St George Girls High School

Trial Higher School Certificate Examination

2005

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Write your Student Number on every page
- All questions may be attempted.
- Begin each question in a new booklet.
- All necessary working must be shown.
- Marks may be deducted for careless or poorly presented work.
- Board-approved calculators may be used.
- A list of standard integrals is included at the end of this paper.
- The mark allocated for each question is listed at the side of the question.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

<u>Question 1</u> – (15 marks) – Start a new booklet **Marks**

a) Find $\int \frac{e^x}{4+e^{2x}}\,dx$ 2

b) By completing the square and using the table of standard integrals find

$$\int \frac{dx}{\sqrt{x^2-4x+9}}$$ 2

c) Evaluate $\int_5^{21} \frac{x}{x+4+2\sqrt{x+4}}\,dx$ using the substitution $x=u^2-4$ 3

d) Using the substitution $t=\tan\frac{x}{2}$, or otherwise, calculate 4

$$\int_0^{\frac{2\pi}{3}} \frac{dx}{13+5\sin x+12\cos x}$$

e) Find $\int e^{2x}\cos x\,dx$ 4

Question 2 – (15 marks) – Start a new booklet **Marks**

a) $z_1 = 3 - 2i$ and $z_2 = 1 - i$ 2

Find in the form $a + ib$ (where a and b are real).

(i) $z_1 \bar{z}_2$ (ii) $\dfrac{i}{z_1}$

b) If $z = 1 - i$ and $w = \sqrt{3} + i$ 6

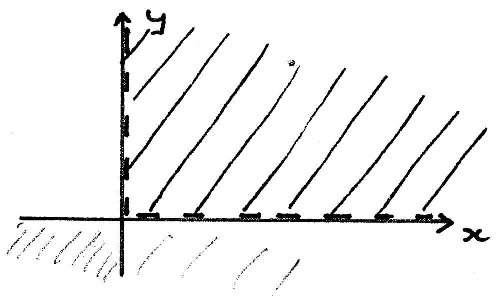
(i) Find $\dfrac{w}{z}$ in the form $a + ib$

(ii) Show that $\arg\left(\dfrac{w}{z}\right) = \dfrac{5\pi}{12}$

(iii) Find the modulus of $\dfrac{w}{z}$

(iv) Hence find the exact value of $\cos\dfrac{5\pi}{12}$

c) It is given that z^2 lies in the first quadrant of the complex plane, as shown in the diagram.



Shade the region in which z can lie. 2

d) Sketch the locus of z if 2

(i) $|z - i| = |z - 2|$ (ii) $\arg(z - i) = \arg(z - 2)$

e) It is given that $z = 2 + i$ is a zero of $P(z) = z^4 - 4z^3 + 20z - 25$ 3

(i) Explain why $2 - i$ is also a zero of $P(z)$

(ii) Hence factorise $P(z)$ over the real numbers (2+i)

Question 3 – (15 marks) – Start a new booklet **Marks**

a) The rectangular hyperbola, H, has equation $x^2 - y^2 = 8$. Write down **5**

(i) the eccentricity

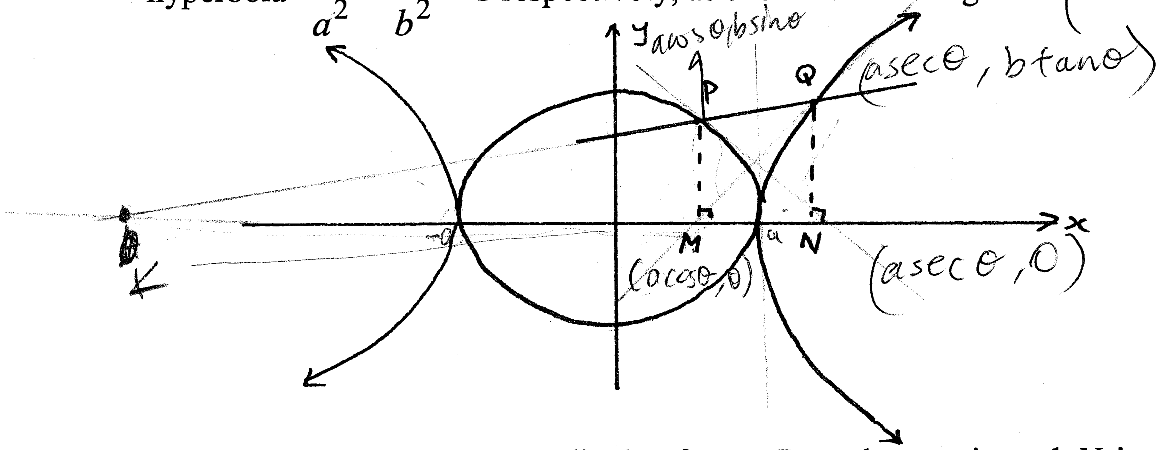
(ii) the coordinates of the foci

(iii) the equations of the directrices

(iv) the equations of the asymptotes and

(v) sketch the curve showing the foci, directrices, asymptotes and any intercepts with the coordinate axes.

b) $P(a\cos\theta, b\sin\theta)$ and $Q(a\sec\theta, b\tan\theta)$ lie on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ respectively, as shown on the diagram. $\left(0<\theta<\dfrac{\pi}{2}\right)$ **10**



M is the foot of the perpendicular from P to the x-axis and N is the foot of the perpendicular from Q to the x-axis. QP meets the x-axis at K. A is the point (a, O).

(i) Given that $\Delta KPM \parallel\!\mid \Delta KQN$, show that $\dfrac{KM}{KN}=\cos\theta$

(ii) Hence show that K has coordinates $(-a, O)$

(iii) Show that the tangent to the ellipse at P has equation $\dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=1$

<u>and</u> deduce that it passes through N.

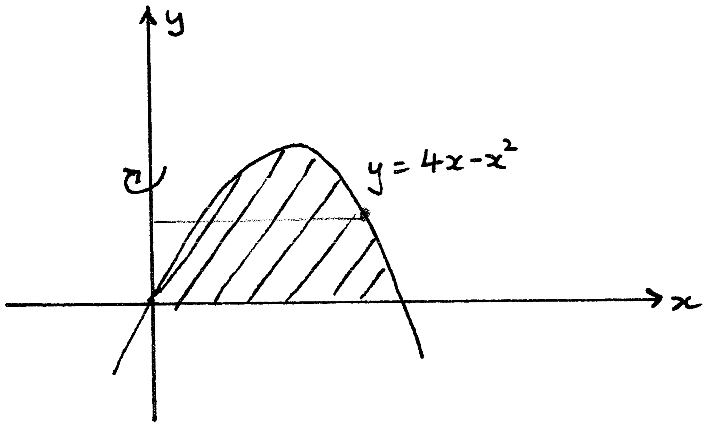
(iv) <u>Given</u> that the tangent to the hyperbola at Q has equation $\dfrac{x\sec\theta}{a}-\dfrac{y\tan\theta}{b}=1$

Show that it passes through M.

(v) Show that the tangents PN, QM and the common tangent at A are concurrent. Find the point of concurrence.

<u>Question 4</u> – (15 marks) – Start a new booklet **Marks**

a) The shaded region is rotated about the y-axis to form a solid of revolution. 4



Using the method of cylindrical shells,

(i) Show that the volume, V, of this solid is given by $V = 2\pi\int_0^4 4x^2 - x^3\,dx$

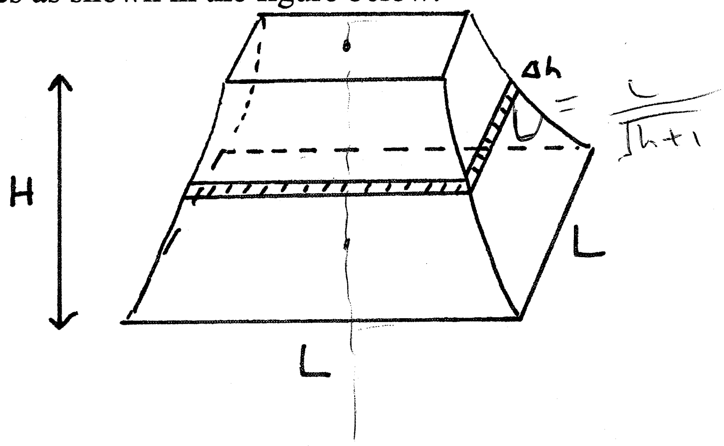
(ii) Hence find the volume of the solid.

b) The base of a solid is a circle of radius 1 unit. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of this solid. 3
(Draw any necessary diagrams).

