St George Girls High School

Trial Higher School Certificate Examination

2011

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Write your Student Number on every page
- Begin each question in a new booklet.
- All necessary working must be shown.
- Marks may be deducted for careless or poorly presented work.
- Board-approved calculators may be used.
- A list of standard integrals is included at the end of this paper.
- The mark allocated for each question is listed at the side of the question.

Total Marks –

- Attempt ALL questions.
- All questions are of equal value.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1 – (15 marks) – Start a new booklet

Marks

a) If $f(x) = \cos x + i \sin x$, show that $\dfrac{f'(x)}{f(x)} = i$ 2

b) Find 3

$$\int \sin^3 x \, dx$$

c) Using integration by parts, or otherwise, find 3

$$\int x \tan^{-1} x \; dx$$

d) (i) Find the remainder when $x^2 + 3$ is divided by $x^2 + 2x - 3$. 1

(ii) Hence, find $\int \frac{x^2+3}{x^2+2x-3} \; dx$ 3

e) If $f(x) = 2 - x^2$, without the use of calculus, sketch $y = \frac{1}{f(x)}$, showing all the asymptotes and points of intersection with the axes. 3

Question 2 – (15 marks) – Start a new booklet

Marks

a) For the function $y = \frac{\log_e(x^2-2)}{1-x}$

(i) State the domain. 1

(ii) Identify all the asymptotes. 2

(iii) Find the x-intercepts. 1

(iv) Sketch the curve without using calculus. 2

b) Find all pairs of real x and y that satisfy $(x + iy)^2 = 9 - 12i$ 3

c) Sketch on the Argand Diagram the locus of a point representing the complex number z if 3

$$1 \le |z| < 2 \text{ and } \frac{\pi}{3} \le \arg \le \pi$$

d) If a complex number z is a zero of $P(x) = a_1 x^n + a_2 x^{n-1} + \ldots + a_{n-1}\, x + a_n$, where $a_1, a_2, \ldots, a_{n-1}, a_n$ are all rational, prove that $\bar{z}$ is also a zero of this polynomial. 3

Question 3 – (15 marks) – Start a new booklet Marks

a) Consider the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$

(i) Determine the eccentricity, the coordinates of the foci (S and S') and the equations of the directices. 3

(ii) Sketch the ellipse showing all important features. 2

(iii) P is a point on the ellipse. Show that $PS+PS'$ is a constant. 2 ~~3~~

(iv) Find the gradient of the tangent at $P(5\cos\theta, 3\sin\theta)$ and, hence, show that the equation of the tangent at P is 3

$$\frac{x\cos\theta}{5}+\frac{y\sin\theta}{3}=1$$

b) (i) Find all the roots of $x^6=-1$ 2

(ii) By considering the three conjugate pairs of roots from part (i), or otherwise, express x^6+1 as a product of three quadratic factors with real coefficients. 2

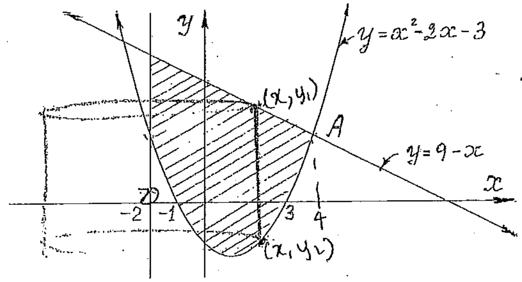
Question 4 – (15 marks) – Start a new booklet Marks

a) The area of a circle $x^2+y^2=r^2$ can be expressed as

$$4\int_0^r \sqrt{r^2-x^2}\,dx$$

Use the substitution $x=r\sin\theta$ to show that the area of the above circle is πr^2. 3

b) The shaded region is bounded by the curve $y=x^2-2x-3$, the line $y=9-x$ and the vertical line $x=-2$, as shown.



