St George Girls High School

Trial Higher School Certificate Examination

2007

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Write your Student Number on every page
- All questions may be attempted.
- Begin each question in a new booklet.
- All necessary working must be shown.
- Marks may be deducted for careless or poorly presented work.
- Board-approved calculators may be used.
- A list of standard integrals is included at the end of this paper.
- The mark allocated for each question is listed at the side of the question.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 – (15 marks) – Start a new booklet **Marks**

a) Evaluate $|3-2i|$ 1

b) Given that $z=x+yi$ where x and y are real numbers, express the following in the form $a+bi$, where a and b are real

(i) $\overline{(3+i)}\,z$ (ii) $\dfrac{z}{5-12i}$ 1, 1

c) On an Argand diagram shade the region containing all points representing the complex number z such that:

$$-\frac{\pi}{6}\le \arg z<\frac{\pi}{3} \quad \text{and} \quad |z|\le 2$$ 3

d) Sketch on separate diagrams the locus specified by:

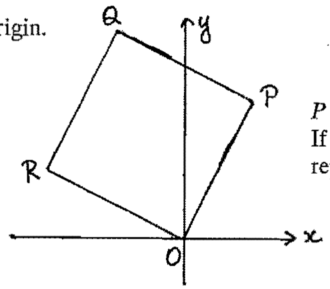
(i) $\arg\left(z-(1+i)\right)=\dfrac{\pi}{6}$ 1

(ii) $\arg(z-4)=\arg(z-2i)$ 2

(iii) $\arg\left(\dfrac{z-2}{z+2}\right)=\dfrac{\pi}{2}$

e) (i) Show that multiplication of a complex number, z, by i can be represented by a rotation of the vector representing z through an angle of $\dfrac{\pi}{2}$ radians about the origin. 2

(ii)



P represents the complex number $2+4i$.
If $OPQR$ is a square, find the complex numbers represented by Q and R. 2

Question 2 – (15 marks) – Start a new booklet **Marks**

a) α, β and γ are the roots of $P(x)=0$ where $P(x)=3x^3+7x^2+9x+1$

Find:

(i) $\alpha^2+\beta^2+\gamma^2$ 2

(ii) $\alpha^3+\beta^3+\gamma^3$ 2

b) Find the value of $\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos\theta}\,d\theta$ 4

c) (i) Find constants A, B, C such that

$$\frac{3x^2+2x+1}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$ 2

(ii) Hence, or otherwise, find $\int \frac{3x^2+2x+1}{(x+1)(x^2+1)}\,dx$ 2

d) Find $\int e^x \sin x\, dx$ 3

Question 3 – (15 marks) – Start a new booklet **Marks**

a) The area represented by the circle $x^2+y^2=a^2$ is rotated about the line $x=a$ $(a>0)$ to form a solid. Use slices to find the volume of the solid. 3

b) When polynomial $P(x)$ is divided by $(x-4)$ the remainder is 3 and when $P(x)$ is divided by $x-3$ the remainder is 1. What is the remainder when $P(x)$ is divided by $x^2-7x+12$? 3

c) The hyperbola, H, has equation $xy=9$

(i) $P\left(3p,\frac{3}{p}\right)$ and $Q\left(3q,\frac{3}{q}\right)$, where $p>0$ and $q>0$, are 2 distinct points on H. Show that the equation of chord PQ is $x+pqy=3(p+q)$. 2

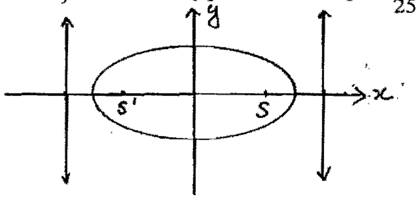
(ii) Show that the equation of the tangent at P is $x+p^2y=6p$. 2

(iii) Find the coordinates of the point of intersection, T, of the tangents at P and Q. 2

(iv) PQ always passes through the point $(0, 9)$. Find the equation of the locus of T. 3

Question 4 – (15 marks) – Start a new booklet **Marks**

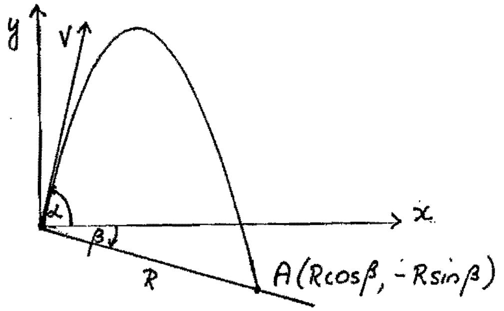
a) $P(5\sin\theta, 3\cos\theta)$ is an arbitrary point on the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ 6



(i) Copy the diagram into your answer booklet, giving the coordinates of the foci, S and S^1 and the equations of the directrices.

(ii) Show that $PS+PS^1$ is independent of the position of P.

b)



A ball is thrown from O with velocity V at acute angle α to the horizontal. It lands at $A(R\cos\beta, -R\sin\beta)$ on a slope inclined at acute angle β to the horizontal as shown above.
It is given that $OA=R$.
The position of the ball at time t is given by

$$x = Vt\cos\alpha$$
$$y = Vt\sin\alpha - \frac{1}{2}gt^2$$

(i) Show that the time taken to reach A is $\frac{R\cos\beta}{V\cos\alpha}$ 1

(ii) Show that $R=\frac{2V^2\cos^2\alpha}{g\cos^2\beta}(\tan\alpha\cos\beta+\sin\beta)$ 2

(iii) If $V=10\sqrt{g}$ and $\alpha=\beta$

(a) find R as a function of α 1

(b) with careful explanation, find the maximum value of R if $0<\alpha\le\frac{\pi}{4}$ 2

c) The area bounded by the curves $y=(x-1)^2$ and $y=x+1$ is rotated about the y-axis to form a solid. Use cylindrical shells to find the volume of this solid. 3

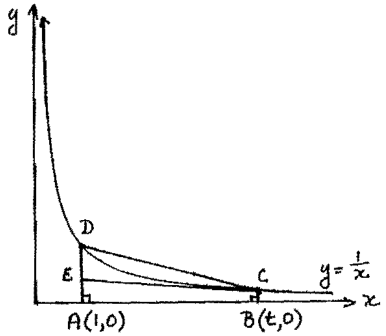
Question 5 – (15 marks) – Start a new booklet **Marks**

a) $z-ki$ (k is real) is a factor of $P(z)=z^4-z^3+9z^2-4z+20$

(i) Find possible values of k 3

(ii) Hence, or otherwise, solve $P(z)=0$ over the complex numbers. 3

b)



