St George Girls' High School

Trial Higher School Certificate Examination

5

2003

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- All questions may be attempted.
- Begin each question on a new page
- All necessary working must be shown.
- Marks may be deducted for careless or poorly presented work.
- Board-approved calculators may be used.
- A list of standard integrals is included at the end of this paper.
- The mark allocated for each question is listed at the side of the question.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 – (15 marks) – Start a new page **Marks**

a) Show that $y = xe^{-x}$ has a stationary point at the point $\left(1, \dfrac{1}{e}\right)$. 2

b) On separate diagrams sketch the following curves. Indicate clearly any turning points, asymptotes and intercepts with the coordinate axes. 9

(i) $y = xe^{-x}$

(ii) $y = x^2e^{-2x}$

(iii) $y = \dfrac{1}{x^2e^{-2x}}$

(iv) $y = \log_e(xe^{-x})$

(v) $y = e^{xe^{-x}}$

c) Solve for x:

$$\frac{3-x}{\sqrt{x-1}} \geq \frac{\sqrt{x-1}}{3-x}$$ 4

Question 2 – (15 marks) – Start a new page **Marks**

a) Evaluate the following definite integrals:

(i) $\int_0^1 \frac{2x}{1+2x}\,dx$ 3

(ii) $\int_2^3 \frac{x+1}{\sqrt{x^2+2x+5}}\,dx$ 3

(iii) $\int_2^4 \frac{dx}{x\sqrt{x-1}}$ 3

b) The cubic equation $x^3-2x+4=0$ has roots α, β, γ.

(i) Prove that the cubic equation with roots $\alpha^2, \beta^2, \gamma^2$ is $x^3-4x^2+4x-16=0$. 3

(ii) Factorise $x^3-4x^2+4x-16$ into linear factors and hence prove that only one of α, β, γ is real. 3

Question 3 – (15 marks) – Start a new page **Marks**

a) Let $P(z) = z^7 - 1$.

(i) Find all the complex roots of $P(z) = 0$. (Call them $z_0, z_1, \ldots, z_6$ leaving all answers in terms of π). 3

(ii) Plot the points representing $z_0, z_1, \ldots, z_6$ on the Argand Diagram.

(iii) Factorise $P(z)$ over the complex numbers. 1

(iv) Factorise $P(z)$ over the real numbers (leave answers in terms of π). 2

(v) Show that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$ 2

b) The area bounded by the curve $y = x(2-x)$ and the x axis is rotated about the y axis. 5

(i) By considering cylindrical shells with generators parallel to the y axis, show that the volume V units ^{3} of the solid so generated is given by $V = \int_0^2 2\pi\, xy\, dx$.

(ii) Hence, determine the volume of this solid.

Question 4 – **(15 marks)** – **Start a new page** **Marks**

a) (i) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ 2

(ii) With reference to an appropriate diagram, show geometrically why the above result is true. 2

(iii) Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ 3

b) If $I_n = \int_0^1 x^n e^{-x}\,dx \quad (n \geq 0)$

(i) Prove that $I_n = n\,I_{n-1} - \frac{1}{e} \quad (n \geq 1)$ 2

(ii) Hence, evaluate $\int_0^1 x^3 e^{-x}\,dx$ 2

c) Using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, calculate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cot\theta \tan\frac{\theta}{2}\,d\theta$ 4

Question 5 – (15 marks) – Start a new page **Marks**

a) Find all pairs of real numbers x and y that satisfy $(x+iy)^2 = -12+16i$ 3

b) Consider the polynomial equation

$$z^3 + (6+i)z^2 + (17+8i)z + 33i + 30 = 0$$

(i) This equation has a root $\alpha = pi$ where p is real. Find the value of p. 3

(ii) If the other roots are β and γ, use the relationships for the sum and product of the roots (or otherwise) to find β and γ.
[Hint: Use your answer from part (a)]. 3

(iii) Show that the points representing α, β, γ on the Argand diagram are the vertices of a right-angled triangle. 1

c) The complex number z and its conjugate $\bar{z}$ satisfy

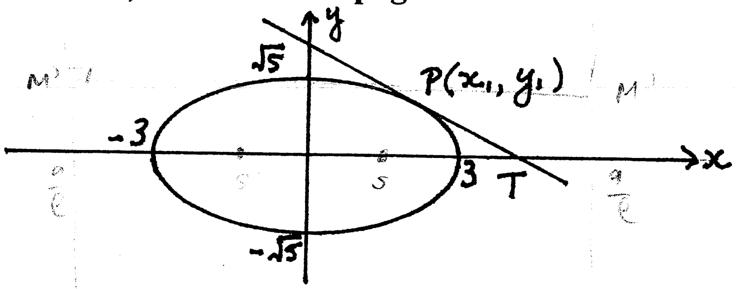
$$(z-\bar{z})^2 + 8(z+\bar{z}) = 16$$ 5

(i) Prove that the point which represents z on the Argand diagram lies on a parabola.

(ii) Sketch the locus of z and hence show that $-\dfrac{\pi}{4} \le \arg z \le \dfrac{\pi}{4}$

Question 6 – (15 marks) – Start a new page **Marks**

a)



The point $P(x_1, y_1)$ lies on the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1 \quad (x_1 \neq 0)$

The tangent to the ellipse at P cuts the x axis at T.

(i) Find the eccentricity, e, of the ellipse. 1

(ii) Find the coordinates of the foci, S and S^1 1

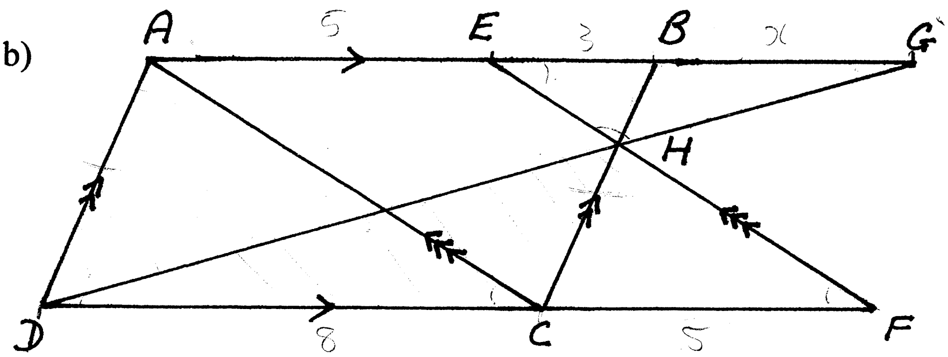
(iii) Find the equations of the directrices. 1

(iv) Show that the equation of the tangent at P is $\frac{x\,x_1}{9}+\frac{y\,y_1}{5}=1$ 2

(v) Hence write down the coordinates of T. 1

(vi) Using the focus-directrix definition of the ellipse, or otherwise, show that $\frac{PS}{PS^1}=\frac{TS}{TS^1}$ 4

b)



NOT TO SCALE 5

$ABCD$ and $AEFC$ are parallelograms. $AE = 5$cm and $EB = 3$cm.