(i) Find the x-coordinate of point A. 1

(ii) The shaded region is rotated about the line $x=-2$. Use the method of cylindrical shells to express the volume of the resulting solid of revolution as an integral. 3

(iii) Evaluate this volume. Leave your answer in terms of π. 2

c) Use implicit differentiation to find the equation of the tangent to the curve $x^5+2x^2y^2+y^3=2$ at the point $(1,-1)$. 3

d) $P(x)=2x^4+9x^3+6x^2-20x-24$ has a zero of multiplicity 3.

(i) Find this zero. 2

(ii) Hence factorise $P(x)$ fully. 1

Question 5 – (15 marks) – Start a new booklet | Marks

a) (i) Prove that | 2

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

(Hint: use substitution $u = a - x$)

(ii) Hence evaluate $\int_0^3 9x^2(3-x)^7\,dx$ | 2

b) Let n be an integer greater or equal to 1. Let $I_n = \int_0^1 x^n e^{-x}\,dx$

(i) Show that $I_{n+1} = (n+1)\,I_n - \frac{1}{e}$ | 3

(ii) Hence evaluate $\int_0^2 x^2 e^{-\frac{x}{2}}\,dx$ Leave your answer in exact form. | 3

c) (i) Show that if $x^4 + px + q = 0$ has a double root α, then | 2

$$\alpha = \left(-\frac{p}{4}\right)^{\frac{1}{3}}$$

(ii) Hence, show that the relationship between p and q is expressed as | 3

$$2^8 q^3 = 3^3 p^4$$

Question 6 – (15 marks) – Start a new booklet | Marks

a) (i) Sketch the curve $y = \frac{1}{x^2+1}$, clearly indicating any turning points and the behaviour of the curve as $x \to \pm\infty$ | 2

(ii) Hence, on the same diagram, sketch $y = \frac{x^2}{x^2+1}$, clearly indicating any turning points, behaviour of the curve as $x \to \pm\infty$ and any points of intersection with the curve in part (i). | 2

b) At 8.10 am, at high tide, the deck of a ship was 1.6 m above the level of a wharf and at 2.30 pm, at low tide, the deck was 2.4 m below the level of the wharf. If the motion of the tide is simple harmonic:

(i) Find when the deck was level with the wharf. | 4

(ii) Find the maximum vertical speed of the deck. | 1

c) (i) Show that $1 + \sin 2x = (\cos x + \sin x)^2$ | 2

(ii) Hence, or otherwise, find all x such that | 3½

$$\cos x + \sin x = 1 + \sin 2x,$$

where $0 \le x \le 2\pi$

Question 7 – (15 marks) – Start a new booklet Marks

a) A body of mass m kg is released from rest and falls vertically with velocity $v\ m\ s^{-1}$ in the medium where the resistance is $\frac{1}{10}v$. After time t seconds the body has fallen a distance of x metres.

(i) Show that the equation of the motion may be written as $\ddot{x} = g - \frac{v}{10m}$ 1

(ii) Show that the terminal velocity is given by $V_t = 10mg$ 1

(iii) Show that the time taken to reach the velocity of half the terminal velocity is $10m \ln 2$ seconds. 3

(iv) Find the distance fallen by the body when $v = \frac{1}{2}V_t$. 2½/3

b) The tangents of the points $P(5p, \frac{5}{p})$ and $Q(5q, \frac{5}{q})$, where $p > 0$ and $q > 0$, on the rectangular hyperbola $xy = 25$ intersect at point T.

