St George Girls High School

Trial Higher School Certificate Examination

2010

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Write your Student Number on every page
- Begin each question in a new booklet.
- All necessary working must be shown.
- Marks may be deducted for careless or poorly presented work.
- Board-approved calculators may be used.
- A list of standard integrals is included at the end of this paper.
- The mark allocated for each question is listed at the side of the question.

Total Marks –

- Attempt ALL questions.
- All questions are of equal value.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 - (15 marks) - Start a new booklet **Marks**

a) Use integration by parts to find 3

$$\int e^x \cos x \; dx$$

b) Use an appropriate substitution to find 3

$$\int \sin^3 x \, cos^4 x \; dx$$

c) Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{dx}{5 + 4\cos x}$$

using the substitution $t = \tan\frac{x}{2}$ 3

d) Show that 2

$$\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \cos 2x \sin x \, dx = 0$$

e) It is given that

$$I_n = \int_1^2 (\log_e x)^n \, dx$$

where n is a non-negative integer.

(i) Prove that $I_n = 2(\log_e 2)^n - n\, I_{n-1}\ (n \geq 1)$ 2

(ii) Hence, or otherwise, find the value of 2

$$\int_1^2 (\log_e x)^3 \, dx$$

Question 2 – (15 marks) – Start a new booklet Marks

a) Given that $z = 2 + 2\sqrt{3}i$ and $w = 1 - i$

(i) write $\frac{z}{w}$ in the form $a + ib$ where a and b are real. 2

(ii) find the square roots of z 3

(iii) find $|z|$ and $\arg(z)$ 2

(iv) express z^3 in modulus-argument form. 2

(v) find $\arg(zw)$ 2

b) If $z_1 = -2 + i$ and $z_2 = 5 + 2i$ show geometrically how to construct the vector that represents $z_1 - z_2$ 1

c) Draw neat sketches in the complex plane of the locus of z.

(i) $|z - 3 + 2i| \leq 1$ 1

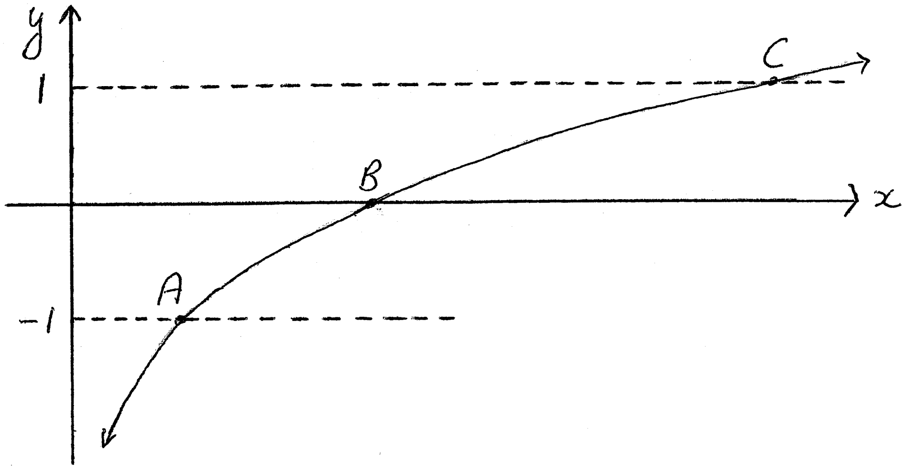
(ii) $\arg(z - 1 - 2i) = \frac{\pi}{4}$ 1

(iii) $\arg(z - 1) - \arg(z + 1) = \frac{\pi}{2}$ 1

Question 3 – (15 marks) – Start a new booklet

Marks

a) The diagram shows the graph of $f(x) = \ln x - 1$



(i) Write down the coordinates of A, B and C. 2

(ii) Draw separate one-third page sketches of the graphs of:

(α) $y = |f(x)|$ 2

(β) $y = \dfrac{1}{f(x)}$ 2

(γ) $y^2 = f(x)$ 2

(δ) $y = e^{f(x)}$ 2

b) $f(x) = \dfrac{7x}{(x^2+3)(x+2)}$

(i) Express $f(x)$ as the sum of partial fractions. 2

(ii) Evaluate 3

$$\int_0^3 f(x)\,dx$$

Question 4 – (15 marks) – Start a new booklet **Marks**

a) If α, β and γ are the roots of the cubic equation $x^3 - 2x + 5 = 0$

(i) Find the value of

(α) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ 2

(β) $\alpha^3 + \beta^3 + \gamma^3$ 2

(ii) Determine the cubic equation whose roots are α^2, β^2 and γ^2 2

b) It is given that $f(x) = x^4 + 5x^3 + 9x^2 + 8x + 4$ has a zero of multiplicity 2. Solve the equation $f(x) = 0$ over

(i) the rational field, Q. 2

(ii) the complex field C. 2

c) A polynomial $P(x)$ is divided by $x^2 - a^2$ (where $a \neq 0$) and the remainder is $px + q$.