In the above sketch of $y=\frac{1}{x}$

- CE is the tangent to $y=\frac{1}{x}$ at C
- C and D lie on the curve $y=\frac{1}{x}$

(i) Prove that E is the point $\left(1, \frac{2t-1}{t^2}\right)$ 2

(ii) By considering the areas of the trapezia $ABCE$ and $ABCD$ show that 2

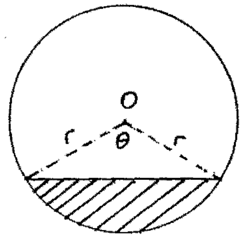
$$\frac{(t-1)(3t-1)}{2t^2}<\int_1^t \frac{1}{x}\,dx<\frac{t^2-1}{2t}$$

(iii) Hence show that $\frac{5}{8}<\ln 2<\frac{3}{4}$ 1

Part c) on next page

Question 5 (cont'd) **Marks**

c)



The area of the shaded segment is one third of the area of the circle.

(i) Show that $3\theta - 2\pi = 3\sin\theta$ 1

(ii) Carefully explain why this equation has a solution $\theta = \alpha$ where $\frac{\pi}{2} < \alpha < \pi$ 1

(iii) With $\alpha = \frac{3\pi}{4}$ as a first approximation, use one application of Newton's method to find a better approximation correct to 1 decimal place. 2

Question 6 – (15 marks) – Start a new booklet **Marks**

a) A body of mass one kilogram is projected vertically upwards from the ground at a speed of 20 metres per second. The particle is under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$, where v is the magnitude of the particle's velocity at that time.

In the following questions take the acceleration due to gravity to be 10 metres per second per second.

(i) While the body is travelling upwards the equation of motion is

$$\ddot{x} = -\left(10 + \frac{1}{40}v^2\right)$$

(a) Taking $\ddot{x} = v\frac{dv}{dx}$, show that the greatest height reached by the particle is $20\log 2$ metres 2

(b) Taking $\ddot{x} = \frac{dv}{dt}$, calculate the time taken to reach this greatest height. 2

(ii) Having reached its greatest height the particle falls to its starting point. The particle is still under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$

(a) Write down the equation of motion of the particle as it falls. 1

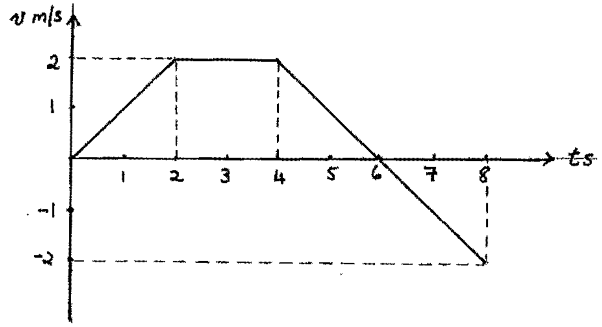
(b) Find the speed of the particle when it returns to its starting point. 3

(c) Express this speed as a percentage of the terminal velocity correct to 3 significant figures. 2

Part b) on next page

Question 6 (cont'd) **Marks**

b)



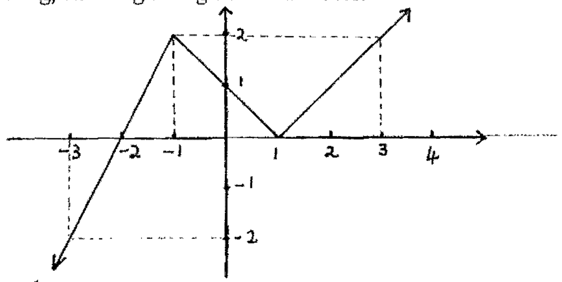
A particle of mass 2kg starts from rest at the origin and moves along the x-axis such that its velocity, v m/s at time t s where $0 \le t \le 8$ is represented by the above graph.

(i) Where is the particle at $t = 8$ s? 2

(ii) For the time period $4 < t \le 8$, find the magnitude and direction of the resultant force on the particle and clearly describe its effect on the particle. 3

Question 7 – (15 marks) – Start a new booklet **Marks**

a) The graph of $y = f(x)$ is shown below. On separate diagrams draw neat sketches of the following, showing all significant features.



(i) $y = \dfrac{1}{f(x)}$ 2

(ii) $y = [f(x)]^2$ 2

(iii) $y = \log_e |f(x)|$ 2

(iv) $y = \tan^{-1} f(x)$ 2

b) $I_n = \int_0^1 \dfrac{x^n}{1+x^2}\, dx$ for $n = 0, 1, 2, 3, \ldots$

(i) Show that $I_n + I_{n+2} = \dfrac{1}{n+1}$ 3

(ii) Find I_0 and I_1 2

(iii) Hence find I_6 2

Question 8 – (15 marks) – Start a new booklet **Marks**

a) Consider the function $f(x)=\dfrac{e^x-1}{e^x+1}$

(i) Show that $f(x)$ is always increasing 2

(ii) Find $f'(0)$ 1

(iii) Sketch $y=f(x)$ showing any asymptotes 2

(iv) Using your graph, or otherwise, find the values of m for which $\dfrac{e^x-1}{e^x+1}=mx$ has 3 real solutions 1

b) A drinking glass having the form of a right circular cylinder of radius a and height h, is filled with water. The glass is slowly tilted over, spilling water out of it, until it reaches the position where the water's surface bisects the base of the glass.

Figure 1 shows this position.

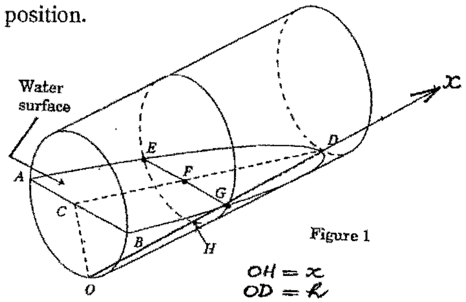


Figure 1

In Figure 1, AB is a diameter of the circular base with centre C, O is the lowest point on the base, and D is the point where the water's surface touches the rim of the glass.