Part c) and d) on next page

Question 4 (cont'd)

c) A stone building of height H metres has the shape of a flat-topped square "pyramid" with curved sides as shown in the figure below. 4



The cross-section at height h metres above the base is a square with sides parallel to the sides of the base and of length l where l is given by $l = \dfrac{L}{\sqrt{h+1}}$

(L is the length of the side of the square base in metres).

(i) Write an expression for the volume of a slice of width Δh at height h metres.

(ii) Hence find the volume of the building in terms of L and H.

d) Prove by Mathematical Induction that $5^n + 2 \times 11^n$ is a multiple of 3 for all positive integers, n. 4

Question 5 – (15 marks) – Start a new booklet **Marks**

a) (i) Find real numbers A, B and C such that $\dfrac{5x^2-7x+15}{(x^2+4)(x-3)} \equiv \dfrac{Ax+B}{x^2+4}+\dfrac{C}{x-3}$ 6

(ii) Hence find $\displaystyle\int_0^2 \frac{5x^2-7x+15}{(x^2+4)(x-3)}\,dx$

b) $P(x)=3x^3+7x+2$ 6

If α, β and γ are the roots of $P(x)=0$

(i) Find the value of $\alpha^3+\beta^3+\gamma^3$

(ii) Form polynomial equations with integer coefficients whose roots are

(A) $\alpha^2, \beta^2, \gamma^2$

(B) $\alpha+\beta,\ \beta+\gamma$ and $\gamma+\alpha$

c) Find the equation of the tangent to the curve $x^3+3xy-y^2=3$ at the point $(1,\ 2)$ 3

Question 6 – (15 marks) – Start a new booklet **Marks**

a) A particle of mass 5kg is acted on by a variable force whose direction is constant and whose magnitude at time t seconds is $\left(3t - 4t^2\right)g$ Newtons. 4

If the particle has an initial velocity of 2m/s in the direction of the force, find its velocity at the end of 1 second.

b) A particle of mass m kg is set in motion with speed u m/s and moves in a straight line before coming to rest. At time t seconds the particle has displacement x metres from its starting point O, velocity v ms^{-1} and acceleration a ms^{-2}. 11

The resultant force acting on the particle directly opposes its motion and has magnitude $m(1+v)$ Newtons.

(i) Show that $a = -(1+v)$

(ii) Find expressions for

(α) x in terms of v

(β) v in terms of t

and (γ) x in terms of t

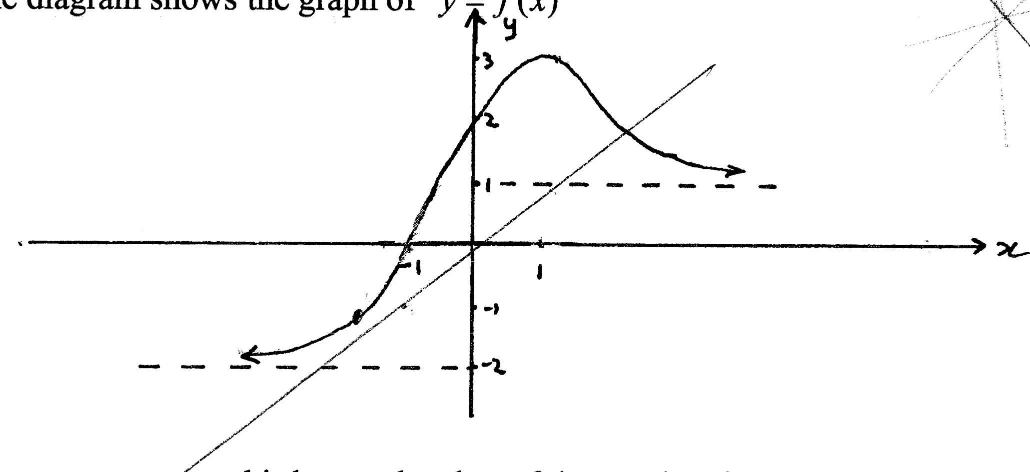
(iii) Show that $x + v + t = u$

(iv) Find the distance travelled and time taken by the particle in coming to rest.

Question 7 – (15 marks) – Start a new booklet

Marks

a) The diagram shows the graph of $y = f(x)$ 8



Draw separate one-third page sketches of the graphs of

(i) $y = f(|x|)$

(ii) $y = \dfrac{1}{f(x)}$

(iii) $y = 2^{f(x)}$

(iv) $y = x\,f(x)$

b) (i) Show that $\int_0^{\frac{\pi}{4}} (\tan x)^{2k} \sec^2 x \, dx = \dfrac{1}{2k+1}$ 7

(ii) By writing $(\sec x)^{2n}$ as $\left(1 + \tan^2 x\right)^n$ show that

$$\int_0^{\frac{\pi}{4}} (\sec x)^{2n+2} dx = \sum_{k=0}^{n} \frac{1}{2k+1} \binom{n}{k}$$

(iii) Hence or otherwise find the value of $\int_0^{\frac{\pi}{4}} (\sec x)^8 dx$

Question 8 – (15 marks) – Start a new booklet **Marks**

a) Let $(3+2x)^{20} = \sum_{r=0}^{20} a_r x^r$ 7

(i) Write an expression for a_r

(ii) Show that $\dfrac{a_{r+1}}{a_r} = \dfrac{40-2r}{3r+3}$

(iii) Hence find the greatest coefficient in the expansion of $(3+2x)^{20}$.
Give your answer in scientific notation correct to 4 significant figures.

b) A projectile is fired from ground level with an initial velocity of V m/s at an angle of elevation of α. The only force acting on the particle is gravity. Acceleration due to gravity is g m/s^2 8

(i) Derive expressions for the horizontal and vertical components of displacement from the point of projection in terms of t, where t is the time in seconds since the projectile was fired.

(ii) Derive an expression for the time of flight, given that the projectile lands at ground level.

(iii) At the instant the projectile is fired a target, which is initially b metres ahead of the point of projection, starts moving along the ground in the same horizontal direction as the projectile is moving. The target is moving at a speed of A m/s.

Show that if the projectile is to hit the target, V and α must satisfy the equation

$$V^2 \sin 2\alpha - 2AV \sin\alpha - bg = 0$$

End of Paper

Solutions To Ext.2. TRIAL HSC 2005.

Q1

a) $I = \int \frac{e^x}{4+e^{2x}}\,dx$

let $u = e^x$

$\therefore du = e^x\,dx$

$\therefore I = \int \frac{du}{4+u^2}$ (2)

$= \frac{1}{2}\tan^{-1}\frac{u}{2} + C$

$= \frac{1}{2}\tan^{-1}\left(\frac{e^x}{2}\right) + C$

b) $\int \frac{dx}{\sqrt{x^2-4x+9}} = \int \frac{dx}{\sqrt{x^2-4x+4+5}}$

$= \int \frac{dx}{\sqrt{(x-2)^2+5}}$ (2)

$= \ln\left|x-2+\sqrt{x^2-4x+9}\right| + C$

c) $\int_5^{21} \frac{x}{x+4+2\sqrt{x+4}}\,dx$ let $x = u^2-4$

$dx = 2u\,du$

when $x=5$, $u=3$

$x=21$, $u=5$

$= \int_3^5 \frac{u^2-4}{u^2+2u}\,2u\,du$

$= \int_3^5 \frac{(u+2)(u-2)\,2\cancel{u}\,du}{\cancel{u}(u+2)}$ (3)

$= \left[u^2-4u\right]_3^5$

$= 25-20-(9-12)$

$= 8$

d) $\int_0^{\frac{2\pi}{3}} \frac{dx}{13+5\sin x+12\cos x}$

let $t = \tan\frac{x}{2}$

$\therefore \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$dx = \frac{2dt}{\sec^2\frac{x}{2}}$