(i) Find, giving brief reasons, the length of AG.

(ii) Find the ratio of the area of ΔCDH to the area of the parallelogram $ABCD$.

Question 7 – (15 marks) – Start a new page **Marks**

a) The acceleration of a particle which is moving along the x axis is given by

$$\frac{d^2x}{dt^2} = 2x^3 - 10x$$

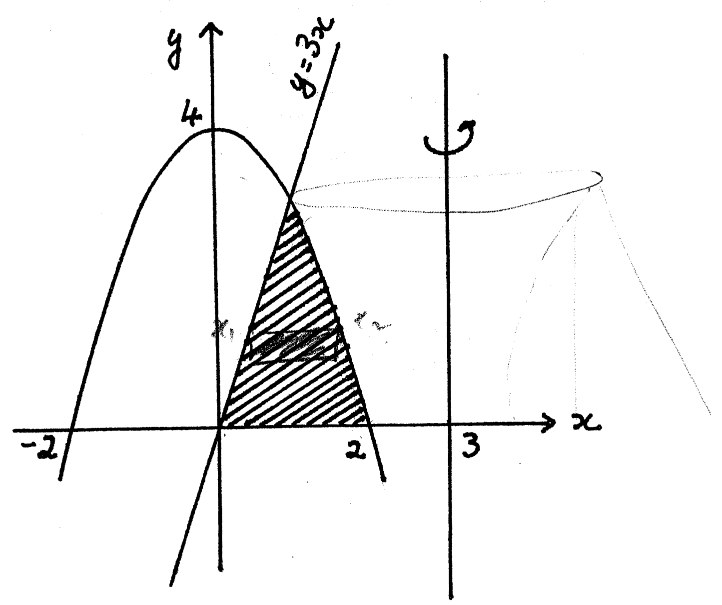
(i) If the particle starts at the origin with velocity u show that its velocity v is given by $v^2 - u^2 = x^4 - 10x^2$ 2

(ii) (α) If $u = 3$ show that the particle oscillates within the interval $-1 \le x \le 1$. 4

(β) Is this an example of simple harmonic motion? Give a clear reason for your answer. 1

(iii) If $u = 6$ carefully describe the motion of the particle. 3

b)



The area enclosed by the curve $y = 4 - x^2$, the line $y = 3x$ and the x axis (for $x \ge 0$) is rotated about the line $x = 3$.

Calculate the volume of the solid generated. 5

Question 8 – (15 marks) – Start a new page **Marks**

a) (i) x, y and z are positive integers. Show that if x is a factor of both y and z then x is a factor of $z - y$. 1

(ii) Show that $2^{2^{k+1}} = \left(2^{2^k}\right)^2$ 1

(iii) $F_n = 2^{2^n} + 1$ defines a set of positive integers called Fermat numbers for $n = 0, 1, 2, ...$ i.e. $F_0 = 2^{2^0} + 1 = 2^1 + 1 = 3$ 5

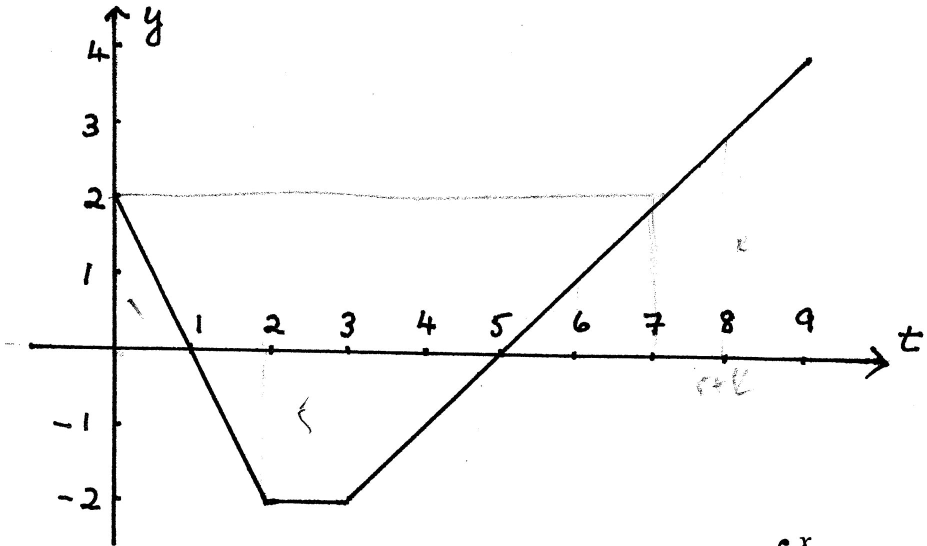
Use mathematical induction to show that

$F_n = F_0 F_1 F_2 .. F_{n-1} + 2$ for $n \geq 1$

(iv) Hence, or otherwise, show that the highest common factor of any two Fermat numbers is 1. 3

[Hint: Let k be a common factor of F_m and F_n, where $m < n$, and use (i) and (iii) to show that $k = 1$]

b) 5



The graph of $y = f(t)$ $0 \leq t \leq 9$ is shown. If $F(x) = \int_0^x f(t)\, dt$ find:

(i) the values of x for which $F(x) = 0$

(ii) the coordinates of any stationary points.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