Figure 2 shows a cross-section of the tilted glass parallel to its base. The centre of this circular section is C' and EFG shows the water level. The section cuts the lines CD and OD of Figure 1 in F and H respectively.

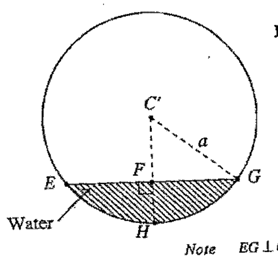


Figure 2

Note $EG \perp C'H$ at F

Question 8 (cont'd) **Marks**

Figure 3 shows the section COD of the tilted glass.

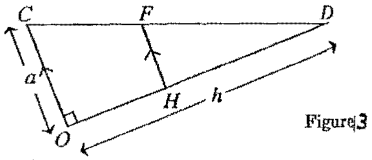


Figure 3

Note $FH \parallel CO$, $CO=a$, and $OD=h$

(i) Use Figure 3 to show that $FH=\dfrac{a}{h}(h-x)$, where $OH=x$ 1

(ii) Use Figure 2 to show that $C'F=\dfrac{ax}{h}$ and $\angle HC'G=\cos^{-1}\left(\dfrac{x}{h}\right)$ 2

(iii) Use (ii) to show that the area of the shaded segment EGH is 2

$$a^2\left[\cos^{-1}\left(\frac{x}{h}\right)-\left(\frac{x}{h}\right)\sqrt{1-\left(\frac{x}{h}\right)^2}\right]$$

(iv) Given that $\int \cos^{-1}\theta\, d\theta=\theta\cos^{-1}\theta-\sqrt{1-\theta^2}$, find the volume of water in the tilted glass of Figure 1. 4

End of Paper

ST GEORGE GIRLS HSC

EXTENSION 2 TRIAL HSC 2007

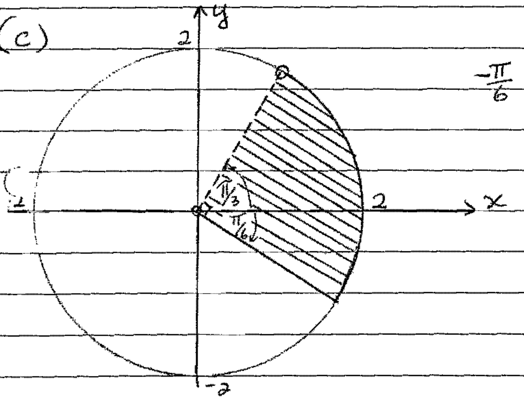
Question 1

(a) $|3-2i| = \sqrt{3^2+(-2)^2}$
$= \sqrt{13}$

(b) (i) $\overline{(3+i)}\,z = (3-i)(x+yi)$
$= 3x+y+(3y-x)i$

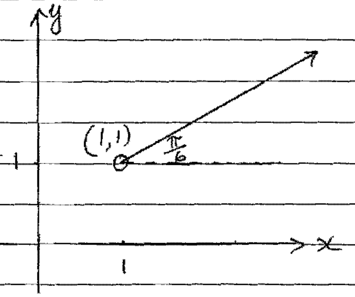
(ii) $\dfrac{z}{5-12i} = \dfrac{(x+yi)}{(5-12i)} \times \dfrac{(5+12i)}{(5+12i)}$
$= \dfrac{5x-12y+i(12x+5y)}{5^2+12^2}$
$= \dfrac{5x-12y}{169} + \dfrac{12x+5y}{169}i$

(c)



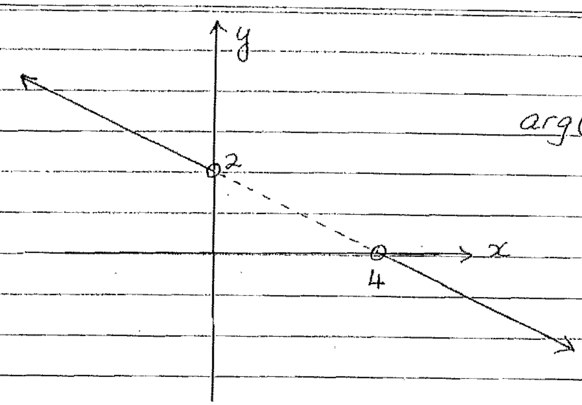
$-\frac{\pi}{6} \le z < \frac{\pi}{3}$ and $|z| \le 2$

(d) (i)



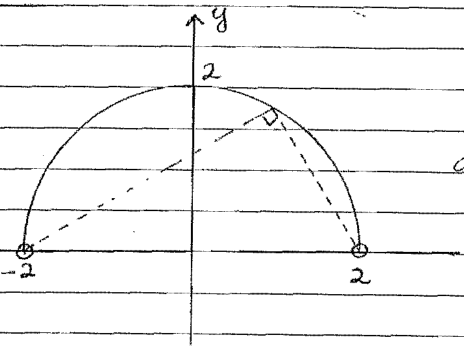
$\arg(z-(1+i)) = \frac{\pi}{6}$

(d) (ii)



$\arg(z-4) = \arg(z-2i)$

(iii)



$\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{2}$

$\arg(z-2) - \arg(z+2) = \frac{\pi}{2}$

e) (i) Let OP, OQ be the position vectors representing z and iz on the Argand diagram

$OP = |z| \quad OQ = |iz| = |i||z| = 1.|z| = OP$

$\arg(iz) = \arg i + \arg z = \frac{\pi}{2} + \arg z$

ie OQ makes an angle of $\frac{\pi}{2}$ with OP

ie The transformation $z \to iz$ is a rotation about O through $\frac{\pi}{2}$ radians (in the anticlockwise direction)

(ii) R represents $i(2+4i) = -4+2i$

Q represents $(2+4i)+(-4+2i) = -2+6i$

Question 2

(a) $P(x) = 3x^3 + 7x^2 + 9x + 1$

(i) $\alpha + \beta + \gamma = -\frac{7}{3}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{9}{3} = 3$

$$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$$
$$= \left(-\tfrac{7}{3}\right)^2 - 2\times 3$$
$$= \frac{49}{9} - 6$$
$$= -\frac{5}{9}$$