$= \frac{2dt}{1+\tan^2\frac{x}{2}}$

$= \frac{2dt}{1+t^2}$

when $x=0$, $t=0$

$x = \frac{2\pi}{3}$, $t = \tan\frac{\pi}{3}$

$= \sqrt{3}$

$$= \int_0^{\sqrt{3}} \frac{\frac{2dt}{1+t^2}}{13+5\times\frac{2t}{1+t^2}+\frac{12(1-t^2)}{1+t^2}}$$

$$= \int_0^{\sqrt{3}} \frac{2dt}{13+13t^2+10t+12-12t^2}$$

$$= \int_0^{\sqrt{3}} \frac{2\,dt}{t^2+10t+25}$$

$$= \int_0^{\sqrt{3}} \frac{2dt}{(t+5)^2}$$

$$= \left[-2(t+5)^{-1}\right]_0^{\sqrt{3}}$$

(4)

$$= -2\left(\frac{1}{5+\sqrt{3}}-\frac{1}{5}\right)$$

$$= \frac{2}{5}-\frac{2}{5+\sqrt{3}}$$

e) $\int e^{2x}\cos x\,dx$.

$$= \int \overset{u}{e^{2x}}\,\overset{v}{\frac{d}{dx}(\sin x)}\,dx$$

$$= e^{2x}\sin x - \int 2e^{2x}.\sin x\,dx$$

$$= e^{2x}\sin x - 2\int e^{2x}\frac{d}{dx}(-\cos x)\,dx$$

$$= e^{2x}\sin x + 2\int \overset{u}{e^{2x}}\frac{d}{dx}(\overset{v}{\cos x})\,dx$$

$$= e^{2x}\sin x + 2\left[e^{2x}\cos x - \int \cos x.2e^{2x}dx\right]$$

$$= e^{2x}\sin x + 2e^{2x}\cos x - 4\int \cos x\,e^{2x}dx.$$

$$\therefore 5\int e^{2x}\cos x\,dx = e^{2x}\sin x + 2e^{2x}\cos x + C$$

$$\therefore \int e^{2x}\cos x\,dx = \frac{1}{5}\left(e^{2x}\sin x + 2e^{2x}\cos x\right) + C$$

(4)

4

Question 2

a) (i) $z_1 = 3-2i \quad , \quad z_2 = 1-i$

$$\therefore z_1 \bar{z}_2 = (3-2i)(1+i)$$
$$= 3 + 3i - 2i + 2 \qquad (1)$$
$$= 5 + i$$

(ii) $$\frac{i}{z_1} = \frac{i}{3-2i} \times \frac{3+2i}{3+2i}$$

$$= \frac{3i-2}{9+4} \qquad (1)$$

$$= \frac{-2}{13} + i\,\frac{3}{13}$$

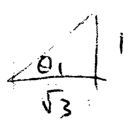
b) (i) $$\frac{w}{z} = \frac{\sqrt{3}+i}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{\sqrt{3} + \sqrt{3}i + i - 1}{1 - i^2} \qquad (1)$$

$$= \frac{\sqrt{3}-1 + i(\sqrt{3}+1)}{2}$$

$$= \frac{\sqrt{3}-1}{2} + i\,\frac{\sqrt{3}+1}{2}$$

(ii) $$\arg\left(\frac{w}{z}\right) = \arg w - \arg z$$

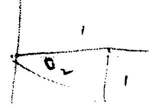


$\tan\theta_1 = \frac{1}{\sqrt{3}}$

$\therefore \theta_1 = \frac{\pi}{6}$

$$= \frac{\pi}{6} - -\frac{\pi}{4} \qquad (2)$$

$$= \frac{2\pi}{12} + \frac{3\pi}{12}$$



$\theta_2 = -\frac{\pi}{4}$

$$= \frac{5\pi}{12}$$

(iii) $\left|\frac{w}{z}\right| = \frac{|w|}{|z|}$

$= \frac{|\sqrt{3}+i|}{|1-i|}$

$= \frac{\sqrt{(\sqrt{3})^2+1^2}}{\sqrt{1^2+(-1)^2}}$ (2)

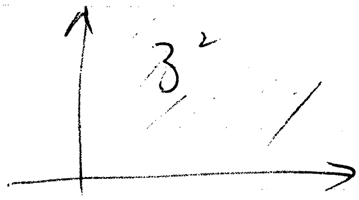
$= \frac{2}{\sqrt{2}} = \sqrt{2}$.

(iv) $\frac{w}{z} = \sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$ from (ii), (iii)

$= \frac{\sqrt{3}-1}{2} + i\,\frac{\sqrt{3}+1}{2}$ from (i)

$\therefore \sqrt{2}\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2}$ (equating real parts)

$\therefore \cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$ (1)

c) 

Let $z = a+ib$.

$\therefore z^2 = a^2-b^2+2abi$

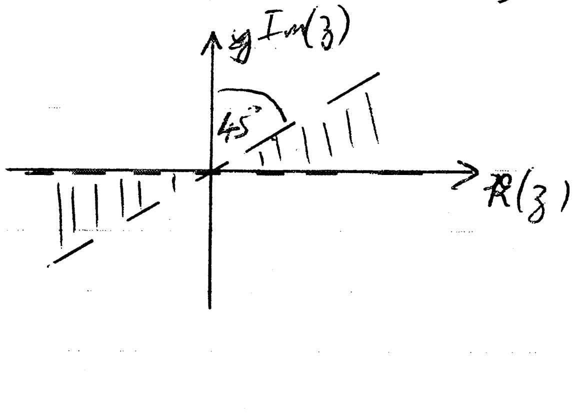
For z^2 to be in 1st quadrant,

$a^2-b^2 > 0$ and $2ab > 0$.

ie $a^2 > b^2$ and $ab > 0$.

$\therefore a > b > 0$ or $a < b < 0$.

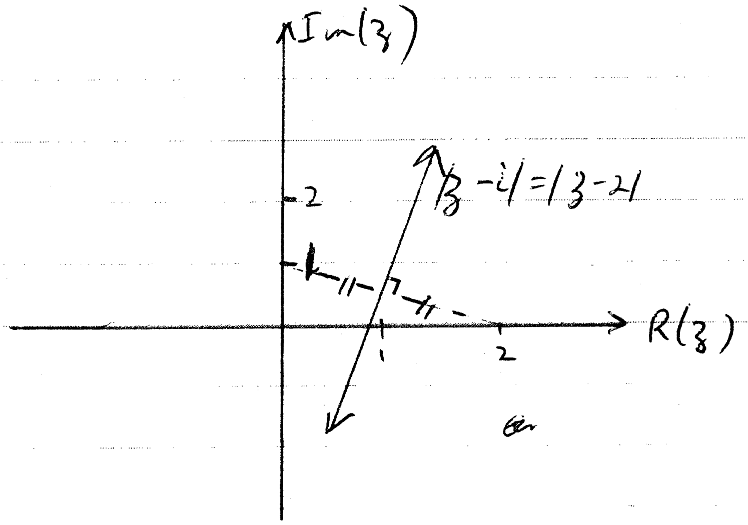
$\therefore a+ib$ (ie z) must lie in this region:



(2)

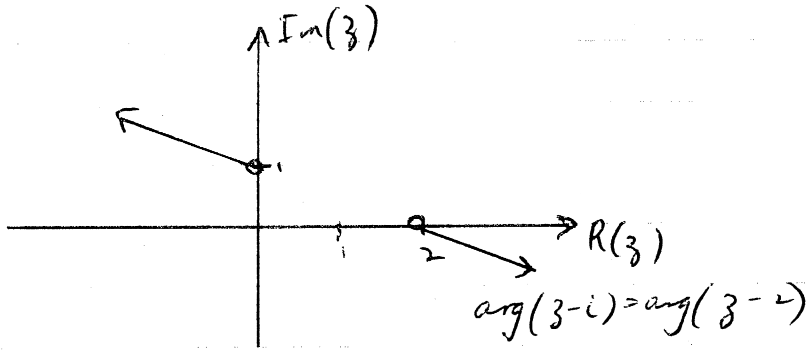
6

d) (i)



(1)

(ii)



(1)

e) (i) Since the coefficients are real, if $2+i$ is a zero, so its its conjugate $2-i$. (conjugate root theorem) (1)

(ii) $[z-(2+i)][z-(2-i)]$ is a factor of $P(z)$

i.e. $(z-2-i)(z-2+i)$ "

i.e. $(z-2)^2+1$ "

i.e. z^2-4z+5 "

$\therefore P(z) = z^4 - 4z^3 + 20z - 25$

$= (z^2-4z+5)(z^2 - 5)$ by inspection [or division of polynomials]

$= (z^2-4z+5)(z+\sqrt{5})(z-\sqrt{5})$

(z^2-4z+5, $\Delta<0$).

