjugates

ie $\bar{a}\,\overline{z^3} + \bar{b}\,\overline{z^2} + \bar{c}\,\bar{z} + \bar{d} = \bar{0}$

ie $a\overline{z^3} + b\overline{z^2} + c\bar{z} + d = 0$ since a,b,c,d are real.

(2) ie $a(\bar{z})^3 + b(\bar{z})^2 + c\bar{z} + d = 0$ since conjugate of products equals products of conjugates

ie $\bar{z}$ satisfies $az^3+bz^2+cz+d=0$ if a,b,c,d are real.

(ii) If $1+2i$ is a root of $x^3-6x^2+13x-20=0$, then $1-2i$ is also a root (from b)(i))

$\therefore$ $(x-1-2i)(x-1+2i)$ is a factor of $x^3-6x^2+13x-20$

ie $(x-1)^2 - (2i)^2$ is factor

ie $x^2 - 2x + 1 + 4$ is factor.

$$
\begin{array}{r|l}
 & x \quad -4 \\
x^2 - 2x + 5 & x^3 - 6x^2 + 13x - 20 \\
 & x^3 - 2x^2 + 5x \\
\hline
 & \quad -4x^2 + 8x - 20 \\
 & \quad -4x^2 + 8x - 20 \\
\hline
 & \qquad 0
\end{array}
$$

ie $x^3 - 6x^2 + 13x - 20 = (x-4)(x-(1+2i))(x-(1-2i))$

(2) ie roots of $x^3 - 6x^2 + 13x - 20 = 0$ are $4, 1+2i, 1-2i$.

c) $P(x) = x^3 - 3x^2 + 2x - 7 = 0$.

(i) We require an equation in x where $x = \frac{1}{\alpha}$, α being a root of $P(x) = 0$

ie $P(\alpha) = 0 \Rightarrow P\left(\frac{1}{x}\right) = 0$

ie $\left(\frac{1}{x}\right)^3 - 3\left(\frac{1}{x}\right)^2 + 2\left(\frac{1}{x}\right) - 7 = 0$

(1) ie $1 - 3x + 2x^2 - 7x^3 = 0$

ie $7x^3 - 2x^2 + 3x - 1 = 0$

(ii) $\alpha + \beta + \gamma = 3$

$\therefore \alpha + \beta = 3 - \gamma$

ie we require equation whose roots are $3-\alpha, 3-\beta, 3-\gamma$.

We need equation in x where $x = 3 - \alpha$.

(2) ie $\alpha = 3 - x$

$\therefore P(\alpha) = 0 \Rightarrow P(3-x) = 0$

ie $(3-x)^3 - 3(3-x)^2 + 2(3-x) - 7 = 0$.

ie $27 - 27x + 9x^2 - x^3 - 27 + 18x - 3x^2 + 6 - 2x - 7 = 0$

ie $x^3 - 6x^2 + 11x + 1 = 0$.

d) $x^3 - 6x^2 + mx + 6 = 0$

Let the roots be $a-d,\ a,\ a+d$.

$\therefore\ a-d+a+a+d = 6$

ie $3a = 6$

$a = 2$.

Also, $a(a-d) + a(a+d) + (a-d)(a+d) = m$.

ie $a^2 - \cancel{ad} + a^2 + \cancel{ad} + a^2 - d^2 = m$

$\therefore\ 3a^2 - d^2 = m$

ie $12 - d^2 = m$.

And $(a-d)a(a+d) = -6$

$\therefore\ a(a^2 - d^2) = -6$

ie $2(4 - d^2) = -6$.

$4 - d^2 = -3$.