(a) $y = xe^{-x}$

$\frac{dy}{dx} = 1.e^{-x} + x.(-e^{-x})$

$= e^{-x}(1-x)$

Stat pts when $\frac{dy}{dx} = 0$

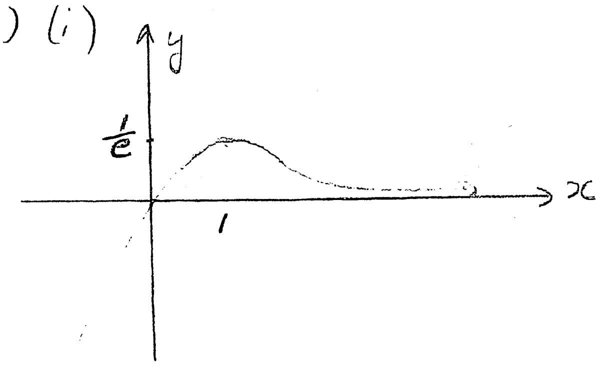
$e^{-x}(1-x) = 0$

$x = 1$

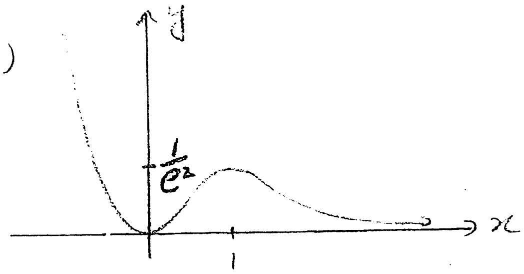
$y = 1.e^{-1} = \frac{1}{e}$

$\therefore (1, \frac{1}{e})$ is a stat pt

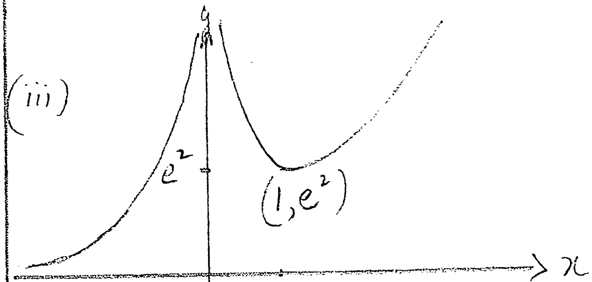
(b) (i)



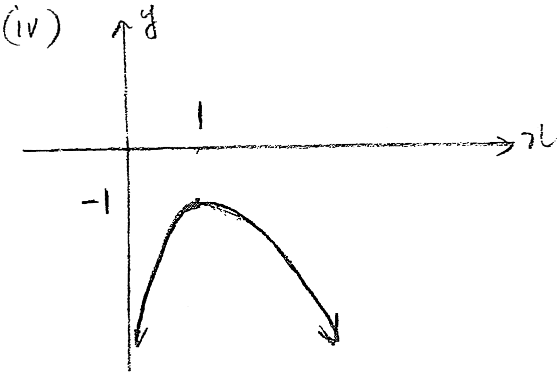
(ii)



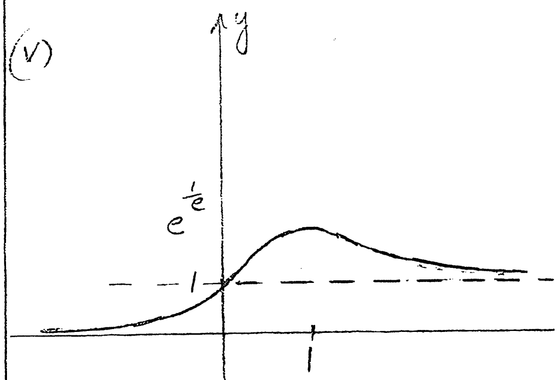
(iii)



(iv)



(v)



(c) $\frac{3-x}{\sqrt{x-1}} \geq \frac{\sqrt{x-1}}{3-x}$ $\quad x > 1$, $x \neq 3$

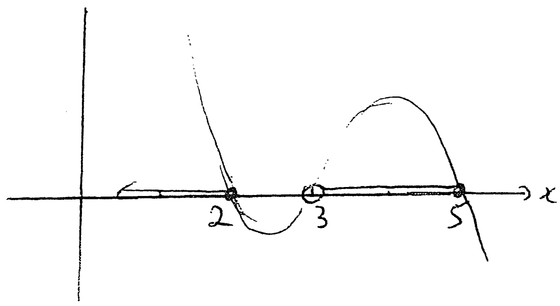
$\frac{3-x}{} \geq \frac{(\sqrt{x-1})^2}{3-x}$

$\frac{3-x}{1} - \frac{x-1}{3-x} \geq 0$

$\frac{9-6x+x^2-x+1}{3-x} \geq 0$ $\quad \left(\times \frac{(3-x)^2}{(3-x)^2}\right)$

$(x^2-7x+10)(3-x) \geq 0$

$(x-2)(x-5)(3-x) \geq 0$



$1 < x \leq 2$ or $3 < x \leq 5$

Question 2

(a) (i) $\int_0^1 \frac{2x}{1+2x}\,dx$

$= \int_0^1 \frac{1+2x-1}{1+2x}\,dx$

$= \int_0^1 1 - \frac{1}{1+2x}\,dx$

$= \left[x - \frac{1}{2}\ln(1+2x)\right]_0^1$

$= \left(1 - \frac{1}{2}\ln 3\right) - \left(0 - \frac{1}{2}\ln 1\right)$

$= 1 - \frac{1}{2}\ln 3$

(ii) $\int_2^3 \frac{x+1}{\sqrt{x^2+2x+5}}\,dx$

$= \frac{1}{2}\int_2^3 (2x+2)(x^2+2x+5)^{-1/2}\,dx$

$= \frac{1}{2}\left[\frac{(x^2+2x+5)^{1/2}}{1/2}\right]_2^3$

$= \left[\sqrt{x^2+2x+5}\right]_2^3$

$= \sqrt{20} - \sqrt{13}$

(iii) $\int_2^4 \frac{dx}{x\sqrt{x-1}}$

$u = \sqrt{x-1}$

$x = u^2+1$

$dx = 2u\,du$

$x=2 \quad u=1$

$x=4 \quad u=\sqrt{3}$

$= \int_1^{\sqrt{3}} \frac{2u\,du}{(u^2+1).u}$

$= 2\int_1^{\sqrt{3}} \frac{du}{u^2+1}$

$= 2\left[\tan^{-1}u\right]_1^{\sqrt{3}}$

$= 2\left(\tan^{-1}\sqrt{3} - \tan^{-1}1\right)$

$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \frac{\pi}{6}$

(b) $x^3-2x+4=0$ has roots α, β, γ.