(over reals) (2)

Q3 CONICS

7

1 a) (i) $e = \sqrt{2}$.

(ii) $x^2 - y^2 = 8$ $\therefore a^2 = b^2 = 8$

$\therefore a = 2\sqrt{2}$.

Foci are $(\pm ae, 0)$

1

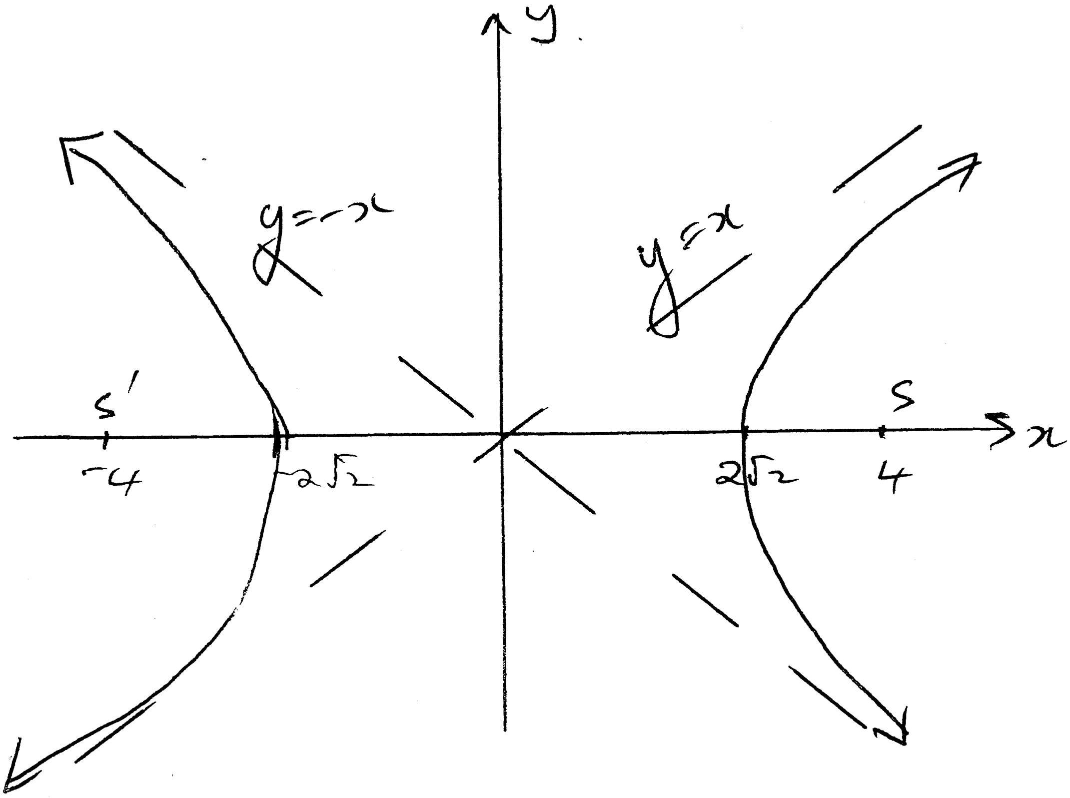
ie $(\pm 4, 0)$

1 (iii). $x = \pm \frac{a}{e}$ $\Rightarrow$ $x = \pm 2$ are the directries

1 (iv). $y = \pm \frac{bx}{a}$ $\Rightarrow$ $y = \pm x$ are the asymptotes.

1

(v).



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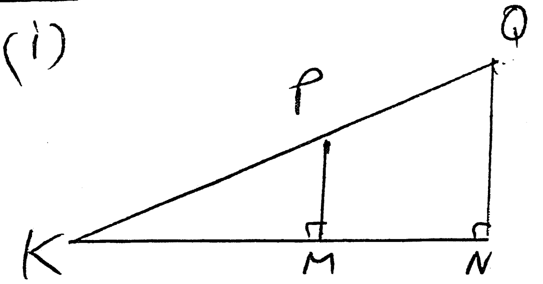
CONICS

b)(a) (i)

(1)



By similar Δ's,

$$\frac{KM}{KN} = \frac{b\sin\theta}{b\tan\theta}$$

$$= \cos\theta$$

(ii) Let K be $(k, 0)$

Using result above: $KM = a\cos\theta - k$

$KN = a\sec\theta - k$.

$$\therefore \frac{a\cos\theta - k}{a\sec\theta - k} = \cos\theta$$

$$\therefore a\cos\theta - k = (a\sec\theta - k)\cos\theta$$

$$= a - k\cos\theta$$

$$a\cos\theta - a = k(1 - \cos\theta)$$

$$a(\cos\theta - 1) = -k(\cos\theta - 1)$$

$$\therefore a = -k.$$

(2)

$\therefore$ K is $(-a, 0)$

(iii) $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$.

Now, $x = a\cos\theta$

$y = b\sin\theta$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$$

$$= \frac{b\cos\theta}{-a\sin\theta}$$

(3)

$\therefore$ equation of tangent at P given by

$$y = -\frac{b\cos\theta}{a\sin\theta} + B \quad (B \text{ const.})$$

ie $b\cos\theta\, x + a\sin\theta\, y = C$ (C const.)

Since P lies on this tangent, it satisfies this equation

i.e. $ab\cos^2\theta + ab\sin^2\theta = c$

$\therefore \quad c = ab.$

$\therefore$ tangent at P has equation

$b\cos\theta\, x + a\sin\theta\, y = ab.$

$$\therefore \quad x\frac{\cos\theta}{a} + y\frac{\sin\theta}{b} = 1 \quad *$$

Does $N\,(a\sec\theta, 0)$ satisfy $*$?

$$a\sec\theta \cdot \frac{\cos\theta}{a} + 0 = 1 \checkmark$$

$\therefore$ N lies on tangent at P.

(iv) M has coords. $(a\cos\theta, 0)$ and it has to satisfy $x\frac{\sec\theta}{a} - y\frac{\tan\theta}{b} = 1$

if it is to lie on tangent at Q.

(α) $\quad LHS = a\cos\theta\,\frac{\sec\theta}{a} - 0$

$= 1$

$= RHS. \quad \checkmark \quad \therefore$ M lies on tangent at Q

(v) Common tangent at A has equation $x = a$

Required to find the point of intersection of $x = a$ with tangent PN.

Put $x = a$ into $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

i.e. $\cos\theta + y\frac{\sin\theta}{b} = 1$

$$\therefore y = b\frac{(1-\cos\theta)}{\sin\theta}$$

$\therefore$ PN meets common tangent at

$\left(a,\ b\left(\frac{1-\cos\theta}{\sin\theta}\right)\right)$

Similarly find point of intersection of $x=a$ with tangent QM.

ie put $x=a$ into

$$\frac{x}{a}\sec\theta - \frac{y\tan\theta}{b} = 1$$

2 $$\Rightarrow \sec\theta - y\frac{\tan\theta}{b} = 1 \qquad (\times\cos\theta)$$

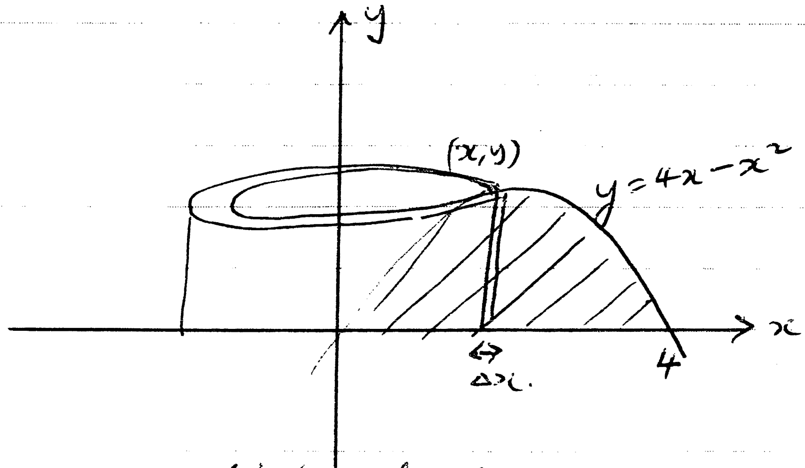
$$\Rightarrow 1 - y\frac{\sin\theta}{b} = \cos\theta$$

$$\text{ie} \quad \cancel{b}\, y = b\left(\frac{1-\cos\theta}{\sin\theta}\right).$$

ie 3 tangents concurrent at $\left(a,\ b\left(\frac{1-\cos\theta}{\sin\theta}\right)\right)$.