(ii) $\left.\begin{array}{l} 3\alpha^3 + 7\alpha^2 + 9\alpha + 1 = 0 \\ 3\beta^3 + 7\beta^2 + 9\beta + 1 = 0 \\ 3\gamma^3 + 7\gamma^2 + 9\gamma + 1 = 0 \end{array}\right\}$ since α, β, γ are roots of $P(x) = 0$

$$\therefore 3(\alpha^3+\beta^3+\gamma^3) + 7(\alpha^2+\beta^2+\gamma^2) + 9(\alpha+\beta+\gamma) + 3 = 0$$
$$3(\alpha^3+\beta^3+\gamma^3) + 7\times -\frac{5}{9} + 9\times -\frac{7}{3} + 3 = 0$$
$$3(\alpha^3+\beta^3+\gamma^3) = \frac{35}{9} + 21 - 3$$
$$\alpha^3+\beta^3+\gamma^3 = 7\tfrac{8}{27} \quad \left(= \frac{197}{27}\right)$$

(b) $\displaystyle\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos\theta}\,d\theta$

$t = \tan\frac{\theta}{2}$

$\theta = 2\tan^{-1}t$

$d\theta = \frac{2}{1+t^2}\,dt$

When $\theta = 0$ $t = 0$

$\theta = \frac{2\pi}{3}$ $t = \sqrt{3}$

$$= \int_0^{\sqrt{3}} \frac{1}{5+4\frac{(1-t^2)}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$
$$= \int_0^{\sqrt{3}} \frac{2}{5+5t^2+4-4t^2}\,dt$$
$$= \int_0^{\sqrt{3}} \frac{2}{9+t^2}\,dt$$

2 (b) (cont) $\displaystyle\int_0^{\sqrt{3}} \frac{2}{9+t^2}\,dt = \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^{\sqrt{3}}$

$$= \frac{2}{3}\left(\tan^{-1}\frac{\sqrt{3}}{3} - \tan^{-1}0\right)$$
$$= \frac{2}{3}\left(\frac{\pi}{6} - 0\right)$$
$$= \frac{\pi}{9}$$

(c) $\dfrac{3x^2+2x+1}{(x+1)(x^2+1)} \equiv \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$

$$3x^2+2x+1 \equiv A(x^2+1) + (Bx+C)(x+1)$$

$x=-1$ $\quad 3-2+1 = 2A$

$A = 1$

Coeff x^2: $\quad 3 = A + B$

$B = 2$

$x = 0$ $\quad 1 = A + C$

$C = 0$

(ii) $\displaystyle\int \frac{3x^2+2x+1}{(x+1)(x^2+1)}\,dx = \int \frac{1}{x+1} + \frac{2x}{x^2+1}\,dx$

$$= \ln|x+1| + \ln(x^2+1) + c$$

(d) $\int e^x \sin x \, dx = \int \frac{d(e^x)}{dx} \sin x \, dx$

$$= e^x \sin x - \int e^x \cos x \, dx$$

$$= e^x \sin x - \int \frac{d(e^x)}{dx} \cos x \, dx$$

$$= e^x \sin x - \left\{ e^x \cos x - \int e^x(-\sin x)\, dx \right\}$$

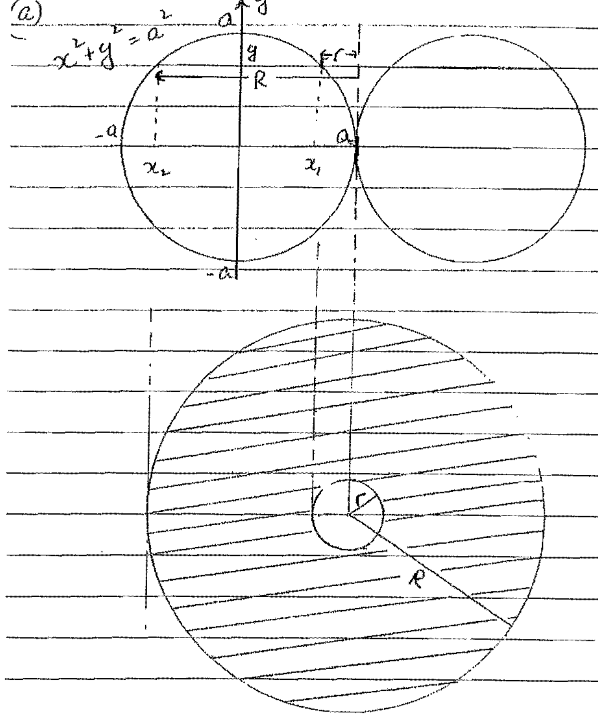
$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$2 \int e^x \sin x \, dx = e^x(\sin x - \cos x) + c$$

$$\int e^x \sin x \, dx = \tfrac{1}{2} e^x (\sin x - \cos x) + C$$

Question 3

(a)



$x_1 = \sqrt{a^2 - y^2}$

$x_2 = -\sqrt{a^2-y^2}$

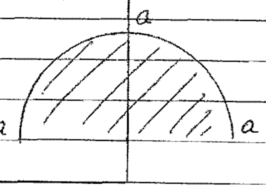
$R = a + \sqrt{a^2-y^2}$

$r = a - \sqrt{a^2-y^2}$

$$A(y) = \pi R^2 - \pi r^2$$
$$= \pi(R-r)(R+r)$$
$$= \pi\left(2\sqrt{a^2-y^2}\right)(2a)$$
$$= 4a\pi\sqrt{a^2-y^2}$$

$\delta V \doteqdot A(y)\,\delta y$

$V = \lim_{\delta y \to 0} \sum_{y=-a}^{a} 4a\pi\sqrt{a^2-y^2}\,\delta y$



$$V = \int_{-a}^{a} A(y)\, dy$$
$$= 4a\pi \int_{-a}^{a} \sqrt{a^2-y^2}\, dy$$
$$= 4a\pi \times \frac{1}{2} \cdot \pi a^2$$
$$= 2a^3\pi^2$$

(b) $P(4) = 3 \qquad P(3) = 1$

Let $P(x) = (x^2 - 7x + 12)Q(x) + ax + b$
$= (x-4)(x-3)Q(x) + ax + b$

$P(4) = 4a + b = 3$ ①
$P(3) = 3a + b = 1$ ②

① − ② $a = 2$
$\therefore 6 + b = 1$
$b = -5$

$\therefore$ Remainder is $2x - 5$

(c) $H: \ xy = 9 \qquad P\left(3p, \frac{3}{p}\right) \quad Q\left(3q, \frac{3}{q}\right)$