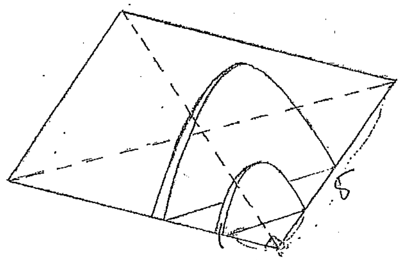
(i) Show that the coordinates of T are $\left(\frac{10pq}{p+q}, \frac{10}{p+q}\right)$. 3

(ii) If the chord PQ produced intersects y-axis at $R(0, 5)$, show that $pq = p + q$. 2

(iii) Hence, find the locus of T and describe it geometrically. 1½/2

Question 8 – (15 marks) – Start a new booklet Marks

a) (i) Show that the area enclosed by the parabola $x^2 = 4ay$ and its latus rectum, $y = a$, is $\frac{8a^2}{3}$ 4



(ii) The base of a solid is a square of side length 5 cm and each cross-section perpendicular to the base and to one of its diagonals is the region enclosed by a parabola and its latus rectum. 3A

Find the volume of this solid.

b) AB is a common chord of two circles. A straight line through B cuts the circles at points E and F. Tangents to the circles at E and F meet at C.

(i) Sketch a neat diagram representing the situation. 1

(ii) Prove that $AECF$ is a cyclic quadrilateral. 3

(iii) Show that if the circles are equal and $EB = BF$ then points A, B and C are collinear. 1/3

End of Paper

Q 1

(a) $f'(x) = -\sin x + i\cos x$

(I) $f'(x) = i(i\sin x + \cos x)$

So $\dfrac{f'(x)}{f(x)} = \dfrac{i(i\sin x + \cos x)}{\cos x + i\sin x}$

$= i$

or (II)

$\dfrac{f'(x)}{f(x)} = \dfrac{-\sin x + i\cos x}{\cos x + i\sin x} \times \left(\dfrac{\cos x - i\sin x}{\cos x - i\sin x}\right)$

$= \dfrac{i(\sin^2 x + \cos^2 x)}{(\cos^2 x + \sin^2 x)}$

$= i$

(b) $\int \sin^3 x\,dx = \int \sin^2 x \cdot \sin x\,dx$

$= \int (1-\cos^2 x)\cdot \sin x\,dx$

$u = \cos x$, $du = -\sin x\,dx$, $= -\int u^2\,du$

$= \int \sin x - \cos^2 x \cdot \sin x\,dx$

$= -\cos x + \frac{1}{3}\cos^3 x + C$

(c) $\int x \cdot \tan^{-1}x\,dx = \left[\tan^{-1}x \cdot \frac{x^2}{2}\right] - \int \frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[x - \tan^{-1}x\right] + C$

$= \frac{1}{2}\tan^{-1}x\,(x^2+1) - \frac{1}{2}x + C$

(d) (i)

$$\begin{array}{r} 1 \\ x^2+2x-3\ \overline{)\ x^2 \qquad +3} \\ -(x^2+2x-3) \\ \hline -2x+6 \end{array}$$

$\therefore x^2+3 = (x^2+2x-3) - (2x-6)$

or $(x^2+2x-3) + (6-2x)$

(ii) (I)

$\int \frac{x^2+3}{x^2+2x-3}\,dx = \int \frac{x^2+2x-3}{x^2+2x-3} + \frac{2(3-x)}{x^2+2x-3}\,dx$

$= \int 1\,dx + 2\int \frac{3-x}{(x+3)(x-1)}\,dx$

$= x + 2\int \frac{\frac{1}{2}}{x-1} - \frac{\frac{3}{2}}{x+3}\,dx$

$= x + 2\left(\frac{1}{2}\ln(x-1) - \frac{3}{2}\ln(x+3)\right) + C$

$= x + \ln(x-1) - 3\ln(x+3) + C$

or $x + \ln\left[\frac{(x-1)}{(x+3)^3}\right] + C$

$\frac{3-x}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$

$3-x = Ax - A + Bx + 3B$

$3-x = (A+B)x + (3B-A)$

So $4B = 2$

$B = \frac{1}{2}$

$\therefore A = -\frac{3}{2}$

II

$\int \frac{x^2+3}{x^2+2x-3}\,dx = \int \frac{x^2+2x-3}{x^2+2x-3} + \frac{2(3-x)}{x^2+2x-3}\,dx$