Let $P(x) = x^3-2x+4$

Eq^n with roots $\alpha^2, \beta^2, \gamma^2$ is

$P(\sqrt{x}) = 0$

$(\sqrt{x})^3 - 2\sqrt{x} + 4 = 0$

$x\sqrt{x} - 2\sqrt{x} + 4 = 0$

$(x-2)\sqrt{x} = -4$

$(x-2)^2.\,x = 16$

$x^3 - 4x^2 + 4x = 16$

$x^3 - 4x^2 + 4x - 16 = 0$

(ii) $x^3 - 4x^2 + 4x - 16$

$= x^2(x-4) + 4(x-4)$

$= (x-4)(x^2+4)$

$= (x-4)(x+2i)(x-2i)$

Since $x^3-4x^2+4x-16=0$ has roots $\alpha^2, \beta^2, \gamma^2$ then

$(x-\alpha^2)(x-\beta^2)(x-\gamma^2)$

$= (x-4)(x+2i)(x-2i)$

Thus, without loss of generality,

$\alpha^2 = 4 \quad \beta^2 = -2i \quad \gamma^2 = 2i$

ie only α can be real, β and γ are non-real complex numbers

Question 3

(a) $P(z) = z^7 - 1$

(i) $z^7 - 1 = 0$

$z^7 = 1 \Rightarrow |z| = 1$

Let $z = \cos\theta + i\sin\theta$

$(\cos\theta + i\sin\theta)^7 = 1$

$\cos 7\theta + i\sin 7\theta = 1$

$\cos 7\theta = 1 \quad \sin 7\theta = 0$

$7\theta = 2k\pi$

$\theta = \frac{2k\pi}{7}$

$k=0 \quad z_0 = \cos 0 + i\sin 0 = 1$

$k=1 \quad z_1 = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$k=2 \quad z_2 = \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}$

$k=3 \quad z_3 = \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7}$

$k=4 \quad z_4 = \cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7}$

$= \cos\frac{6\pi}{7} - i\sin\frac{6\pi}{7}$

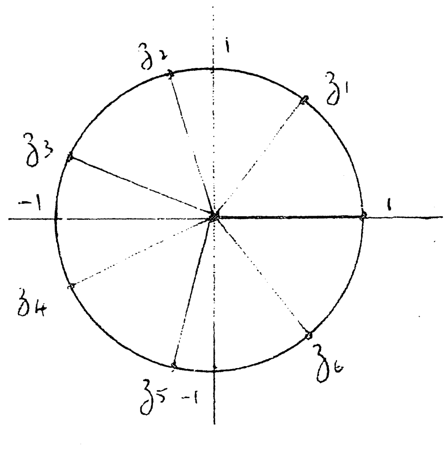
$k=5 \quad z_5 = \cos\frac{10\pi}{7} + i\sin\frac{10\pi}{7}$

$= \cos\frac{4\pi}{7} - i\sin\frac{4\pi}{7}$

$k=6 \quad z_6 = \cos\frac{12\pi}{7} + i\sin\frac{12\pi}{7}$

$= \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}$

(ii)



(iii) $P(z) = (z-1)(z-z_1)(z-z_2)(z-z_3)(z-z_4) \times (z-z_5)(z-z_6)$

where $z_1, \dots, z_6$ are indicated in (i)

(iv)

$(z-z_1)(z-z_6) = z^2 - (z_1+z_6)z + z_1 z_6$

$= z^2 - 2\cos\frac{2\pi}{7} z + 1$

Similarly we find $(z-z_2)(z-z_5)$ and $(z-z_3)(z-z_4)$

$\therefore P(z) =$

$(z-1)(z^2 - 2\cos\frac{2\pi}{7}z + 1)(z^2 - 2\cos\frac{4\pi}{7} + 1) \times (z^2 - 2\cos\frac{6\pi}{7} + 1)$

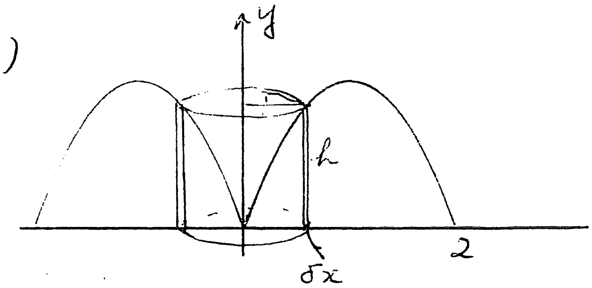
(iv) $z_0 + z_1 + \dots + z_6 = \frac{-\text{coeff } z^6}{\text{coeff } z^7} = 0$

$\therefore 1 + 2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = 0$

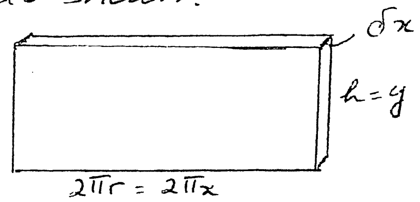
$2(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}) = -1$

$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$

(b)



When the cylindrical shell of radius $r = x$, height $h = y$ and width δx is cut parallel to y axis and "flattened out" its volume δV is approx that of a rectangular prism as shown:



$\delta V \doteq 2\pi x y \delta x$

$V = \lim_{\delta x \to 0} \sum_{x=0}^{2} 2\pi x y \delta x = \int_0^2 2\pi x y \, dx$