(i) Grad $PQ = \dfrac{\frac{3}{p} - \frac{3}{q}}{3p - 3q}$
$= \dfrac{3(q-p)}{pq} \times \dfrac{1}{3(p-q)}$
$= -\dfrac{1}{pq}$

$\therefore$ Eqⁿ of PQ is $y - \frac{3}{p} = -\frac{1}{pq}(x - 3p)$

$pqy - 3q = -x + 3p$

$x + pqy = 3p + 3q$
$= 3(p+q)$

(ii) $y = \frac{9}{x}$
$\frac{dy}{dx} = -9x^{-2}$

At P $\frac{dy}{dx} = -\frac{9}{9p^2} = \frac{-1}{p^2}$

$\therefore$ Eqⁿ of tangent at P is

$y - \frac{3}{p} = -\frac{1}{p^2}(x - 3p)$

$p^2y - 3p = -x + 3p$

$x + p^2y = 6p$

(iii) Tangents at P and Q:
$x + p^2y = 6p$ — ①
$x + q^2y = 6q$ — ②
Meet at T:
① − ② $(p^2 - q^2)y = 6(p-q)$
$(p-q)(p+q)y = 6(p-q)$
$y = \dfrac{6}{p+q} \qquad (p \neq q)$

$x = 6p - p^2 \times \dfrac{6}{p+q}$
$= \dfrac{6p^2 + 6pq - 6p^2}{p+q}$
$= \dfrac{6pq}{p+q}$

$\therefore$ T has coordinates $\left(\dfrac{6pq}{p+q}, \dfrac{6}{p+q}\right)$

(iv) Since PQ passes through $(0, 9)$

$$0 + 9pq = 3(p+q)$$
$$3pq = p+q$$

For T: $x = \dfrac{6pq}{p+q}$ $\qquad y = \dfrac{6}{p+q}$

$$= \frac{2.3pq}{p+q}$$
$$= \frac{2(p+q)}{p+q}$$
$$= 2$$

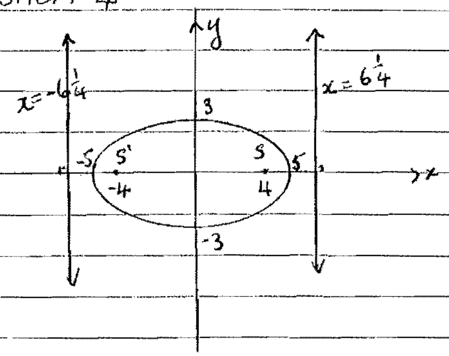
Since $p > 0$ and $q > 0$ $p+q > 0$ and so $y > 0$

Tangents intersect below the curve : When $x = 2$ $y = \frac{9}{2}$

$\therefore$ Locus of T is $x = 2$ $\quad 0 < y < \frac{9}{2}$

Question 4

(a)



$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

$$a^2 = 25$$
$$b^2 = 9$$
$$= a^2(1-e^2)$$
$$9 = 25(1-e^2)$$
$$e^2 = 1 - \frac{9}{25}$$
$$= \frac{16}{25}$$
$$e = \frac{4}{5} \quad (e > 0)$$

$$ae = 5 \times \frac{4}{5} = 4$$
$$\frac{a}{e} = \frac{5}{4/5} = \frac{25}{4}$$

$S(4,0) \quad S'(-4,0)$

Directrices $x = \pm \frac{25}{4}$

(ii) Let M, M' be feet of perpendiculars from P to the directrices

$$PS = e\,PM = \frac{4}{5} PM$$
$$PS' = e\,PM' = \frac{4}{5} PM'$$

$$PS + PS' = \frac{4}{5}(PM + PM')$$
$$= \frac{4}{5} \times MM'$$
$$= \frac{4}{5} \times 2 \times \frac{25}{4}$$
$$= 10$$

ie $PS + PS'$ is independent of P.

(b) $x = Vt\cos\alpha$

$y = Vt\sin\alpha - \frac{1}{2}gt^2$

(i) At A $\quad x = R\cos\beta$

$\therefore R\cos\beta = Vt\cos\alpha$

$\therefore t = \dfrac{R\cos\beta}{V\cos\alpha}$

(ii) $$-R\sin\beta = V.\frac{R\cos\beta}{V\cos\alpha}.\sin\alpha - \frac{1}{2}g.\frac{R^2\cos^2\beta}{V^2\cos^2\alpha}$$

$$= R\tan\alpha\cos\beta - \frac{gR^2\cos^2\beta}{2V^2\cos^2\alpha} \qquad (\div R$$

$$\frac{Rg\cos^2\beta}{2V^2\cos^2\alpha} = \tan\alpha\cos\beta + \sin\beta$$

$$R = \frac{2V^2\cos^2\alpha}{g.\cos^2\beta}(\tan\alpha\cos\beta + \sin\beta)$$

(iii) If $V = 10\sqrt{g}$ $\quad \alpha = \beta$

(a) $$R = \frac{2\times 100g\cos^2\alpha}{g\cos^2\alpha}(\tan\alpha\cos\alpha + \sin\alpha)$$

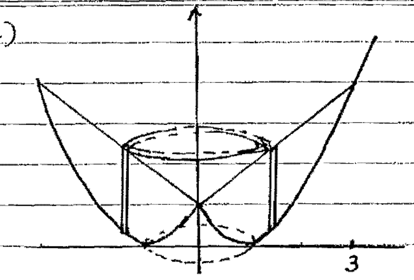
$$= 200(\sin\alpha + \sin\alpha)$$

$$= 400\sin\alpha$$

(b) Since $\sin\alpha$ is an increasing function for $0 < \alpha \le \frac{\pi}{4}$, maximum value of R for this domain occurs when $\alpha = \frac{\pi}{4}$

$\therefore$ Max $R = 400\sin\frac{\pi}{4} = 400\times\frac{1}{\sqrt{2}} = 200\sqrt{2}$

(c)