Question 4

a) (i)



Take a strip of thickness Δx parallel to y-axis as shown – rotate about Y-axis

$$\Delta V = \pi(R^2 - r^2)h$$
$$= \pi\left((x+\Delta x)^2 - x^2\right)y$$
$$\doteqdot 2\pi x \Delta x y$$

[$(\Delta x)^2$ is disregarded - too insignificant]

(3)

$\therefore \Delta V \doteqdot 2\pi x(4x - x^2)\Delta x$ since $y = 4x - x^2$

$$\therefore V = \lim_{\Delta x \to 0} \sum_{x=0}^{4} 2\pi(4x^2 - x^3)\Delta x$$
$$= 2\pi \int_0^4 4x^2 - x^3 \, dx$$

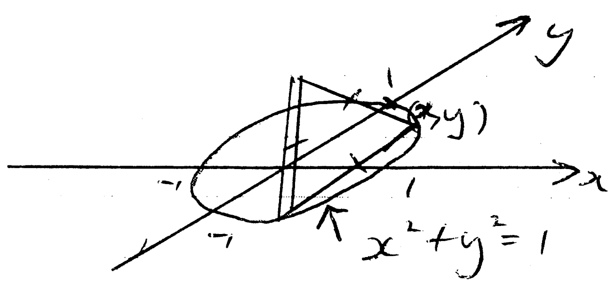
(ii) $\therefore V = 2\pi \left[\frac{4}{3}x^3 - \frac{x^4}{4}\right]_0^4$

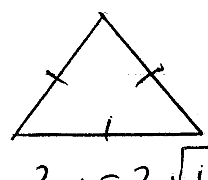
$$= 2\pi\left(\frac{4}{3} \times 64 - 64 - 0\right)$$

(1)

$$= 2\pi \times \frac{64}{3}$$
$$= \frac{128\pi}{3} \text{ units}^3.$$

b)





$2y = 2\sqrt{1-x^2}$

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(b) Area of each triangular cross-section
$= "\frac{1}{2} ab \sin C"$
$= \frac{1}{2} \cdot 2\sqrt{1-x^2} \cdot 2\sqrt{1-x^2} \cdot \sin 60°$

$= 2(1-x^2) \cdot \frac{\sqrt{3}}{2}$

$= \sqrt{3}(1-x^2)$

$\therefore \Delta V = \sqrt{3}(1-x^2)\,\Delta x$

$\therefore V = \lim_{\Delta x \to 0} \sum_{x=-1}^{1} \sqrt{3}(1-x^2)\,\Delta x$

$= \sqrt{3}\int_{-1}^{1} 1-x^2 \, dx$

$= 2\sqrt{3}\int_{0}^{1} 1-x^2 \, dx$

(b) (3)

$= 2\sqrt{3}\left[x - \frac{x^3}{3}\right]_0^1$

$= 2\sqrt{3} \times \frac{2}{3}$

$= \frac{4}{\sqrt{3}}$

$= \frac{4\sqrt{3}}{3}$ units^3.

c(i) Consider a slice of thickness Δh at height h.

Area of cross-section $= l^2$

2) $= \frac{L^2}{h+1}$ m^2

$\therefore$ Volume of slice $= \frac{L^2}{h+1}\Delta h$ m^3

(ii) $\therefore V = \lim_{\Delta h \to 0} \sum_{h=0}^{H} \frac{L^2}{h+1}\Delta h$ m^3.

$= L^2\int_0^H \frac{1}{h+1}\, dh$

$$= \left[L^2 \ln(h+1)\right]_0^H$$

(2) $$= L^2 \ln(H+1).$$

d) Step 1. Show true for $n=1$.
$5^1 + 2(11)^1 = 27$, which is a multiple of 3

Step 2. Let $n=k$ be a value for which the result is true.

ie $5^k + 2(11)^k = 3M, \quad M \in J.$

Consider now $n=k+1$.

$$\begin{aligned}
&5^{k+1} + 2(11)^{k+1} \\
&= 5 \times 5^k + 2 \times 11^k \times 11 \\
&= 5(5^k + 2 \times 11^k) + 6 \times 2 \times 11^k \\
&= 5(3M) + 3(4(11)^k) \\
&= 3\left[15 + 4 \times 11^k\right] \\
&= 3N, \quad N \in J.
\end{aligned}$$

(4)

ie if result holds for $n=k$, it also holds for $n=k+1$.

Step 3
Since result holds for $n=1$, it must also hold for $n=1+1=2$ (from step 2), and hence for $n=3$ etc
ie result true for all integral $n \geq 1$.

Question 5

a) (i) $\dfrac{5x^2-7x+15}{(x^2+4)(x-3)} \equiv \dfrac{Ax+B}{x^2+4} + \dfrac{C}{x-3}$

$\therefore 5x^2-7x+15 \equiv (Ax+B)(x-3) + C(x^2+4)$

let $x=3$

$45-21+15 = 13C$

$\therefore C=3$

let $x=0$

(3)

$15 = -3B + 4C$

$3B = 12-15$

$\therefore B=-1$

coeff. of x^2:

$5 = A+C$

$\therefore A=2$

(ii) $\displaystyle\int_0^2 \frac{5x^2-7x+15}{(x^2+4)(x-3} \, dx = \int_0^2 \frac{2x-1}{x^2+4} \, dx + \frac{3}{x-3} \, dx$

$= \left[\ln(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + 3\ln|x-3|\right]_0^2$

$= \ln 8 - \frac{1}{2}\tan^{-1}1 + 3\ln|-1|$
$\quad - \ln 4 + \frac{1}{2}\tan^{-1}0 - 3\ln|-3|$

$= \ln 8 - \frac{1}{2}\cdot\frac{\pi}{4} + 0 - \ln 4 + 0 - 3\ln 3$

(3)

$= \ln\frac{8}{4\times 27} - \frac{\pi}{8}$

$= \ln\frac{2}{27} - \frac{\pi}{8}$

b)(i) $P(x) = 3x^3 + 7x + 2$

$P(\alpha) = P(\beta) = P(\gamma) = 0$

$\therefore \quad 3\alpha^3 + 7\alpha + 2 = 0$

$3\beta^3 + 7\beta + 2 = 0$

$3\gamma^3 + 7\gamma + 2 = 0$

Adding gives

$3(\alpha^3 + \beta^3 + \gamma^3) + 7(\alpha + \beta + \gamma) + 6 = 0$

(2)

$3(\alpha^3 + \beta^3 + \gamma^3) + 7 \times 0 + 6 = 0$

$\therefore \alpha^3 + \beta^3 + \gamma^3 = -2.$

(ii) (A) We want an equation in x such that $x = \alpha^2$

ie $\alpha = \sqrt{x}$

Now, $P(\alpha) = 0$

$\therefore P(\sqrt{x}) = 0$ is required equation.

ie $3(\sqrt{x})^3 + 7\sqrt{x} + 2 = 0.$

$3x\sqrt{x} + 7\sqrt{x} + 2 = 0.$

$\sqrt{x}(3x + 7) = -2$

(2)