$= \int 1 - \frac{2x+2-8}{x^2+2x-3}\,dx$

$= \int 1\,dx - \int \frac{2x+2}{x^2+2x-3}\,dx + \int \frac{8}{(x+3)(x-1)}\,dx$

$= x - \ln(x^2+2x-3) + \int \frac{2}{x-1} - \frac{2}{x+3}\,dx$

$= x - \ln(x+3)(x-1) + 2\ln(x-1) - 2(x-3) + C$

$= x + \ln\left[\frac{(x-1)^2}{(x+3)(x-1)(x-3)^2}\right] + C$

$= x + \ln\left[\frac{(x-1)}{\dots}\right] + C$

$\frac{8}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$

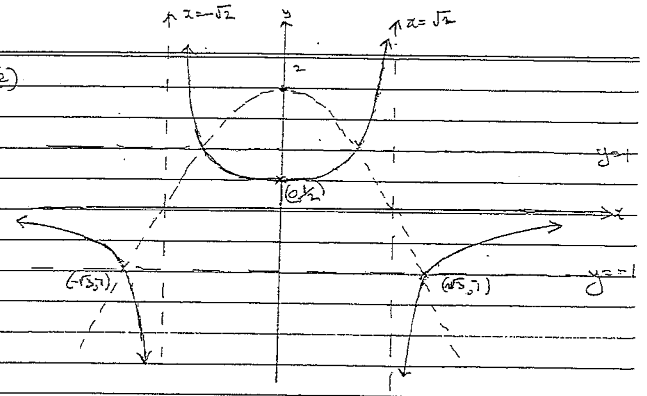
$\therefore (A+B)x + 3B - A = 8$

So $4B = 8$

$B = 2$

$\therefore A = -2$

(e)



Question 2

(a) $y = \dfrac{\log_e(x^2-2)}{1-x}$

(i) domain: $x^2-2>0$

$x^2>2$

$x>\sqrt{2}$ or $x<-\sqrt{2}$, x is real

(ii) Vertical asymptotes:

$x^2-2=0$ $x=\sqrt{2}$ and $x=-\sqrt{2}$

Since $x=1$ is inside the interval $-\sqrt{2}\le x\le\sqrt{2}$, it is not an asymptote.

Horizontal asymptote:

$\lim\limits_{x\to\infty} \dfrac{\log_e(x^2-2)}{1-x} = 0^-$, since x dominates $\log_e x$

$\lim\limits_{x\to-\infty} \dfrac{\log_e(x^2-2)}{1-x} = 0^+$, since x dominates $\log_e x$

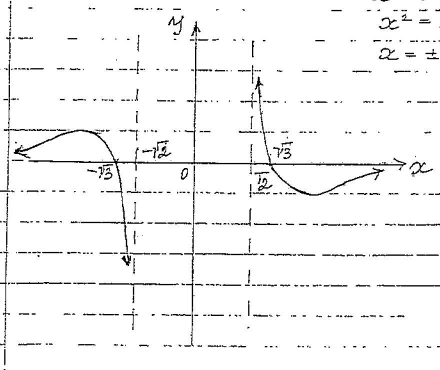
$\therefore y=0$ is a horizontal asymptote

(iii) x-intercepts: $\log(x^2-2)=0$

$x^2-2=1$

$x^2=3$

$x=\pm\sqrt{3}$



As $x\to-\sqrt{2}$

$\log_e(x^2-2)\to-\infty$

$1-x\to 1+\sqrt{2}\ (>0)$

$\therefore y\to-\infty$

As $x\to+\sqrt{2}$

$\log_e(x^2-2)\to-\infty$

$1-x\to 1-\sqrt{2}\ (<0)$

$\therefore y\to+\infty$

(b) $(x+iy)^2 = 9-12i$

$x^2 - y^2 = 9$

$2xy = -12$

$y = -\frac{6}{x}$

$x^2 - \left(-\frac{6}{x}\right)^2 = 9$

$x^4 - 36 = 9x^2$

$x^4 - 9x^2 - 36 = 0$

$(x^2-12)(x^2+3) = 0$

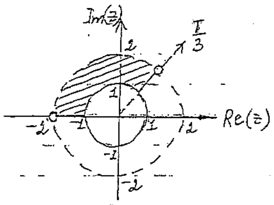
$x = \pm\sqrt{12} = \pm 2\sqrt{3}$, since x, y are real $x^2 \neq -3$