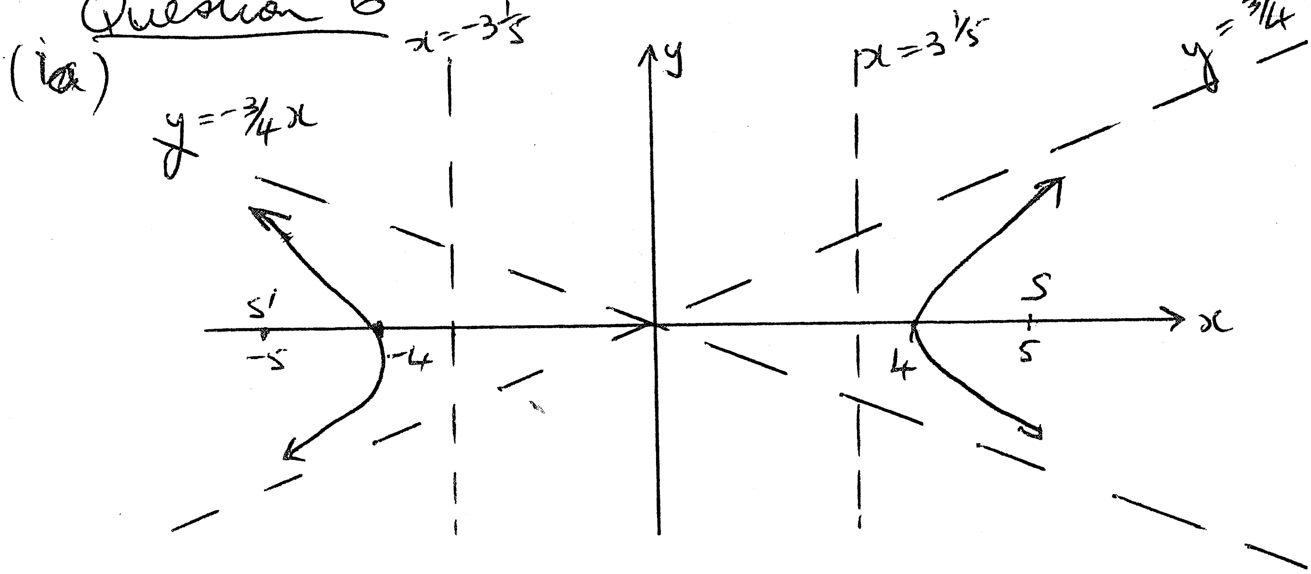
$d^2 = 7$.

$d = \pm\sqrt{7}$. $\Rightarrow m = 5$.

$\therefore$ roots are $2-\sqrt{7},\ 2,\ 2+\sqrt{7}$.

Question 6

(ia)



$\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

Vertices $(\pm 4, 0)$

$a = 4 \qquad b = 3$

$b^2 = a^2(e^2 - 1)$

$9 = 16(e^2 - 1)$

$\therefore \frac{9}{16} = e^2 - 1$

$e^2 = \frac{25}{16}$

$e = \frac{5}{4}$

$\therefore$ foci are $(\pm ae, 0)$

i.e. $(5, 0)$ and $(-5, 0)$.

directrices are $x = \pm \frac{a}{e}$

$= \pm 4/5/4$

$= \pm 16/5$.

(6)

Asymptotes are $y = \pm \frac{bx}{a}$

$= \pm \frac{3x}{4}$.

(ii) $x = 4\sec\theta$, $y = 3\tan\theta$.

$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} / \frac{dx}{d\theta}$

$= \frac{3\sec^2\theta}{4\sec\theta\tan\theta}$

(3). 1

$= \frac{3}{4}\operatorname{cosec}\theta$

$\therefore$ tangent is $y - 3\tan\theta = \frac{3}{4}\operatorname{cosec}\theta(x - 4\sec\theta)$

and normal is $y - 3\tan\theta = -\frac{4}{3}\sin\theta(x - 4\sec\theta)$

(iii) Putting $x = 0$ in the tangent gives

$y = 3\tan\theta - 3\operatorname{cosec}\theta\sec\theta$

$= \frac{3}{\sin\theta\cos\theta}(\sin^2\theta - 1)$

$= \frac{-3}{\sin\theta\cos\theta} \cdot \cos^2\theta$

$= -3\cot\theta$

$\therefore$ T is $(0, -3\cot\theta)$.