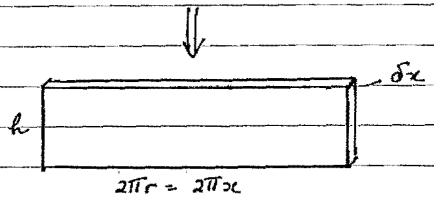
$y = x+1$ meets $y = (x-1)^2$

$x^2 - 2x + 1 = x + 1$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0, 3$



$r = x$

$h = (x+1) - (x-1)^2$

$= 3x - x^2$

$$\delta V \doteqdot 2\pi r h\,\delta x$$

$$= 2\pi x(3x - x^2)\delta x$$

$$V = \lim_{\delta x\to 0}\sum_{x=0}^{3} 2\pi x(3x - x^2)\delta x$$

$$= \int_0^3 2\pi x(3x - x^2)\,dx$$

$$= 2\pi\int_0^3 (3x^2 - x^3)\,dx$$

$$= 2\pi\left[x^3 - \frac{x^4}{4}\right]_0^3$$

$$= 2\pi\left\{\left(27 - \frac{81}{4}\right) - 0\right\}$$

$$= \frac{27\pi}{2}$$

Question 5

(a) $P(z) = z^4 - z^3 + 9z^2 - 4z + 20$

(i) $P(ki) = 0$

$(ki)^4 - (ki)^3 + 9(ki)^2 - 4(ki) + 20 = 0$

$k^4 - k^3(-i) + 9(-k^2) - 4ki + 20 = 0$

Equating real parts gives

$k^4 - 9k^2 + 20 = 0$

$(k^2-4)(k^2-5) = 0$

$(k-2)(k+2)(k-\sqrt{5})(k+\sqrt{5}) = 0$

$k = 2, -2, \sqrt{5}, -\sqrt{5}$

Equating imaginary parts gives

$k^3 - 4k = 0$

$k(k^2-4) = 0$

$k = 0, 2, -2$

$\therefore \{2, -2, \sqrt{5}, -\sqrt{5}\} \cap \{0, 2, -2\} = \{2, -2\}$

$\therefore k = 2, -2$

(ii) $\therefore (z-2i)(z+2i) \mid P(z)$

$\therefore z^4 - z^3 + 9z^2 - 4z + 20 = (z^2+4)(z^2 - z + 5)$

$\therefore$ If $P(z) = 0$

$z = \pm 2i, \dfrac{1 \pm \sqrt{1-4\times1\times5}}{2}$

$= \pm 2i, \dfrac{1 \pm i\sqrt{19}}{2}$

(b) (i) $y = \dfrac{1}{x}$

$\dfrac{dy}{dx} = -\dfrac{1}{x^2}$

At $C(t, \frac{1}{t})$ $\dfrac{dy}{dx} = -\dfrac{1}{t^2}$

Eq^n of tangent at C is

$y - \dfrac{1}{t} = -\dfrac{1}{t^2}(x-t)$

$t^2y - t = -x + t$

$t^2y = -x + 2t$

When $x = 1$ $t^2y = -1 + 2t$

$y = \dfrac{2t-1}{t^2}$

$\therefore$ E has coords $\left(1, \dfrac{2t-1}{t^2}\right)$

(ii) Area ABCE $= \dfrac{(t-1)}{2} \times \left(\dfrac{2t-1}{t^2} + \dfrac{1}{t}\right) = \left[\dfrac{AB}{2}(AE+BC)\right]$

$= \dfrac{(t-1)}{2} \times \dfrac{(2t-1+t)}{t^2}$

$= \dfrac{(t-1)(3t-1)}{2t^2}$

Area ABCD $= \dfrac{(t-1)}{2}\left(1 + \dfrac{1}{t}\right) = \left[\dfrac{AB}{2}(AD+BC)\right]$

$= \dfrac{(t-1)}{2}\dfrac{(t+1)}{t}$

$= \dfrac{t^2-1}{2t}$

Area under curve from D to C $= \displaystyle\int_1^t \frac{1}{x}\,dx$

Area of ABCE $<$ area under curve $<$ area of ABCD

$$\frac{(t-1)(3t-1)}{2t^2} < \int_1^t \frac{1}{x}\,dx < \frac{t^2-1}{2t}$$

(iii) Let $t = 2$

$$\frac{(2-1)(3\times 2-1)}{2\times 2^2} < \int_1^2 \frac{1}{x}\,dx < \frac{2^2-1}{2\times 2}$$

$$\frac{5}{8} < \Big[\ln x\Big]_1^2 < \frac{3}{4}$$

$$\frac{5}{8} < \ln 2 - \ln 1 < \frac{3}{4}$$

$$\frac{5}{8} < \ln 2 < \frac{3}{4}$$

(c) (i) Shaded area $= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

$$\therefore \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{3}\pi r^2 \qquad \left(\times \frac{6}{r^2}\right)$$

$$3\theta - 3\sin\theta = 2\pi$$

$$3\theta - 2\pi = 3\sin\theta$$

(ii) Let $f(\theta) = 3\theta - 2\pi - 3\sin\theta$

$$f\left(\tfrac{\pi}{2}\right) = \frac{3\pi}{2} - 2\pi - 3\sin\frac{\pi}{2}$$

$$= -\frac{\pi}{2} - 3$$

$$< 0$$

$$f(\pi) = 3\pi - 2\pi - 3\sin\pi$$

$$= \pi - 0$$

$$= \pi$$

$$> 0$$

$\therefore$ Since $f\left(\frac{\pi}{2}\right) < 0$, $f(\pi) > 0$ and $f(\theta)$ is continuous $f(\theta) = 0$ has a solution $\theta = \alpha$ where $\frac{\pi}{2} < \alpha < \pi$