$x(9x^2 + 42x + 49) = 4$

$9x^3 + 42x^2 + 49x - 4 = 0.$

(B) $\alpha + \beta = \alpha + \beta + \gamma - \gamma = -\gamma$

$\beta + \gamma = \alpha + \beta + \gamma - \alpha = -\alpha.$

$\gamma + \alpha = \alpha + \beta + \gamma - \beta = -\beta.$

(2) $P(-x) = 0$ has roots $-\alpha, -\beta, -\gamma$

ie $3(-x)^3 + 7(-x) + 2 = 0$

$-3x^3 - 7x + 2 = 0.$

ie $3x^3 + 7x - 2 = 0.$

c) $x^3 + 3xy - y^2 = 3$

Differentiate wrt x gives

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$$3x^2 + 3y + 3x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0.$$

$$3x^2 + 3y = (2y - 3x)\frac{dy}{dx}.$$

3

$$\frac{dy}{dx} = \frac{3(x^2+y)}{2y-3x}.$$

at $(1, 2)$, $\frac{dy}{dx} = \frac{3(1+2)}{4-3}$

$$= 9$$

$\therefore$ eqn of tangent is

$$y - 2 = 9(x-1)$$

$$y = 9x - 7.$$

Question 6

a) $m = 5$.

$$F = m\frac{dv}{dt} = 5\frac{dv}{dt} = (3t - 4t^2)g$$

$$\therefore \frac{dv}{dt} = \frac{3t-4t^2}{5}g$$

$$v = \frac{g}{5}\int 3t - 4t^2 \, dt.$$

$$= \frac{g}{5}\left(\frac{3}{2}t^2 - \frac{4}{3}t^3\right) + c.$$

when $t = 0$, $v = 2$

$\therefore\ 2 = 0 + c$

ie $c = 2$

(4)

$$\therefore v = \frac{g}{5}\left(\frac{3}{2}t^2 - \frac{4}{3}t^3\right) + 2$$

when $t = 1$, $v = \frac{g}{5}\left(\frac{3}{2} - \frac{4}{3}\right) + 2$

$$= \frac{g}{30} + 2.$$

ie particle travelling at $\left(2 + \frac{g}{30}\right)$ m/s at end of first second.

b)

18 Question 6 (cont.)

Mechanics question.

(1) b) (i) By Newton's 2nd law, $m\ddot{x} = -m(1+v)$

$\therefore a = -(1+v)$

(ii) (α) $a = v\dfrac{dv}{dx} = -(1+v)$

$\therefore \dfrac{dv}{dx} = -\dfrac{1+v}{v}$

$\dfrac{dx}{dv} = \dfrac{-v}{1+v}$

$$\begin{array}{r} -1 \\ v+1 \overline{)\,-v} \\ \underline{-v-1} \\ 1 \end{array}$$

$= -1 + \dfrac{1}{v+1}$

$\therefore x = \int -1 + \dfrac{1}{v+1}\, dv$

$= -v + \ln(1+v) + C$

(2) at $t=0$, $x=0$, $v=u$

$\therefore 0 = -u + \ln(1+u) + C$

$\therefore C = u - \ln(1+u)$

$\therefore x = -v + \ln(1+v) + u - \ln(1+u)$

$= u - v + \ln\left(\dfrac{1+v}{1+u}\right)$

(β) $a = \dfrac{dv}{dt} = -(1+v)$

$\dfrac{dt}{dv} = -\dfrac{1}{1+v}$

$\therefore t = -\int \dfrac{dv}{1+v}$

(2)

$= -\ln(1+v) + C$

when $t=0$, $v=u$

$\therefore 0 = -\ln(1+u) + C$

$C = \ln(1+u)$

$\therefore t = \ln\left(\dfrac{1+u}{}\right)$

$$\therefore e^{t} = \frac{1+u}{1+v}$$

$$1+v = (1+u)e^{-t}$$

$$v = (1+u)e^{-t} - 1.$$

(γ) $$v = \frac{dx}{dt} = (1+u)e^{-t} - 1$$

$$\therefore x = \int (1+u)e^{-t} - 1 \, dt$$

$$= -(1+u)e^{-t} - t + K.$$

(2)

when $t=0$, $x=0$

$$\therefore 0 = -(1+u)e^{0} + K.$$

$$\therefore K = 1+u.$$

$$\therefore x = -(1+u)e^{-t} - t + 1+u$$

(iii). $x = u - v + \ln\left(\frac{1+v}{1+u}\right)$ * from (ii)(α)

from (ii)(β) $v = (1+u)e^{-t} - 1$

$$\therefore \frac{v+1}{u+1} = e^{-t}$$

(2)

$$\therefore e^{t} = \frac{u+1}{v+1}.$$

$$\therefore t = \ln\left(\frac{u+1}{v+1}\right)$$

$$\therefore -t = \ln\left(\frac{v+1}{u+1}\right).$$

$\therefore$ from *, $x = u - v - t$.

$$\therefore x + v + t = u.$$

(iv) $x = u - v + \ln\left(\frac{1+v}{1+u}\right).$

when ~~x=0~~ $v=0$, $x = u + \ln\left(\frac{1}{1+u}\right)$

$$= u - \ln(1+u)$$

ie particle travels $\{u - \ln(1+u)\}$ metres before coming to rest.

Also, $v = (1+u)e^{-t} - 1$

$\therefore$ when $v = 0$, $0 = (1+u)e^{-t} - 1$.

$\therefore \dfrac{1}{u+1} = e^{-t}$.

(2)

$u + 1 = e^{t}$

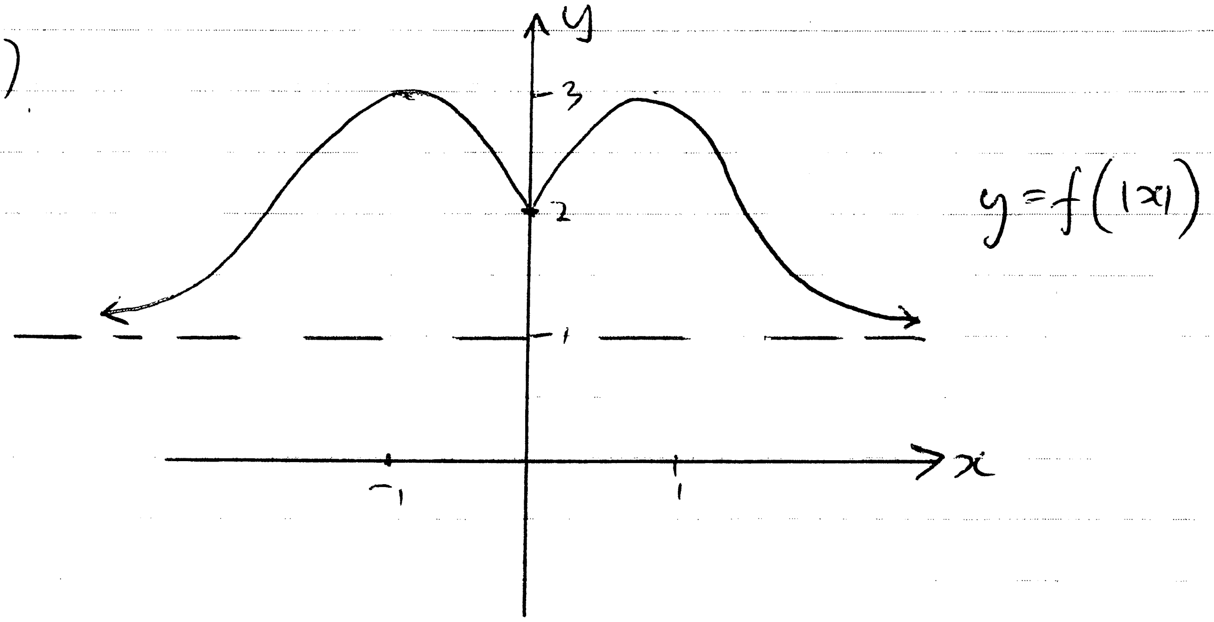
$t = \ln(1+u)$

ie particle takes $\ln(1+u)$ seconds to come to rest.