Putting $x = 0$ in the normal gives

$$x = 3\tan\theta + \frac{16}{3}\sin\theta\sec\theta$$

$$= \frac{25}{3}\tan\theta$$

2

$\therefore$ N is $\left(0, \frac{25}{3}\tan\theta\right)$.

(iv a) The centre, C, of the required circle is the midpt of NT.

i.e. C is $\left(0, \frac{1}{6}(25\tan\theta - 9\cot\theta)\right)$

The radius of the circle is $\frac{1}{2}NT$

$$= \frac{1}{6}(25\tan\theta + 9\cot\theta) \quad *$$

But $CS^2 = \frac{1}{36}(25\tan\theta - 9\cot\theta)^2 + 25$.

(S = focus (0, 5))

$$= \frac{1}{36}(25^2\tan^2\theta + 9^2\cot^2\theta - 450 + 900)$$

4

$$= \frac{1}{36}(25\tan\theta + 9\cot\theta)^2$$

= the square of the radius from *

i.e. the circle passes through the focus, S.

Similarly, the circle passes through the other focus S'.

(b)

Q7 Solutions

a) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x+\sin x}\, dx$

let $t = \tan\frac{x}{2}$.

$\therefore \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$.

$dx = \frac{2\,dt}{\sec^2\frac{x}{2}}$

$= \frac{2\,dt}{1+\tan^2\frac{x}{2}}$

$= \frac{2\,dt}{t^2+1}$.

$\sin x = \frac{2t}{1+t^2}$. using "t" results.

$\cos x = \frac{1-t^2}{1+t^2}$

When $x=0$ $t=0$

$x=\frac{\pi}{2}$, $t=\tan\frac{\pi}{4}=1$.

$$\therefore I = \int_0^1 \frac{\frac{2\,dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}$$

$$= \int_0^1 \frac{2\,dt}{1+t^2+1-t^2+2t}.$$

$$= 2\int_0^1 \frac{dt}{2t+2}.$$

$$= \int_0^1 \frac{dt}{t+1} \qquad (3)$$

$$= \left[\ln(t+1)\right]_0^1$$

$$= \ln 2 - \ln 1$$

$$= \ln 2$$

b) (i) $I_n = \int_1^e x(\ln x)^n \, dx$.

$= \int_1^e (\ln x)^n \cdot \frac{d}{dx}\frac{1}{2}x^2 \, dx$.

$= \left[\frac{1}{2}x^2 \cdot (\ln x)^n\right]_1^e - \int_1^e \frac{1}{2}x^2 \cdot \frac{d}{dx}(\ln x)^n \, dx$.

$= \frac{1}{2}e^2 \cdot 1 - 0 - \frac{1}{2}\int_1^e x^2 \cdot n(\ln x)^{n-1} \cdot \frac{1}{x} \, dx$.

(3) $= \frac{e^2}{2} - \frac{n}{2}\int_1^e x(\ln x)^{n-1} \, dx$

$= \frac{e^2}{2} - \frac{n}{2} I_{n-1}$. $\ast$

(ii) $\int_1^e x(\ln x)^3 \, dx = I_3$.

Now, $I_3 = \frac{e^2}{2} - \frac{3}{2} I_2$. from $\ast$

& $I_2 = \frac{e^2}{2} - \frac{2}{2} I_1$.

Now, ~~$I_1 = \int_1^e x(\ln x)^1 \, dx$~~

$I_1 = \frac{e^2}{2} - \frac{1}{2} I_0$.

Now, $I_0 = \int_1^e x(\ln x)^0 \, dx$

$= \left[\frac{x^2}{2}\right]_1^e = \frac{e^2}{2} - \frac{1}{2}$.

(2) $\therefore I_1 = \frac{e^2}{2} - \frac{1}{2}\left(\frac{e^2}{2} - \frac{1}{2}\right)$

$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}$

$= \frac{e^2}{4} + \frac{1}{4}$.