(iii) $f(\theta) = 3\theta - 2\pi - 3\sin\theta$

$$f'(\theta) = 3 - 3\cos\theta$$

$$f\left(\tfrac{3\pi}{4}\right) = \frac{9\pi}{4} - 2\pi - 3\sin\frac{3\pi}{4}$$

$$= \frac{\pi}{4} - \frac{3}{\sqrt{2}}$$

$$f'\left(\tfrac{3\pi}{4}\right) = 3 - 3\cos\frac{3\pi}{4}$$

$$= 3 + \frac{3}{\sqrt{2}}$$

Let α_1 be the second approximation

$$\alpha_1 = \alpha - \frac{f(\alpha)}{f'(\alpha)}$$

$$= \frac{3\pi}{4} - \frac{\frac{\pi}{4} - \frac{3}{\sqrt{2}}}{3 + \frac{3}{\sqrt{2}}}$$

$$= 2.61704\ldots$$

$$= 2.6 \text{ (1 decimal place)}$$

Question 6

(a) $$\ddot{x} = -\left(10 + \frac{v^2}{40}\right)$$

$$v\frac{dv}{dx} = -\left(10 + \frac{v^2}{40}\right)$$

$$\frac{dv}{dx} = -\frac{(400 + v^2)}{40v}$$

$$\frac{dx}{dv} = -\frac{40v}{400 + v^2}$$

Let H be the greatest height

$$H = \int_{20}^{0} \frac{-40v}{400+v^2}\,dv$$

$$= 20\int_{0}^{20} \frac{2v}{400+v^2}\,dv$$

$$= 20\left[\log_e(400+v^2)\right]_0^{20}$$

$$= 20\{\log_e 800 - \log_e 400\}$$

$$= 20\log_e 2$$

$\therefore$ Greatest height is $20\log_e 2$

(b) $$\frac{dv}{dt} = -\left(10 + \frac{v^2}{40}\right)$$

$$\frac{dt}{dv} = -\frac{40}{400+v^2}$$

Let T be the time taken to reach H

$$T = \int^{0} \frac{-40}{400+v^2}\,dv$$

$$T = \int_0^{20} \frac{40}{400+v^2}\,dv$$

$$= 40 \times \frac{1}{20}\left[\tan^{-1}\frac{v}{20}\right]_0^{20}$$

$$= 2\left\{\tan^{-1}1 - \tan^{-1}0\right\}$$

$$= 2\left\{\frac{\pi}{4} - 0\right\}$$

$$= \frac{\pi}{2}$$

$\therefore$ Time taken is $\frac{\pi}{2}$ seconds

(ii) (a) $\ddot{x} = g - \frac{1}{40}v^2$

$$= 10 - \frac{v^2}{40}$$

$$= \frac{400 - v^2}{40}$$

(b) $v\frac{dv}{dx} = \frac{400-v^2}{40}$

$$\frac{dv}{dx} = \frac{400-v^2}{40v}$$

$$\frac{dx}{dv} = \frac{40v}{400-v^2}$$

$$H = \int_0^V \frac{40v}{400-v^2}\,dv$$ where V is speed wh returns to starting point

$$20\log_e 2 = -20\int_0^V \frac{-2v}{400-v^2}\,dv$$

$$= -20\left[\log_e(400-v^2)\right]_0^V$$

$$= -20\left\{\log_e(400-V^2) - \log_e 400\right\}$$

$$20\log_e 2 = 20\log_e \frac{400}{400-V^2}$$

$$2 = \frac{400}{400-V^2}$$

$$800 - 2V^2 = 400$$

$$2V^2 = 400$$

$$V^2 = 200$$

$$V = 10\sqrt{2}$$

(c) $\ddot{x} = 10 - \frac{v^2}{40}$

Terminal velocity, V_T, occurs as $\ddot{x} \to 0$

ie $10 - \frac{V_T^2}{40} = 0$

$$V_T^2 = 400$$

$$V_T = 20$$

$$\frac{V}{V_T} = \frac{10\sqrt{2}}{20}\cancel{\times 100}$$

$$= \frac{\sqrt{2}}{2}$$

$$V = \frac{\sqrt{2}}{2}V_T$$

$$= 70.7\% \text{ of } V_T$$

(b) (i) $x = 6$

(ii) $\ddot{x} = -1 \ (ms^{-2})$

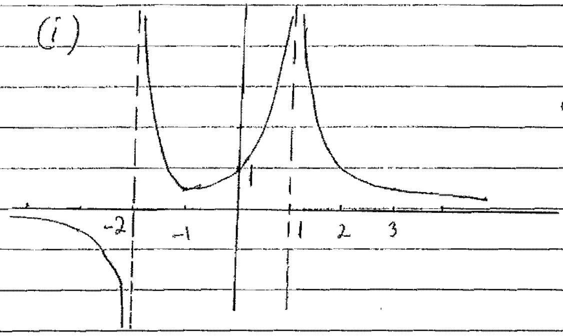
$m\ddot{x} = 2x - 1 \ N$

$\therefore$ Resultant force is 2 N in negative direction

Slows particle down from 2m/s ($t=4$) until it stops, at $t=6$ turns around and moves in negative direction at increasing speed.

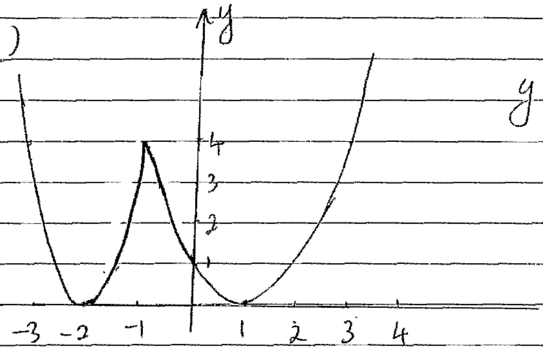
Question 7.

(a) (i)



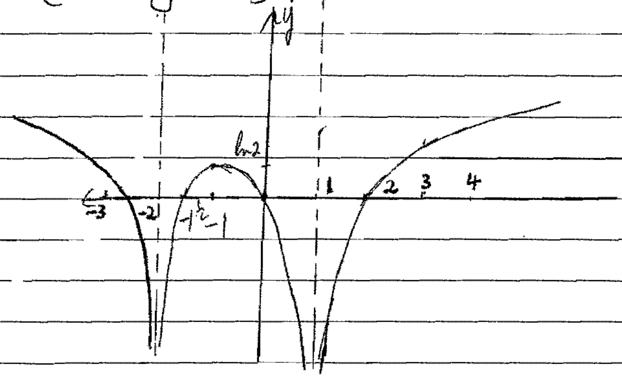
$y = \dfrac{1}{f(x)}$

(ii)