(ii) $xy = x \, x(2-x) = 2x^2 - x^3$

$V = 2\pi \int_0^2 2x^2 - x^3 \, dx$

$= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2$

$= 2\pi \left(\frac{16}{3} - \frac{16}{4} - 0 \right)$

Volume $= \frac{8\pi}{3}$ units^3

Question 4

(a) (i) $\int_0^a f(a-x)\,dx$

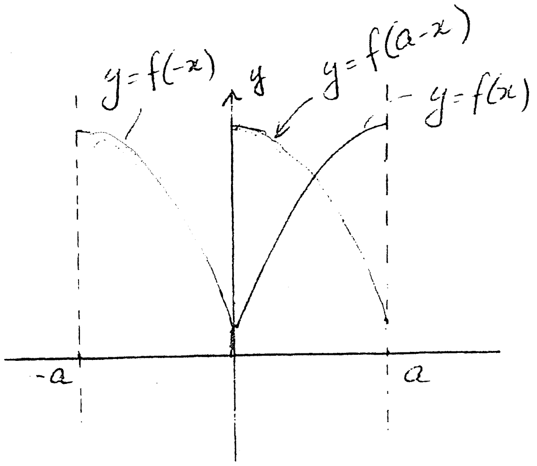
Let $u = a-x$
$du = -dx$
$dx = -du$
$x=0 \quad u=a$
$x=a \quad u=0$

$= \int_a^0 f(u).(-1)\,du$

$= \int_0^a f(u)\,du$

$= \int_0^a f(x)\,dx$

(ii)



If $y=f(x)$ $0 \le x \le a$ is reflected in the y axis we get
$y = f(-x)$ $\quad -a \le x \le 0$
If this is then translated 'a' units to the right we get
$y = f(-(x-a))$ $\quad 0 \le x \le a$
$= f(a-x)$ $\quad 0 \le x \le a$

From the graph it can be seen that the area under $y=f(x)$ from $x=0$ to $x=a$ ($\int_0^a f(x)\,dx$) equals the area under $y=f(-x)$ from $x=-a$ to $x=0$ and this is in turn equal to the area under $y=f(a-x)$ from $x=0$ to $x=a$ (ie $\int_0^a f(a-x)\,dx$)

(iii) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

$= \int_0^{\pi/2} \frac{\sqrt{\sin(\frac{\pi}{2}-x)}}{\sqrt{\sin(\frac{\pi}{2}-x)}+\sqrt{\cos(\frac{\pi}{2}-x)}}\,dx$ (by (i))

$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$

Now

$\int_0^{\frac{\pi}{2}} \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} + \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} \right) dx$

$= \int_0^{\pi/2} 1\,dx$

$= \left[x \right]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

Since $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}$

$= \frac{\pi}{4}$

(b) $I_n = \int_0^1 x^n e^{-x}\,dx \quad n \ge 0$

(i) $= \int_0^1 \frac{d(e^{-x})}{dx} . x^n\,dx$

$= \left[-e^{-x} . x^n \right]_0^1 - \int_0^1 -e^{-x} . nx^{n-1}\,dx$

$= (-e^{-1} \times 1 - 0) + n\int_0^1 x^{n-1} e^{-x}\,dx$

$= n I_{n-1} - \frac{1}{e}$

(ii) $\int_0^1 x^3 e^{-x}\,dx = I_3$

$= 3I_2 - \frac{1}{e}$

$= 3(2I_1 - \frac{1}{e}) - \frac{1}{e}$

$= 6(1.I_0 - \frac{1}{e}) - \frac{4}{e}$

$= 6I_0 - \frac{10}{e}$

$I_0 = \int_0^1 e^{-x}\,dx$

$= \left[-e^{-x} \right]_0^1$

$= -e^{-1} - -e^0$

$= 1 - \frac{1}{e}$

$\int_0^1 x^3 e^{-x}\,dx = 6(1-\frac{1}{e}) - \frac{10}{e}$

$= 6 - \frac{16}{e}$

Question 5

(c) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cot\theta \tan\frac{\theta}{2} \, d\theta$

$t = \tan\frac{\theta}{2}$ $\qquad \theta = 2\tan^{-1}t$

$d\theta = \frac{2}{1+t^2}\,dt$

$\theta = \frac{\pi}{3} \quad t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$\theta = \frac{\pi}{2} \quad t = \tan\frac{\pi}{4} = 1$

$\cot\theta = \frac{1-t^2}{2t}$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cot\theta . \tan\frac{\theta}{2} \, d\theta$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1-t^2}{2t} . t . \frac{2}{1+t^2} \, dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1-t^2}{1+t^2} \, dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2-(1+t^2)}{1+t^2} \, dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{1+t^2} - 1 \, dt$$

$$= \left[2\tan^{-1}t - t\right]_{1/\sqrt{3}}^{1}$$

$$= \left(2\tan^{-1}1 - 1\right) - \left(2\tan^{-1}\tfrac{1}{\sqrt{3}} - \tfrac{1}{\sqrt{3}}\right)$$

$$= \left(2\times\frac{\pi}{4} - 1\right) - \left(2\times\frac{\pi}{6} - \frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6} - 1 + \frac{1}{\sqrt{3}}$$