when $x = 2\sqrt{3}$ $y = -\frac{6}{2\sqrt{3}} = -\frac{6\sqrt{3}}{6} = -\sqrt{3}$

when $x = -2\sqrt{3}$ $y = -\frac{6}{-2\sqrt{3}} = \sqrt{3}$

$\therefore x = 2\sqrt{3}, y = -\sqrt{3}$ or $x = -2\sqrt{3}, y = \sqrt{3}$

(c)



(d) $P(x) = a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \dots + a_{n-1}x + a_n$, since z is zero:

$a_1z^n + a_2z^{n-1} + a_3z^{n-2} + \dots + a_{n-1}z + a_n = 0$

Taking conjugates of LHS and RHS

$\overline{a_1z^n + a_2z^{n-1} + a_3z^{n-2} + \dots + a_{n-1}z + a_n} = \overline{0}$

$\overline{a_1z^n} + \overline{a_2z^{n-1}} + \overline{a_3z^{n-2}} + \dots + \overline{a_{n-1}z} + \overline{a_n} = 0$, since $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$ and $\overline{0} = 0$

$\overline{a_1}\,\overline{z^n} + \overline{a_2}\,\overline{z^{n-1}} + \overline{a_3}\,\overline{z^{n-2}} + \dots + \overline{a_{n-1}}\,\overline{z} + \overline{a_n} = 0$, since $\overline{z_1z_2} = \overline{z_1}\,\overline{z_2}$ and $\overline{a} = a$ if a is real

$a_1\overline{z}^n + a_2\overline{z}^{n-1} + a_3\overline{z}^{n-2} + \dots + a_{n-1}\overline{z} + a_n = 0$

since $\overline{z^n} = (\overline{z})^n$

$\therefore P(\overline{z}) = 0$, as required.

QUESTION 3:

(a) $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ (i) $b^2 = a^2(1-e^2)$ $\therefore e^2 = 1 - \frac{b^2}{a^2}$

$e^2 = 1 - \frac{9}{25}$

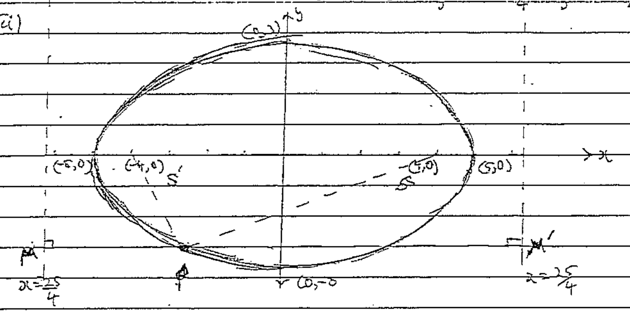
$e = \frac{4}{5}$

- $S(ae, 0)$; $S'(-ae, 0)$ since $a > b$

$S(4,0)$; $S'(-4,0)$

- Directrices $x = \pm\frac{a}{e}$, $x = \frac{25}{4}$; $x = -\frac{25}{4}$

(ii)



(iii) Let $P(x,y)$ lie on the ellipse

Let M and M' be foot of perpendicular from P to directrix.