$\therefore I_2 = \frac{e^2}{2} - I_1$

$= \frac{e^2}{2} - \left(\frac{e^2}{4} + \frac{1}{4}\right)$

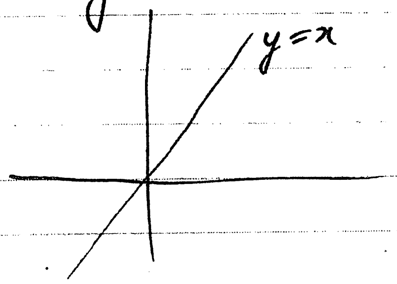
$= \frac{e^2}{4} - \frac{1}{4}$.

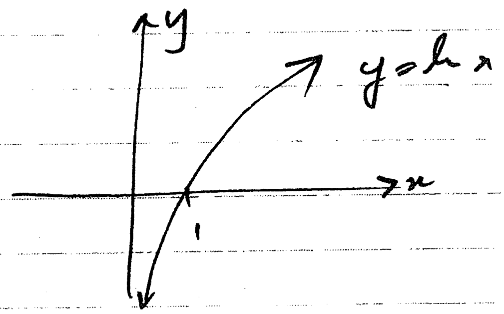
$\therefore I_3 = \frac{e^2}{2} - \frac{3}{2}\left(\frac{e^2}{4} - \frac{1}{4}\right)$

$= \frac{e^2}{2} - \frac{3e^2}{8} + \frac{3}{8}$.

$= \frac{e^2}{8} + \frac{3}{8}$.

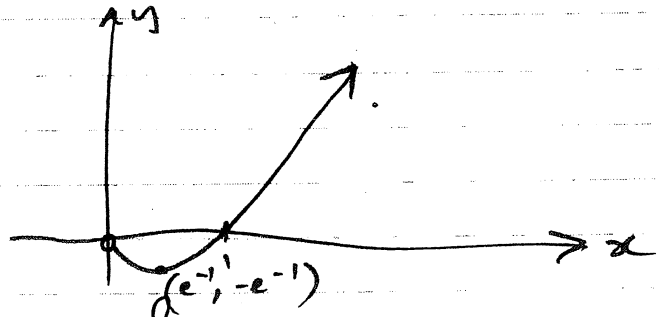
c) (1) $y = x \ln x$ D : $x > 0$.





$y = 0$ when $x = 1$.
$y < 0$ when $0 < x < 1$.
as $x \to \infty$, $y \to \infty$.
as $x \to 0^+$, $y \to 0^-$ (sub. values for small x)

(2)



$y = x \ln x$

$\therefore y' = x \cdot \frac{1}{x} + \ln x \cdot 1$

$= 1 + \ln x$.

$y' = 0 \Rightarrow \ln x + 1 = 0$

$\therefore x = e^{-1}$ is st. pt.

$y'' = \frac{1}{x}$

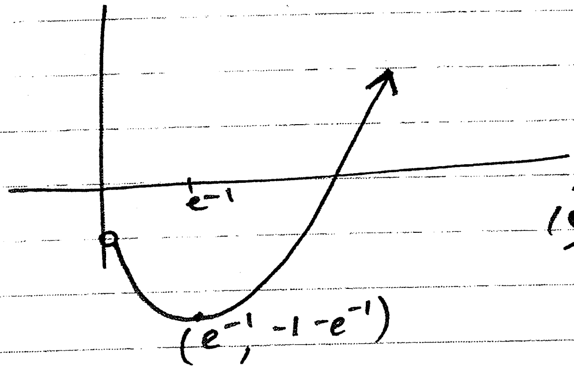
$= e > 0$ at $x = e^{-1}$

$\therefore$ Min. tpt at $(e^{-1}, -e^{-1})$.

(ii) Consider $y = x\ln x - 1$

ie move $y = x\ln x$ down 1 unit.

ie



ie only 1 sol'n to

$x\ln x - 1 = 0$

or $x\ln x = 1$.

If $x = \sqrt{e}$, $x\ln x = \frac{1}{2}\sqrt{e}$

$\approx .82\ldots$

< 1.