Question 7

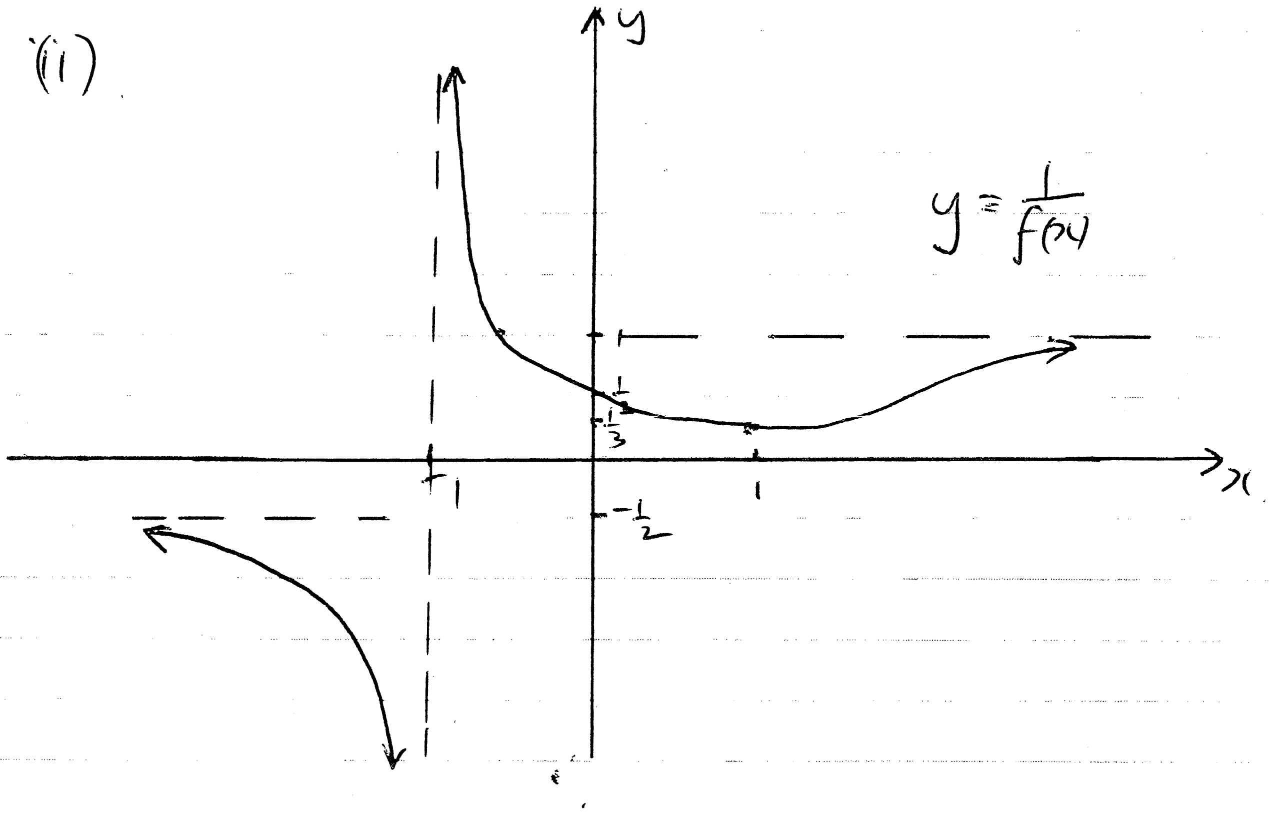
a)

(i)



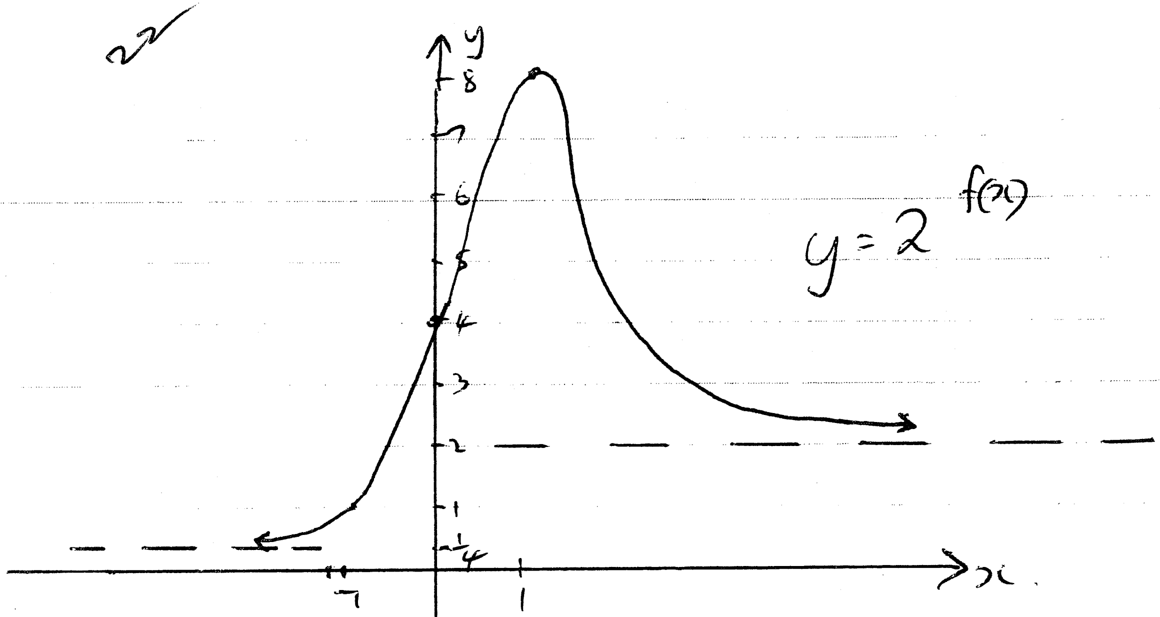
(2)

(ii)



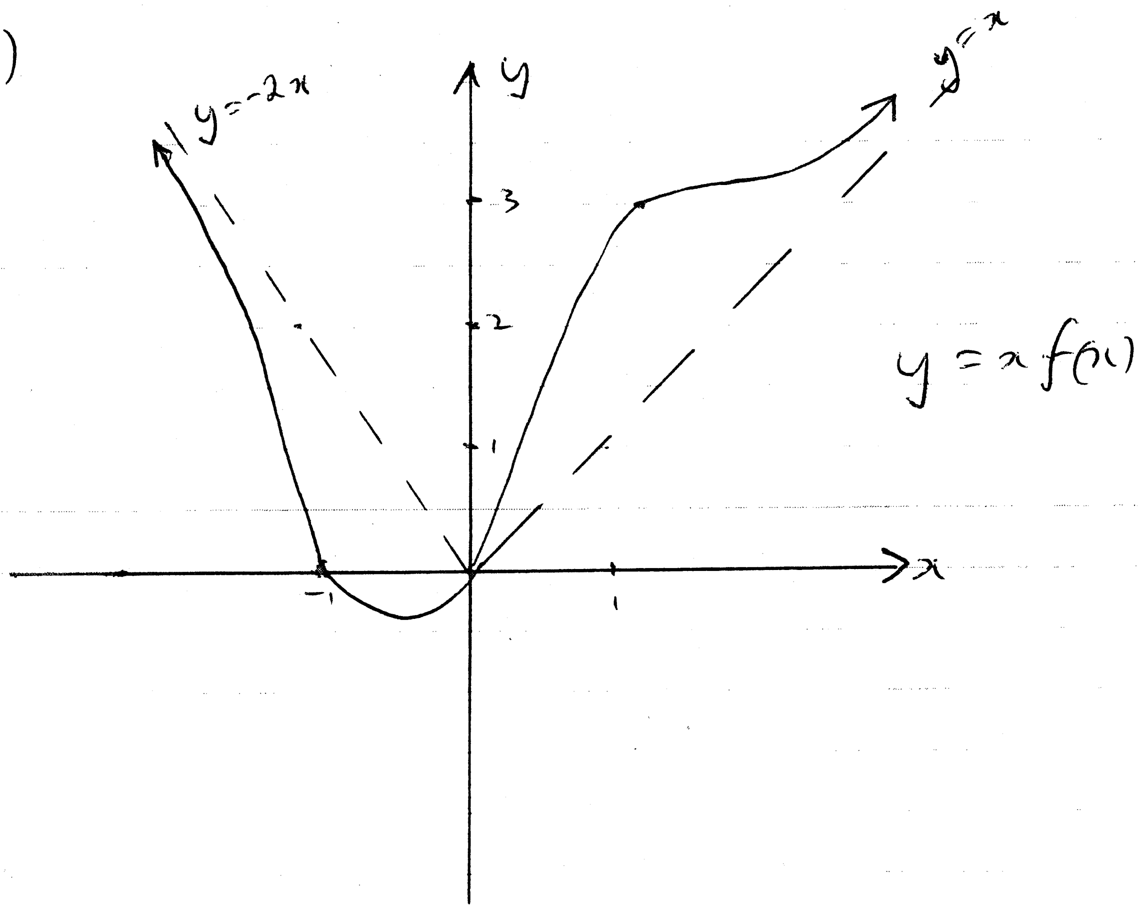
(2)