(a) $(x+iy)^2 = -12+16i \quad (x, y \text{ real})$

$x^2 - y^2 + 2ixy = -12+16i$

$x^2 - y^2 = -12 \quad — ①$

$2xy = 16$

$y = \frac{8}{x} \quad — ②$

Subst ② in ①

$x^2 - \frac{64}{x^2} = -12$

$x^4 + 12x^2 - 64 = 0$

$(x^2+16)(x^2-4) = 0$

$x = 2, -2 \quad (x \text{ real})$

$y = 4, -4$

$x = 2 \; y = 4$ or $x = -2, \; y = -4$

(b) $z^3 + (6+i)z^2 + (17+8i)z + 33i + 30 = 0$

(i) $\alpha = pi$ is a root

$(pi)^3 + (6+i)(pi)^2 + (17+8i)pi + 33i + 30 = 0$

$-p^3 i - 6p^2 - ip^2 + 17pi - 8p + 33i + 30 = 0$

Equating real and imag. parts

$-p^3 - p^2 + 17p + 33 = 0 \quad — ①$

$-6p^2 - 8p + 30 = 0 \quad — ②$

From ② $\quad 3p^2 + 4p - 15 = 0$

$(3p-5)(p+3) = 0$

$p = 5/3, -3$

Test in ①

$p \neq 5/3$ since 5×33, 3×1

$p = -3 \quad LHS = -(-3)^3 - (-3)^2 + 17\times -3 + 33$

$= 27 - 9 - 51 + 33$

$= 0 \quad = RHS$

$\therefore \alpha = -3i$

(ii) $\alpha + \beta + \gamma = -(6+i)$

$-3i + \beta + \gamma = -6 - i$

$\beta + \gamma = -6 + 2i \quad — ①$

$\alpha\beta\gamma = -(33i + 30)$

$-3i\beta\gamma = -3i(11 - 10i)$

$\beta\gamma = 11 - 10i \quad — ②$

From ① $\gamma = -6+2i-\beta$

Subst in ②

$\beta(-6+2i-\beta) = 11-10i$

$(-6+2i)\beta - \beta^2 = 11-10i$

$\beta^2 + (6-2i)\beta + 11-10i = 0$

$\beta^2 + (6-2i)\beta + (3-i)^2 = -11+10i + (3-i)^2$

$(\beta + (3-i))^2 = -11+10i+9-6i-1$

$= -3+4i$

$\beta + (3-i) = \pm\sqrt{-3+4i} \quad (*)$

$= \pm(1+2i)$

$\beta = -3+i+1+2i, \ -3+i-1-2i$

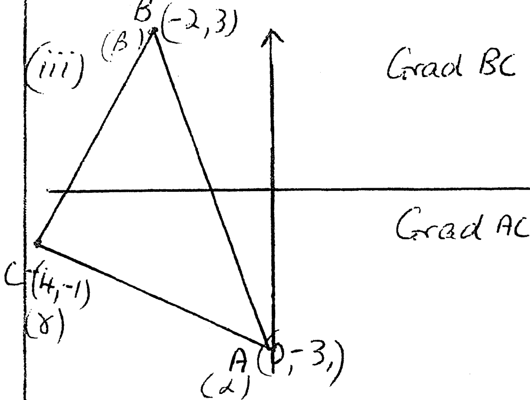
$= -2+3i, \ -4-i$

$* \ \sqrt{-12+16i} = 2\sqrt{-3+4i}$

$\pm(2+4i) = 2\sqrt{-3+4i}$

$\therefore \sqrt{-3+4i} = \pm(1+2i)$

(iii)



$\text{Grad } BC = \dfrac{3--1}{-2--4}$

$= \dfrac{4}{2}$

$= 2$

$\text{Grad } AC = \dfrac{-1--3}{-4-0}$

$= \dfrac{2}{-4}$

$= -\dfrac{1}{2}$

$\text{Grad } BC \times \text{Grad } AC = 2 \times -\frac{1}{2} = -1$

$\therefore A, B, C$ vertices of right-angled Δ

(c) $(z-\bar{z})^2 + 8(z+\bar{z}) = 16$

Let $z = x+iy$

$\therefore \bar{z} = x-iy$

$z-\bar{z} = 2iy$

$z+\bar{z} = 2x$

$\therefore$ Eqn becomes

$(2iy)^2 + 8.2x = 16$

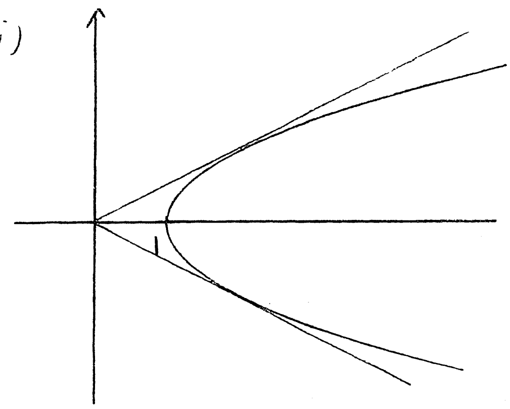
$-4y^2 + 16x = 16$

$16x = 16 + 4y^2$

$x = 1 + \dfrac{y^2}{4}$

which is a parabola

(ii)



Maximum and minimum values of $\arg z$ occur when $y = mx$ is a tangent to $x = 1 + \dfrac{y^2}{4}$

$y = mx \quad$ — ①

$x = 1 + \dfrac{y^2}{4} \quad$ — ②

Subst ① in ②

$x = 1 + \dfrac{(mx)^2}{4}$

$4x = 4 + m^2x^2$

$m^2x^2 - 4x + 4 = 0$

Line is a tangent when quadratic has equal roots ie $\Delta = 0$

$(-4)^2 - 4 \times m^2 \times 4 = 0$

$16 - 16m^2 = 0$

$m^2 = 1$

$m = \pm 1$

$\tan\theta = \pm 1$

$\theta = \pm\dfrac{\pi}{4}$

$\therefore -\dfrac{\pi}{4} \le \arg z \le \dfrac{\pi}{4}$

Question 6

(a) $\frac{x^2}{9} + \frac{y^2}{5} = 1$

$a^2 = 9 \qquad b^2 = 5$

$a = 3 \qquad b = \sqrt{5}$

(i) $b^2 = a^2(1-e^2)$

$5 = 9(1-e^2)$

$e^2 = \frac{4}{9}$

$e = \frac{2}{3} \quad (e>0)$

(ii) $ae = 3 \times \frac{2}{3} = 2$

$S(2,0) \quad S'(-2,0)$

(iii) $\frac{a}{e} = \frac{3}{2/3} = \frac{9}{2}$

Directrices: $x = \frac{9}{2}, \quad x = -\frac{9}{2}$

(iv) $\frac{x^2}{9} + \frac{y^2}{5} = 1$

$\frac{2x}{9} + \frac{2y}{5}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{9} \times \frac{5}{2y}$

$= -\frac{5x}{9y}$

At $P(x_1,y_1)$ $\frac{dy}{dx} = -\frac{5x_1}{9y_1}$

Eq^n of tangent at P is

$y - y_1 = -\frac{5x_1}{9y_1}(x-x_1)$

$\frac{yy_1}{5} - \frac{y_1^2}{5} = -\frac{xx_1}{9} + \frac{x_1^2}{9}$

$\frac{xx_1}{9} + \frac{yy_1}{5} = \frac{x_1^2}{9} + \frac{y_1^2}{5}$

$= 1$ (since $P(x_1,y_1)$ lies on ellipse)