The defn $\frac{PS}{PM} = e$ & $\frac{PS'}{PM'} = e$

So $PS + PS' = e\,PM + e\,PM'$

$= e\,(PM + PM')$

$= e \times 2\frac{a}{e}$

$= 2a$

Here, $PS + PS' = 2 \times 5$

$= 10$ constant

(iv) $\frac{2x}{25} + \frac{2y}{9}\cdot\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{9x}{25y}$

at $P(5\cos\theta, 3\sin\theta)$ gradient $m = \frac{-9 \times 5\cos\theta}{25 \times 3\sin\theta}$

$m = -\frac{3\cos\theta}{5\sin\theta}$

Equation of tangent is

$y - 3\sin\theta = \frac{-3\cos\theta}{5\sin\theta}(x - 5\cos\theta)$

$5y\sin\theta - 15\sin^2\theta = -3x\cos\theta + 15\cos^2\theta$

$3x\cos\theta + 5y\sin\theta = 15(\sin^2\theta + \cos^2\theta)$ $\div 15$

$\therefore \frac{x\cos\theta}{5} + \frac{y\sin\theta}{3} = 1$

(b) Let $z = \cos\theta + i\sin\theta$ be a solution to $x^6 + 1 = 0$ ie $x^6 = -1$

Now $|z| = 1$, $(\cos\theta + i\sin\theta)^6 = -1$

$|-1| = 1$ $\cos 6\theta + i\sin 6\theta = -1 + 0i$ [De Moivre]

Equate Real & Imaginary parts.

$\cos 6\theta = -1$ & $\sin 6\theta = 0$

$\therefore 6\theta = 2n\pi + \pi$ $6\theta = 2n\pi + \pi$

$\theta = \frac{\pi}{6}(2n+1)$

$n=0$; $\theta = \frac{\pi}{6}$ $\rightarrow z_1 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$, $z_1 = \frac{\sqrt{3}}{2} + i\frac{1}{2}$

$n=1$; $\theta = \frac{\pi}{2}$ $\rightarrow z_2 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, $z_2 = i$

$n=2$; $\theta = \frac{5\pi}{6}$ $\rightarrow z_3 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$, $z_3 = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$

$n=3$; $\theta = \frac{7\pi}{6}$ $\rightarrow z_4 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$, $z_4 = -\frac{\sqrt{3}}{2} - i\frac{1}{2}$

$n=4$; $\theta = \frac{3\pi}{2}$ $\rightarrow z_5 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$, $z_5 = -i$

$n=5$; $\theta = \frac{11\pi}{6}$ $\rightarrow z_6 = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$, $z_6 = \frac{\sqrt{3}}{2} - i\frac{1}{2}$

Note: $z_6 = \overline{z_1}$; $z_5 = \overline{z_2}$; $z_4 = \overline{z_3}$ Conjugate pairs

(a) $x^6 + 1 = (x-z_1)(x-z_2)(x-z_3)(x-z_4)(x-z_5)(x-z_6)$

$= (x-z_1)(x-\overline{z_1})(x-z_2)(x-\overline{z_2})(x-z_3)(x-\overline{z_3})$

$= [x^2 - 2\,\mathrm{Re}(z_1)x + |z_1|^2][x^2 - 2\,\mathrm{Re}(z_2)x + |z_2|^2][x^2 - 2\,\mathrm{Re}(z_3)x + |z_3|^2]$

$= (x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1)(x^2 + 1)$

Question 4

(a) Area $= 4\int_0^r \sqrt{r^2 - x^2}\, dx$

Let $x = r\sin\theta$
$\therefore dx = r\cos\theta\, d\theta$
When $x = 0$, $\theta = 0$
$x = r$, $\theta = \frac{\pi}{2}$

$\therefore$ Area $= 4\int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2\theta}\; r\cos\theta\, d\theta$

$= 4\int_0^{\frac{\pi}{2}} r\sqrt{1-\sin^2\theta}\; r\cos\theta\, d\theta$

$= 4r^2\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$

$= 4r^2\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\, d\theta$

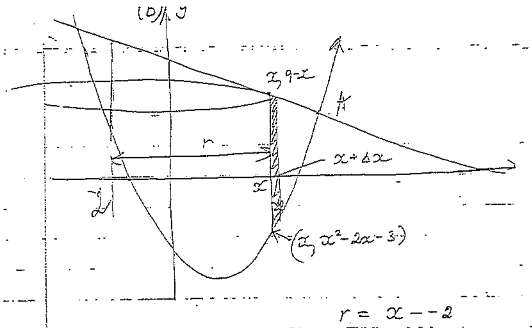
$= 2r^2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$