(2)

If $x = e$, $e\ln e = e > 1$

$\therefore$ root to $x\ln x - 1 = 0$ lies between $\sqrt{e}$ & e.

(iii) $a_2 = a_1 - \dfrac{f(a_1)}{f'(a_1)}$ — Newton's Method

$f(x) = x\ln x - 1$

$f'(x) = 1 + \ln x$

$= a_1 - \dfrac{(a_1\ln a_1 - 1)}{1 + \ln a_1}$

(2)

$= \dfrac{a_1 + a_1\ln a_1 - a_1\ln a_1 + 1}{1 + \ln a_1}$

$= \dfrac{1 + a_1}{1 + \ln a_1}$

(iv) $e \doteqdot 2.7$

$\sqrt{e} \doteqdot 1.6$

$\therefore$ Take $a_1 = 2$.

(1)

$\therefore a_2 = \dfrac{1+2}{1+\ln 2} = \dfrac{3}{1+\ln 2} \doteqdot 1.7718483$

$\therefore a_3 = \dfrac{1+a_2}{1+\ln a_2} \doteqdot \dfrac{2.77184\ldots}{1.572023254}$

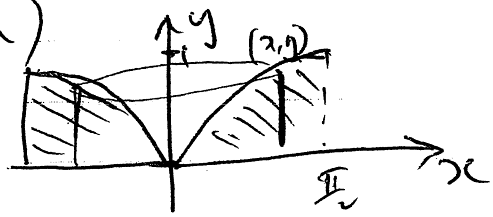
Discussion??

$\doteqdot 1.76$.

Q7(iv) (continued) This root appr... We need successive approximations be the root to 2 dec. places. where the 2 decimal places after the decimal point are fixed & only further decimal places vary.
eg 1.7654... & 1.7657...

Question 8

a)



(i) $\Delta V = 2\pi x . y . \Delta x$.

$\therefore V = \lim_{\Delta x \to 0} \sum_{x=0}^{\frac{\pi}{2}} \Delta V$.

$= \int_0^{\frac{\pi}{2}} 2\pi x y \, dx$.

$= 2\pi \int_0^{\frac{\pi}{2}} x \sin x \, dx$.

$= 2\pi \int_0^{\frac{\pi}{2}} x . \frac{d}{dx}(-\cos x) \, dx$.

$= 2\pi \left[-x\cos x\right]_0^{\frac{\pi}{2}} - 2\pi \int_0^{\frac{\pi}{2}} (-\cos x) . 1 \, dx$.

4

$= 2\pi (0-0) + 2\pi \left[\sin x\right]_0^{\frac{\pi}{2}}$.

$= 2\pi (1-0)$

$= 2\pi$ units^3.

(ii) $\frac{d}{dy}\left[y(\sin^{-1}y)^2 + 2(\sin^{-1}y)\sqrt{1-y^2} - 2y\right]$

$= (\sin^{-1}y)^2 + y . 2\sin^{-1}y . \frac{1}{\sqrt{1-y^2}} + \cancel{2\sqrt{1-y^2} \frac{1}{\sqrt{1-y^2}}}$

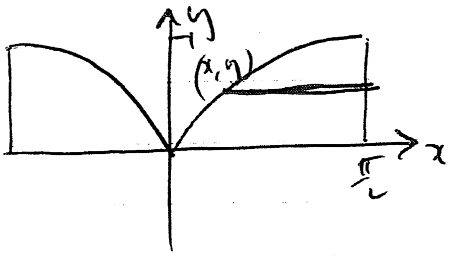
$+ 2\sin^{-1}y . \frac{1}{2}(1-y^2)^{-\frac{1}{2}}(-2y) - \cancel{2}$

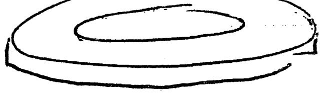
2

$= (\sin^{-1}y)^2 + \frac{2y\sin^{-1}y}{\sqrt{1-y^2}} - \frac{2y\sin^{-1}y}{\sqrt{1-y^2}} + 2 - 2$.