$\frac{xx_1}{9} + \frac{yy_1}{5} = 1$

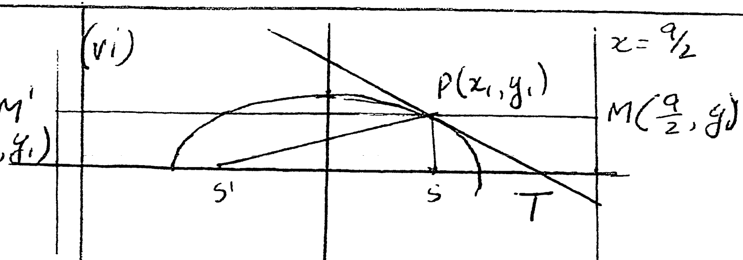
(v) When $y = 0$

$\frac{xx_1}{9} = 1$

$x = \frac{9}{x_1}$

$\therefore T$ has coords $\left(\frac{9}{x_1}, 0\right)$

(vi)



Let $M(\frac{9}{2}, y_1)$ and $M'(-\frac{9}{2}, y_1)$ be the feet of the perpendiculars from P to the directrices

$\frac{PS}{PM} = \frac{PS'}{PM'} (= e)$ (focus-directrix definition)

$\therefore \frac{PS}{PS'} = \frac{PM}{PM'} = \frac{\frac{9}{2} - x_1}{\frac{9}{2} + x_1}$

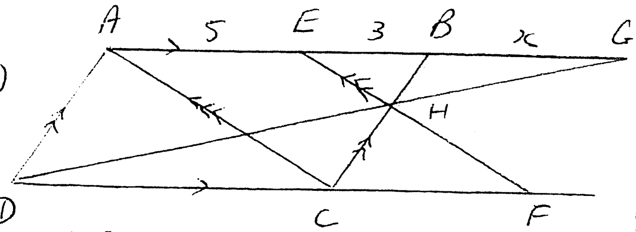
$= \frac{9-2x_1}{9+2x_1}$

Now $\frac{TS}{TS_1} = \frac{\frac{9}{x_1} - 2}{\frac{9}{x_1} + 2}$

$= \frac{9-2x_1}{9+2x_1}$

$= \frac{PS}{PS'}$

(b)



Let $GB = x$

In ΔBAC

$\frac{BH}{HC} = \frac{BE}{EA}$ (line parallel to one side of Δ divides other 2 sides in same ratio)

$= \frac{3}{5}$

$\therefore \frac{BH}{BC} = \frac{3}{8}$

In ΔGAD, ΔGBH

$B\hat{G}H$ is common

$G\hat{A}D = G\hat{B}H$ (corresp $\angle$s equal $BH \parallel AD$)

$\Delta GAD \parallel\!\mid \Delta GBH$ (equiangular)

$\therefore \frac{GA}{GB} = \frac{AD}{BH} = \frac{BC}{BH}$ ($BC = AD$ opp sides of parallelogram)

$\frac{x+8}{x} = \frac{8}{3}$

$3x + 24 = 8x$

$24 = 5x$

$x = 4.2$

$\therefore AG = 12.2\text{ cm}$

(ii) Area $\Delta CDH = \frac{1}{2} \times CH \times DC \sin D\hat{C}H$

Area $ABCD = BC \times CD \sin D\hat{C}H$

$$\frac{\text{Area } \Delta CDH}{\text{Area } ABCD} = \frac{\frac{1}{2} \cdot CH \cdot DC \cdot \sin D\hat{C}H}{BC \times DC \cdot \sin D\hat{C}H}$$

$$= \frac{\frac{1}{2} CH}{BC}$$

$$= \frac{1}{2} \times \frac{5}{8}$$

$$= \frac{5}{16}$$

Question 7

(a) (i) $\frac{d(\frac{1}{2}v^2)}{dx} = 2x^3 - 10x$

$\frac{1}{2}v^2 = \frac{2x^4}{4} - \frac{10x^2}{2} + c$

When $x = 0$ $v = u$

$\frac{1}{2}u^2 = c$

$\frac{1}{2}v^2 = \frac{x^4}{2} - \frac{10x^2}{2} + \frac{u^2}{2}$

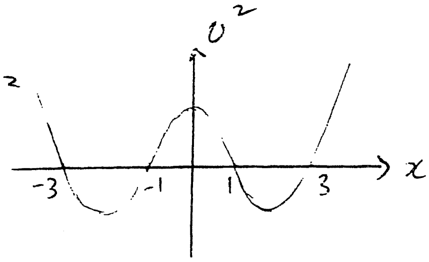
$v^2 - u^2 = x^4 - 10x^2$

(ii) (α) If $u = 3$ then $v^2 - 9 = x^4 - 10x^2$

$v^2 = x^4 - 10x^2 + 9$

$= (x^2 - 1)(x^2 - 9)$

$= (x-1)(x+1)(x-3)(x+3)$



Since $v^2 \geq 0$ then $(x-1)(x+1)(x-3)(x+3) \geq 0$

$\therefore$ $x \leq -3$ or $-1 \leq x \leq 1$ or $x \geq 3$

Since particle starts at $x = 0$ with $v = 3$ it then moves to the right (positive direction). At $x = 1$ $v = 0$ and $\ddot{x} = -8$ and so particle will then move to the left until it reaches $x = -1$ where $v = 0$ and $\ddot{x} = 8$. This means particle will then move to the right until $v = 0$ again at $x = 1$. Thus particle oscillates between $x = 1$ and $x = -1$ ie within the interval $-1 \leq x \leq 1$

(β) Not SHM since $\ddot{x} \neq -n^2(x - b)$

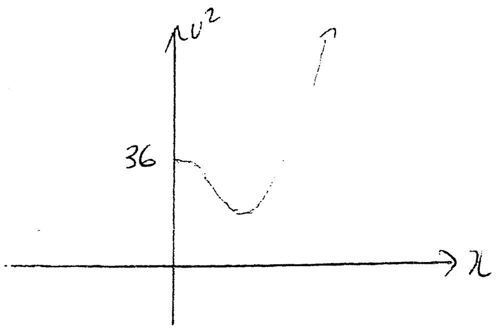
(iii) If $u = 6$ then $v^2 - 36 = x^4 - 10x^2$