$= 2r^2\left[\left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right]$

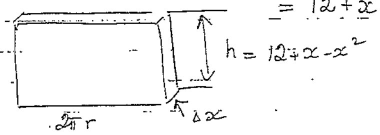
$= 2r^2\left(\frac{\pi}{2} - 0\right)$

$= \pi r^2$, as required.

(b)



$r = x - -2$
$= x + 2$
$h = (9-x) - (x^2 - 2x - 3)$
$= 12 + x - x^2$



$\Delta V = 2\pi r h\,\Delta x$
$= 2\pi(x+2)(12+x-x^2)\Delta x$
$= 2\pi(12x + x^2 - x^3 + 24 + 2x - 2x^2)\Delta x$
$= 2\pi(14x - x^2 - x^3 + 24)\Delta x$

$V \approx \sum_{x=-2}^{4} 2\pi(14x - x^2 - x^3 + 24)\Delta x$

$V = 2\pi\int_{-2}^{4}(14x - x^2 - x^3 + 24)\, dx$

$= 2\pi\left[7x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + 24x\right]_{-2}^{4}$

$= 2\pi\left(\left(112 - \frac{64}{3} - 64 + 96\right) - \left(28 + \frac{8}{3} - 4 - 48\right)\right)$

$= 2\pi\left(\frac{368}{3} + \frac{64}{3}\right)$

$= 288\pi\ u^3$

(c)

$x^5 + 2x^2y^2 + y^3 = 2$

$5x^4 + \left(y^2 \times 4x + 2x^2 \times 2y \times \frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0.$

$5x^4 + 4xy^2 + \frac{dy}{dx}(4x^2y + 3y^2) = 0$

$\frac{dy}{dx} = -\frac{(5x^4 + 4xy^2)}{4x^2y + 3y^2}$

At $P(1,-1)$ $\frac{dy}{dx} = -\frac{5+4}{-4+3} = \frac{9}{1} = 9$

Tangent has gradient 9.

$y + 1 = 9(x-1)$
$y + 1 = 9x - 9$
$y = 9x - 10$ or $9x - y - 10 = 0$.

(d) $P(x) = 2x^4 + 9x^3 + 6x^2 - 20x - 24$
$P'(x) = 8x^3 + 27x^2 + 12x - 20$
$P''(x) = 24x^2 + 54x + 12$
$P''(x) = 0$ if triple root
$24x^2 + 54x + 12 = 0$
$4x^2 + 9x + 2 = 0$
$(4x+1)(x+2) = 0$
$x = -\frac{1}{4}$ or $x = -2$

$P'(-2) = -64 + 108 - 24 - 20 = 0$
$P'(-\frac{1}{4}) = -\frac{8}{64} + \frac{27}{16} - 3 - 20 \neq 0$
$\therefore x = -2$ is the tripple root
$P(x) = (x+2)^3(2x + c)$.
$8c = -24$
$c = -3 \Rightarrow P(x) = (x+2)^3(2x-3)$

Question 5:

(a) (i) let $u = a - x$ then $dx = -du$
when $x = a$, $u = 0$
$x = 0$, $u = a$

so $\int_0^a f(a-x)\, dx = \int_a^0 f(u) \cdot -du$

$= -\int_a^0 f(u)\, du$

$= \int_0^a f(u)\, du$

This is $\int_0^a f(x)\, dx$ [since "u" any dummy variable]

(ii) $\int_0^3 9x^2(3-x)^7\, dx = \int_0^3 9(3-x)^2\,[3-(3-x)]^7\, dx$ from (i)