$= (\sin^{-1}y)^2$.

(iii) P.T.O.





$$\Delta V = \pi(R^2 - r^2)\Delta y$$
$$= \pi\left(\left(\frac{\pi}{2}\right)^2 - x^2\right)\Delta y$$

$$\therefore V = \lim_{\Delta y \to 0} \sum_{y=0}^{1} \Delta V$$

$y = \sin x$
$\therefore x = \sin^{-1} y$

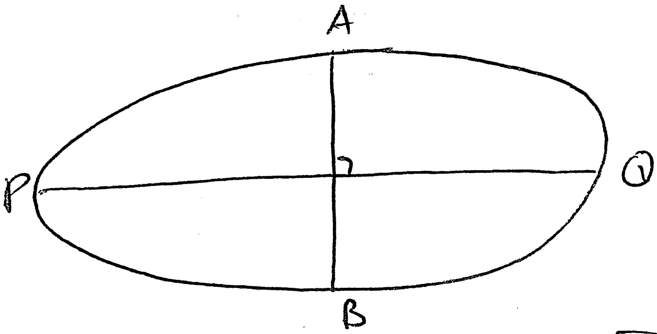
$$= \pi \int_0^1 \left(\frac{\pi}{2}\right)^2 - x^2 \, dy$$
$$= \pi \int_0^1 \frac{\pi^2}{4} - (\sin^{-1} y)^2 \, dy$$
$$= \pi \left[\frac{\pi^2}{4} y - y(\sin^{-1} y)^2 - 2(\sin^{-1} y)\sqrt{1-y^2} + 2y\right]_0^1$$

(from a)(ii))

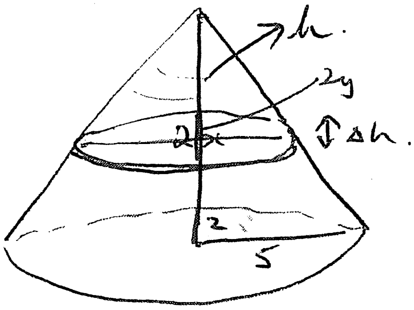
4

$$= \pi\left(\frac{\pi^2}{4} - 1\left(\frac{\pi}{2}\right)^2 - 2 \cdot \frac{\pi}{2} \cdot 0 + 2 - 0\right)$$
$$= 2\pi \text{ units}^3 \quad \text{as before in (i).}$$

b)



$PQ = 10$
$AB = 4$



Take a vertical axis through centre of base – let h be distance from base ie $h = 10$ is apex of cone

Consider a thin slice parallel to base – cross-sectional area ~~that~~ πxy and thickness Δ

ie $\Delta V = \pi xy \, \Delta h$. ie x = length of major semi-axis of

Now, $\frac{x}{5} = \frac{y}{2} = \frac{10-h}{10}$ y = length of minor semi-axis

$\therefore \Delta V = \pi \left(\frac{10-h}{2}\right) \cdot \frac{(10-h)}{5} \Delta h$.

$\therefore V = \lim_{\Delta h \to 0} \pi \sum_{h=0}^{10} \frac{(10-h)^2}{10} \Delta h$.

$= \frac{\pi}{10} \int_0^h (10-h)^2 \, dh$

$= \frac{\pi}{10} \left[-\frac{1}{3}(10-h)^3 \right]_0^h$

$= \frac{\pi}{10} \left(0 + \frac{1}{3} \cdot 10^3 \right)$

$= \frac{100\pi}{3}$ units^3

(or take h as distance from top of cone

$\Rightarrow \frac{x}{5} = \frac{h}{10} = \frac{y}{2}$

$\therefore \Delta V = \pi x y \, \Delta h$
$= \pi \frac{h}{2} \cdot \frac{h}{5} \Delta h$.

5

$\Rightarrow V = \frac{\pi}{10} \int_0^{10} h^2 \, dh$.

$= \frac{100}{3} \pi \; u^3$)