(i) Show that $p = \dfrac{1}{2a}\,[P(a) - P(-a)]$ and $q = \dfrac{1}{2}\,[P(a) + P(-a)]$ 3

(ii) Find the remainder when $P(x) = x^n - a^n$, for n a positive integer, is divided by $x^2 - a^2$. 2

Question 5 – (15 marks) – Start a new booklet **Marks**

a) (i) Sketch the graph of $y = 2\sin x + 1$ for $0 \le x \le 2\pi$ **2**

(ii) Find the value of **3**

$$\int_0^{2\pi} |2\sin x + 1| \; dx$$

b) (i) Using the expansions of $\sin(A + B)$ and $\sin(A - B)$ show that **2**

$$\sin X + \sin Y = 2\sin\left(\frac{X+Y}{2}\right)\cos\left(\frac{X-Y}{2}\right)$$

(ii) Hence, or otherwise, find the general solution for **3**

$$\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta = 0$$

c) (i) Find constants A and B such that **2**

$$A(3\sin x + 2\cos x) + B(3\cos x - 2\sin x) = 8\sin x + 14\cos x$$

(ii) Hence find the exact value of **3**

$$\int_0^{\frac{\pi}{2}} \frac{8\sin x + 14\cos x}{3\sin x + 2\cos x} \; dx$$

Question 6 – (15 marks) – Start a new booklet **Marks**

a) Consider the hyperbola with equation

$$\frac{x^2}{9}-\frac{y^2}{4}=1$$

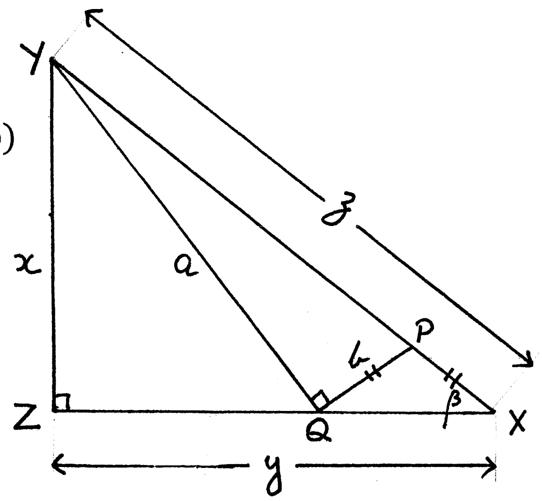
(i) Find the eccentricity of the hyperbola. 1

(ii) Write down the coordinates of the foci, the equations of the directrices and the equations of the asymptotes. 3

(iii) Find the equation of the tangent to the hyperbola at the point $P(3\sec\theta, 2\tan\theta)$ 2

(iv) The tangent at P meets the asymptotes at the points A and B. Show that $PA = PB$. 3

b)



ΔXYZ is as shown on the diagram with $\angle XZY = 90°$ and $x < y < z$.

P is a point on XY and Q is a point on XZ such that $\angle YQP = 90°$ and $PQ = PX$

Let $QY = a,\ PQ = PX = b,\ \angle ZXY = \beta$

(i) Prove $\Delta XYZ ||| \Delta YQZ$ 2

(ii) Show that $a = \dfrac{xz}{y}$ 1

(iii) Explain why $\angle QPY = 2\beta$ 1

(iv) Show that $b = \dfrac{z(y^2-x^2)}{2y^2}$ 2

<u>Question 7</u> – (15 marks) – Start a new booklet

Marks

a) If $m > 0$ show that $m + \frac{1}{m} \geq 2$ 2

b) If $P(3p^2, 2p^3)$ and $Q(3q^2, 2q^3)$ are 2 points on the curve with parametric equations

$$x = 3t^2 \quad y = 2t^3$$

(i) Show that the equation of the tangent to the curve at P is $px - y = p^3$ 2

(ii) Find the coordinates of T, the point of intersection of the tangents at P and Q 2