$v^2 = x^4 - 10x^2 + 25 + 11$

$= (x^2 - 5)^2 + 11$

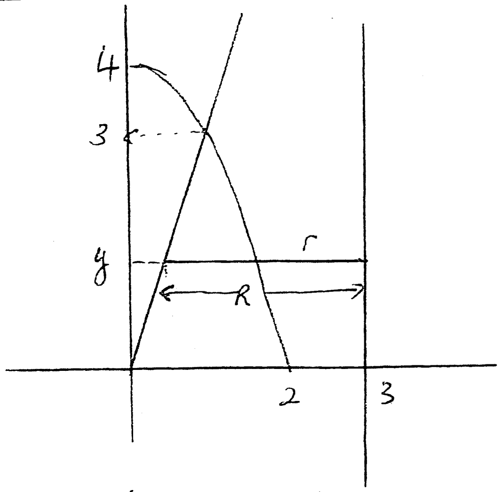
$\therefore$ $v^2 > 0$ for all x in domain

$\therefore$ Particle will never stop



Moves to the right with decreasing velocity until it passes $x = \sqrt{5}$ (with velocity $\sqrt{11}$) and continues to the right with increasing velocity

(b)



$y = 4 - x^2$

$y = 3x$

$x^2 + 3x - 4 = 0$

$(x+4)(x-1) = 0$

$x = -4, 1$

$y = -12, 3$

Let $A(y)$ be the area of the cross-section at height y

$A(y) = \pi R^2 - \pi r^2$ $\qquad R = 3 - \frac{y}{3}$ $\qquad y = 3x$

$\qquad r = 3 - \sqrt{4-y}$ $\qquad y = 4 - x^2$

$R^2 = \left(3 - \frac{y}{3}\right)^2 = 9 - 2y + \frac{y^2}{9}$

$r^2 = \left(3 - \sqrt{4-y}\right)^2 = 9 - 6\sqrt{4-y} + 4 - y$

$A(y) = \pi\left[\left(9 - 2y + \frac{y^2}{9}\right) - \left(9 - 6\sqrt{4-y} + 4 - y\right)\right]$

$= \pi\left(\frac{y^2}{9} + y + 6\sqrt{4-y} - 4\right)$

$V = \int_0^3 \pi\left(\frac{y^2}{9} \cancel{+} - y + 6\sqrt{4-y} - 4\right) dy$

$= \pi\left[\frac{y^3}{27} \cancel{+} - \frac{y^2}{2} + \frac{6(4-y)^{3/2}}{\frac{3}{2} \times -1} - 4y\right]_0^3$

$= \pi\left\{\left(\frac{27}{27} \cancel{+} - \frac{9}{2} - 4 \times 1 - 12\right) - \left(-4 \times 4^{3/2}\right)\right]$

$= \pi\left\{\cancel{\frac{4}{4}} -3\frac{1}{2} - 16 + 32\right\}$

$= \cancel{\frac{43\pi}{2}} \quad 12\frac{1}{2}\pi.$

OR cylindrical shells.

Question 8

(a)(i) if x is a factor of both y & z then $y = ax$ and $z = bx$ where a, b are integers

$z - y = bx - ax = (b-a)x$

ie x is a factor of $z - y$

(ii) $2^{2^{k+1}} = 2^{2^k \times 2^1} = \left(2^{2^k}\right)^2$

(iii) $F_n = 2^{2^n} + 1 \quad n = 0, 1, 2, \ldots$

Rtp $F_n = F_0 F_1 \ldots F_{n-1} + 2 \quad n \geq 1$

$F_0 = 3$

$n = 1 \quad F_1 = 2^{2^1} + 1$

$= 2^2 + 1$

$= 5$

$= 3 + 2$

$= F_0 + 2$

$\therefore$ Proposition is true for $n = 1$

Let k be a value for which proposition is true $(k \geq 1)$

ie $F_k = F_0 F_1 \ldots F_{k-1} + 2$

Aim to show that proposition is then true for $n = k+1$

ie $F_{k+1} = F_0 F_1 \ldots F_k + 2$

$F_{k+1} = 2^{2^{k+1}} + 1$

$= \left(2^{2^k}\right)^2 + 1$

$= (F_k - 1)^2 + 1$

$= F_k^2 - 2F_k + 1 + 1$

$= F_k(F_k - 2) + 2$

$= F_k(F_0 F_1 \ldots F_{k-1}) + 2$ (by inductive hypothesis)

$= F_0 F_1 \ldots F_{k-1} F_k + 2$

which is of the required form

$\therefore$ If proposition is true for $n = k$ it is also true for $n = k+1$. Since it is true for $n = 1$ it is also true for $n = 1+1 = 2$ and hence by induction it is true for all positive integers

(iv) If $m < n$ then

$F_n = F_0 F_1 \ldots F_m \ldots F_{n-1} + 2$

Let k be the hcf of F_n and F_m

$\therefore$ k is a factor of F_n and $F_0 F_1 \ldots F_m \ldots F_{n-1}$

Hence k is a factor of their difference

ie k is a factor of 2

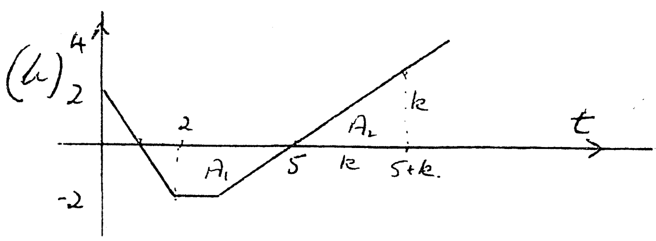
ie $k = 1$ or 2

But $F_n = 2^{2^n} + 1$ is odd

$\therefore k \neq 2$

ie $k = 1$

(b)



(i) $F(x) = 0$ when $x = 0, 2, 5+k$

where k is such that $A_1 = A_2$

$A_1 = \frac{1}{2}(1+3) \times 2 = 4$

$A_2 = \frac{1}{2}k^2$

$\therefore k^2 = 8$

$k = \sqrt{8} = 2\sqrt{2}$

ie $x = 0, 2, 5 + 2\sqrt{2}$

(ii) Stationary points at $x = 1$ and $x = 5$

$F(1) = \frac{1}{2} \times 1 \times 2 = 1$ $\qquad F(5) = -4$

$\therefore$ $(1, 1)$ and $(5, -4)$ are stationary points