(iii)



②

(iv)



②

b) (i) $\int_0^{\frac{\pi}{4}} (\tan x)^{2k} \sec^2 x \, dx$

$= \left[\frac{[\tan x]^{2k+1}}{2k+1} \right]_0^{\frac{\pi}{4}}$

$$= \frac{\left(\tan\frac{\pi}{4}\right)^{2k+1}}{2k+1} - \frac{(\tan 0)^{2k+1}}{2k+1}$$

2 $$= \frac{1^{2k+1}}{2k+1} - 0$$

$$= \frac{1}{2k+1}$$

(ii) $$\int_0^{\frac{\pi}{4}} (\sec x)^{2n+2}\,dx$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2 x)^n \sec^2 x\,dx$$

$$= \int_0^{\frac{\pi}{4}} (1+\tan^2 x)^n \sec^2 x\,dx$$

$$= \int_0^{\frac{\pi}{4}} \left[\binom{n}{0} + \binom{n}{1}\tan^2 x + \binom{n}{2}\tan^4 x + \dots + \binom{n}{n}\tan^{2n} x\right] \times \sec^2 x\,dx$$

3 $$= \int_0^{\frac{\pi}{4}} \sum_{k=0}^{n} \binom{n}{k} \tan^{2k} x \cdot \sec^2 x\,dx$$

$$= \sum_{k=0}^{n} \binom{n}{k} \int_0^{\frac{\pi}{4}} \tan^{2k} x \sec^2 x\,dx$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2k+1} \quad \text{from (i)}$$

(iii) $$\int_0^{\frac{\pi}{4}} (\sec x)^8\,dx$$

$$= \int_0^{\frac{\pi}{4}} \cancel{\sec^6 x \sec^2} x\,dx \qquad 2n+2=8 \quad \therefore n=3$$

$$= \sum_{k=0}^{3} \binom{3}{k} \frac{1}{2k+1}$$

2 $$= \binom{3}{0}\times\frac{1}{1} + \binom{3}{1}\times\frac{1}{3} + \binom{3}{2}\times\frac{1}{5} + \binom{3}{3}\times\frac{1}{7}$$

$$= 1 + 3\times\frac{1}{3} + 3\times\frac{1}{5} + \frac{1}{7}$$

$$= 2\tfrac{1}{7} + \tfrac{3}{5} = 2\tfrac{26}{35}.$$

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Question 8

a) (i) $(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r$

$\therefore (3+2x)^{20} = \sum_{r=0}^{20} {}^{20}C_r 3^{20-r} (2x)^r$

① $= \sum_{r=0}^{20} {}^{20}C_r 3^{20-r} 2^r x^r$

$\therefore a_r = {}^{20}C_r 3^{20-r} 2^r$

(ii) $\dfrac{a_{r+1}}{a_r} = \dfrac{{}^{20}C_{r+1} 3^{20-r-1} 2^{r+1}}{{}^{20}C_r 3^{20-r} 2^r}$

$= \dfrac{\dfrac{20!}{(r+1)!(20-r-1)!}}{\dfrac{20!}{r!(20-r)!}} \times \dfrac{2}{3}$

$= \dfrac{r!(20-r)!}{(r+1)!(20-r-1)!} \times \dfrac{2}{3}$

③ $= \dfrac{20-r}{r+1} \times \dfrac{2}{3}$

OR ②,④? $= \dfrac{40-2r}{3r+3}$

(iii) Consider $\dfrac{a_{r+1}}{a_r} > 1$

ie $\dfrac{40-2r}{3r+3} > 1$

ie $40-2r > 3r+3$

$5r < 37$

$r < 7\frac{2}{5}$

$\therefore$ when $r = 1, 2 .. 7$, $a_{r+1} > a_r$.

ie $a_8 > a_7 > a_6 \ldots > a_1$

Also, when $r > 7\frac{2}{5}$, $a_{r+1} < a_r$.
ie when $r = 8, 9 \ldots$, $a_r > a_{r+1}$.

ie $a_8 > a_9 > a_{10} \ldots$

ie a_8 is greatest coefficient.

(3)

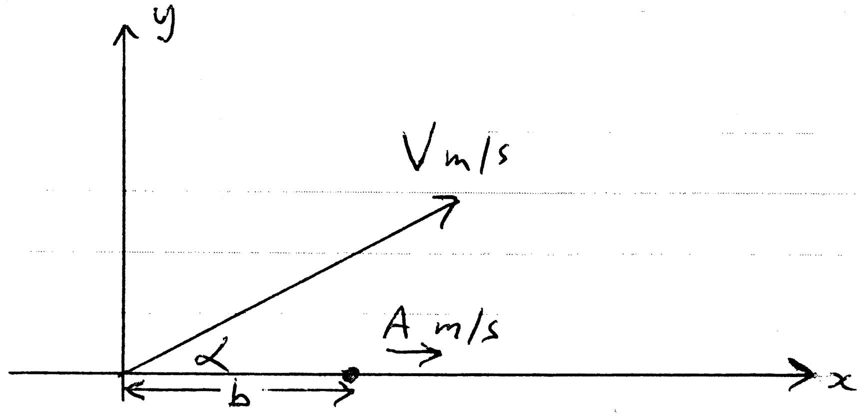
Now, $a_r = {}^{20}C_r\, 3^{20-r}\, 2^r$

$\therefore a_8 = {}^{20}C_8\, 3^{20-8} \cdot 2^8$

$= \frac{20!}{8!\,12!} \times 3^{12} \times 2^8$

$\doteq 1.714 \times 10^{13}$ (to 4 sig figs)

b)



(i) initial conditions: when $t=0$; $x=0, y=0$,
$\dot{x} = V\cos\alpha$, $\dot{y} = V\sin\alpha$, $\ddot{x} = 0$, $\ddot{y} = -g$.

Horizontally:

$\ddot{x} = 0$	$\therefore x = V\cos\alpha\, t + C_2$
$\therefore \dot{x} = C_1$	when $t=0$, $x=0$
when $t=0$, $\dot{x} = V\cos\alpha$	$\therefore C_2 = 0$
$\therefore \dot{x} = V\cos\alpha$	$\therefore x = V\cos\alpha\, t$

Vertically:

$$\ddot{y} = -g$$

$$\therefore \dot{y} = -gt + C_3$$

when $t=0$, $\dot{y} = V\sin\alpha$

$$\therefore C_3 = V\sin\alpha$$

$$\therefore \dot{y} = V\sin\alpha - gt$$

(3) $$\therefore y = V\sin\alpha\, t - \frac{gt^2}{2} + C_4$$

when $t=0$, $y=0$

$$\therefore C_4 = 0$$

$$\therefore y = V\sin\alpha\, t - \frac{gt^2}{2}$$

(ii) When particle hits ground, $y=0$.

$$\therefore t\left(V\sin\alpha - \frac{gt}{2}\right) = 0$$

ie $t = 0$ or $\dfrac{2V\sin\alpha}{g}$.

(2) ie time of flight $= \dfrac{2V\sin\alpha}{g}$.

(iii) Range of projectile $= V\cos\alpha \times$ time of flight

$$= V\cos\alpha \cdot \frac{2V\sin\alpha}{g}$$

$$= \frac{V^2\sin 2\alpha}{g}$$

The distance travelled by the target during the time of flight equal

$$A \times \frac{2V\sin\alpha}{g}$$

$\therefore$ Displacement of target from the origin is

$$b + A \times \frac{2V\sin\alpha}{g}$$

This must equal the range of the projectile.

(3) i.e. $$\frac{V^2 \sin 2\alpha}{g} = b + \frac{2AV\sin\alpha}{g}.$$

i.e. $$V^2 \sin 2\alpha - 2AV\sin\alpha - bg = 0.$$

QED.