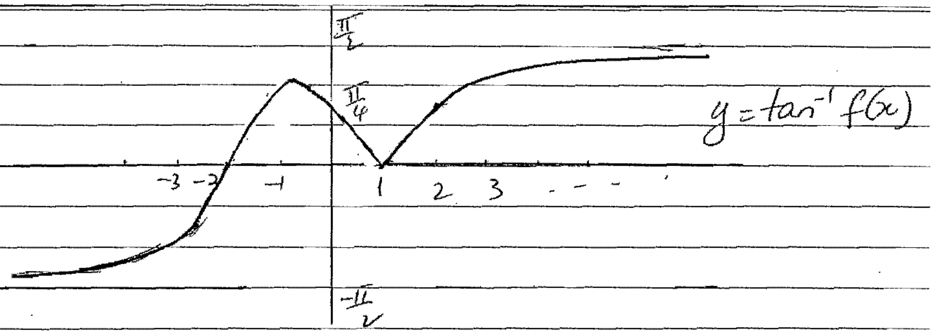


$y = [f(x)]^2$

(iii) $y = \log_e |f(x)|$



(iv)



(b) $I_n = \int_0^1 \frac{x^n}{1+x^2}\,dx$

$$I_n + I_{n+2} = \int_0^1 \frac{x^n}{1+x^2} + \frac{x^{n+2}}{1+x^2}\,dx$$

$$= \int_0^1 \frac{x^n(1+x^2)}{1+x^2}\,dx$$

$$= \int_0^1 x^n\,dx$$

$$= \left[\frac{x^{n+1}}{n+1}\right]_0^1$$

$$= \left(\frac{1}{n+1} - \frac{0}{n+1}\right)$$

$$= \frac{1}{n+1}$$

(ii) $I_0 = \int_0^1 \frac{1}{1+x^2}\,dx$

$$= \left[\tan^{-1} x\right]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0 \quad = \frac{\pi}{4}$$

$$I_1 = \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= \frac{1}{2}\left[\ln(1+x^2)\right]_0^1$$

$$= \frac{1}{2}(\ln 2 - \ln 1)$$

$$= \frac{1}{2}\ln 2$$

(iii) $I_0 + I_2 = \frac{1}{1}$

$$I_2 = 1 - \frac{\pi}{4}$$

$$I_2 + I_4 = \frac{1}{3}$$

$$I_4 = \frac{1}{3} - \left(1 - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{4} - \frac{2}{3}$$

$$I_4 + I_6 = \frac{1}{5}$$

$$I_6 = \frac{1}{5} - \left(\frac{\pi}{4} - \frac{2}{3}\right)$$

$$= \frac{13}{15} - \frac{\pi}{4}$$

Question 8

(a) $f(x) = \dfrac{e^x - 1}{e^x + 1}$

(i) $f'(x) = \dfrac{e^x(e^x+1) - e^x(e^x-1)}{(e^x+1)^2}$

$= \dfrac{2e^x}{(e^x+1)^2}$

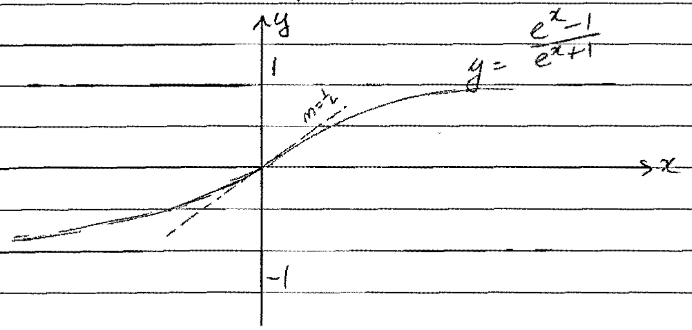
$> 0 \quad \forall x$ since $e^x > 0$

$\therefore f(x)$ is always increasing

(ii) $f'(0) = \dfrac{2e^0}{(e^0+1)^2} = \dfrac{2}{(1+1)^2} = \dfrac{1}{2}$

(iii) As $x \to \infty$ $f(x) \to 1^-$

$x \to -\infty$ $f(x) \to -1^+$



(iv) 3 real solutions whenever $0 < m < \frac{1}{2}$

(b) (i) $\dfrac{FH}{CO} = \dfrac{DH}{DO}$ (corresp sides in similar Δs FHD & COD)

$\dfrac{FH}{a} = \dfrac{h-x}{h}$

$FH = \dfrac{a(h-x)}{h}$

(ii) $C'F = a - FH$

$= a - \dfrac{a(h-x)}{h}$

$= \dfrac{ah - ah + ax}{h}$

$= \dfrac{ax}{h}$

$\cos(\angle HC'G) = \dfrac{C'F}{C'G}$

$= \dfrac{ax}{h} \times \dfrac{1}{a}$

$= \dfrac{x}{h}$

$\angle HC'G = \cos^{-1}\left(\dfrac{x}{h}\right)$

(iii)

Area of EGH $= \frac{1}{2}a^2\theta - \frac{1}{2}a^2\sin\theta$ where $\theta = 2 \times H\hat{C'}G$

$= \frac{1}{2}a^2 . 2\cos^{-1}\left(\frac{x}{h}\right) - \frac{1}{2}a^2 . 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= a^2\cos^{-1}\left(\frac{x}{h}\right) - a^2\sqrt{1-\left(\frac{x}{h}\right)^2} . \frac{x}{h}$

$= a^2\left[\cos^{-1}\left(\frac{x}{h}\right) - \frac{x}{h}\sqrt{1-\left(\frac{x}{h}\right)^2}\right]$

(IV) Volume $= \int_0^h a^2\left[\cos^{-1}\left(\frac{x}{h}\right) - \frac{x}{h}\sqrt{1-\left(\frac{x}{h}\right)^2}\right]dx$

Let $\theta = \frac{x}{h}$ $\quad dx = h\,d\theta$

When $x=0$ $\quad \theta = 0$

$x = h$ $\quad \theta = 1$

$$V = \int_0^1 a^2\left(\cos^{-1}\theta - \theta\sqrt{1-\theta^2}\right)h\,d\theta$$

$$= a^2h\left[\theta\cos^{-1}\theta - \sqrt{1-\theta^2} + \frac{1}{2}\frac{(1-\theta^2)^{3/2}}{3/2}\right]_0^1$$

$$= a^2h\left\{\left(1\cos^{-1}1 - 0 + 0\right) - \left(0 - 1 + \frac{1}{3}\times 1\right)\right\}$$

$$= a^2h\left\{(0-0+0) + \left(\frac{2}{3}\right)\right\}$$

$$= \frac{2a^2h}{3}$$