(iii) If the tangents at P and Q make angles of θ and $\frac{\pi}{2} - \theta$ with the positive x axis show that $pq = 1$ 2

(v) Hence find the equation of the locus of T. 3

c) (i) Sketch the curve $y = 4 + 3x - x^2$ 1

(ii) The area bounded by the curve $y = 4 + 3x - x^2$ and the x axis is rotated about the line $x = -2$. Use the method of cylindrical shells to find the volume of the solid generated. 3

Question 8 – (15 marks) – Start a new booklet

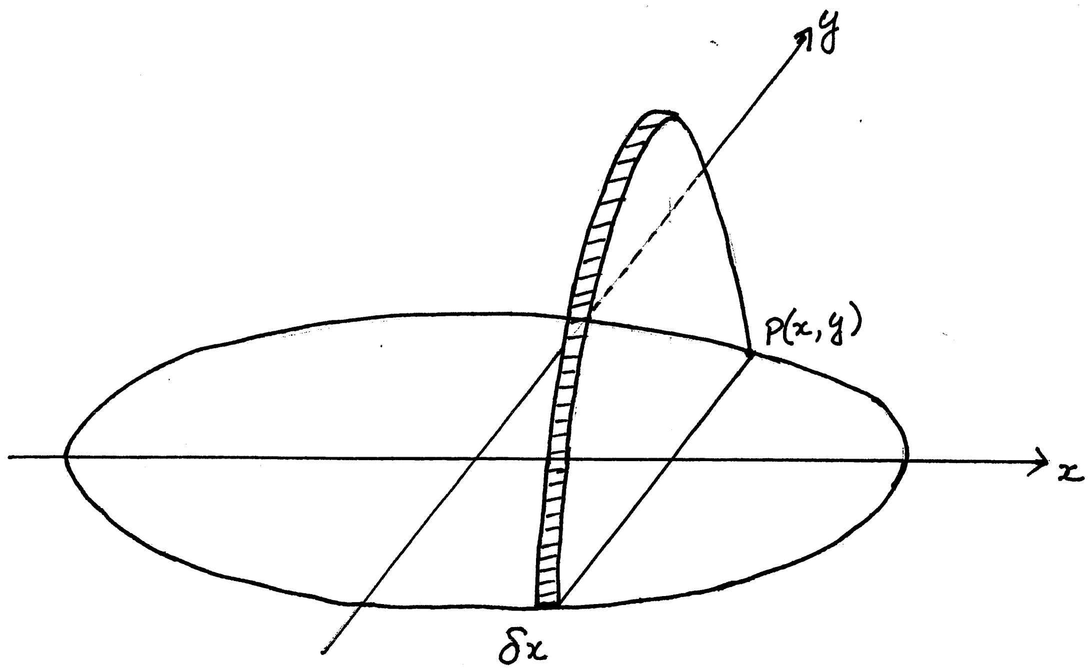
Marks

a) Show that the area, A, enclosed by the parabola $x^2 = 4ay$ and its latus rectum is given by **2**

$$A = \frac{8a^2}{3}$$

b) A solid figure has the ellipse $\frac{x^2}{36} + \frac{y^2}{9} = 1$ as its base in the $x - y$ plane.

Each cross-section perpendicular to the x axis is a parabola with latus rectum in the $x - y$ plane.



(i) Show that the area of the cross-section for each x value is $\frac{36-x^2}{6}$ **3**
[Hint: Use part (a)]

(ii) Hence find the volume of this solid. **2**

Question 8 (cont'd)

Marks

c) Two particles, A and B, move in the same <u>vertical</u> line in a medium whose resistance is proportional to the velocity of the particle.

Particle A is projected upwards from ground level with initial velocity u and, at the same instant, the particle B falls from rest from a height, h.

(i) The equation of motion for particle A is $\ddot{x} = -g - kv$ where g is the acceleration due to gravity and k is a positive constant.

Show that its height, x, above ground level at time t is given by **6**

$$x = \frac{g + ku}{k^2}[1 - e^{-kt}] - \frac{gt}{k}$$

(ii) It can be shown that the height of particle B above the ground at time t is given by

$$x_B = h - \frac{gt}{k} - \frac{ge^{-kt}}{k^2} + \frac{g}{k^2}$$

(There is no need to prove this)

Prove that the particles will meet at time, T, where **2**

$$T = \frac{1}{k}\log_e\left(\frac{u}{u - kh}\right)$$

End of Paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$