$= \int_0^3 9(9 - 6x + x^2)\cdot x^7\, dx$

$= 9\int_0^3 9x^7 - 6x^8 + x^9\, dx$

$= 9\left[\frac{9}{8}x^8 - \frac{2}{3}x^9 + \frac{1}{10}x^{10}\right]_0^3$

$= 9\left[3^8\left(\frac{9}{8} - 2 + \frac{1}{10}\cdot 3^2\right) - 0\right]$

$= 3^{10}\left(\frac{45 - 80 + 36}{40}\right)$

$= \frac{1}{40}\,3^{10}$ $\boxed{\frac{59049}{40}}$

(b) (i) $I_n = \int_0^1 x^n e^{-x}\, dx$

(I) $I_n = \left[e^{-x}\frac{x^{n+1}}{n+1}\right]_0^1 - \int \frac{x^{n+1}}{n+1} \cdot -e^{-x}\, dx$

$I_n = \frac{1}{n+1}\cdot e^{-1} + \frac{1}{n+1}\int x^{n+1}\cdot e^{-x}\, dx$

$\therefore\ (n+1)\, I_n = \frac{1}{e} + I_{n+1}$

So $\quad I_{n+1} = (n+1)\, I_n - \frac{1}{e}$

OR (II) now $\quad I_{n+1} = \int_0^1 x^{n+1} . e^{-x}\, dx$

$= \left[x^{n+1} . -e^{-x} \right]_0^1 - \int (n+1) x^n . -e^{-x}\, dx$

$I_{n+1} = -e^{-1} + (n+1) \int x^n . e^{-x}\, dx$

$I_{n+1} = (n+1)\, I_n - \frac{1}{e}$

(ii)(I) let $u = \frac{x}{2}$, $2u = x$ $\quad$ So $\quad I_2 = \int_0^2 x^2 . e^{-\frac{x}{2}}\, dx$

then $2du = dx$ $\quad$ Sub. $\quad I_2 = \int_0^1 (2u)^2 . e^{-u} 2du$

when $x = 0,\ u = 0$

$x = 2,\ u = 1$ $\quad = 8 \int_0^1 u^2 . e^{-u}$

Now $\quad U_2 = \int_0^1 u^2 e^{-u}\, du$

$= 2U_1 - \frac{1}{e}$

$= 2\left[U_0 - \frac{1}{e}\right] - \frac{1}{e}$

So $\quad U_0 = \int_0^1 e^{-u}\, du$ $\quad = 2\left[\left(1 - \frac{1}{e}\right) - \frac{1}{e}\right] - \frac{1}{e}$

$= \left[-e^{-u}\right]_0^1$ $\quad = 2 - \frac{5}{e}$

$= -\frac{1}{e} + 1$

Gives $\quad I_2 = 8\, U_2$

$= 8\left[2 - \frac{5}{e}\right]$

$= \frac{16e - 40}{e}$

OR (II) $\quad I_2 = \int_0^2 x^2 . e^{-\frac{x}{2}}\, dx$ $\quad u = x^2$; $dv = e^{-\frac{x}{2}}$; $du = 2x$; $v = -2e^{-\frac{x}{2}}$

So $\quad I_2 = \left[x^2 . -2e^{-\frac{x}{2}}\right]_0^2 - \int_0^2 2x . -2 . e^{-\frac{x}{2}}\, dx$

$= \left(-\frac{8}{e}\right) + 4 \int_0^2 x . e^{-\frac{x}{2}}\, dx$

$= \left(-\frac{8}{e}\right) + 4\left[\left[x . -2e^{-\frac{x}{2}}\right]_0^2 - \int_0^2 -2e^{-\frac{x}{2}}\, dx\right]$

$= -\frac{8}{e} + 4\left[\left(-\frac{4}{e}\right) + 2\left[-2 . e^{-\frac{x}{2}}\right]_0^2\right]$

$= -\frac{8}{e} + 4\left[-\frac{4}{e} + 2\left(-\frac{2}{e} + 2\right)\right]$

$= -\frac{40}{e} + 16 \quad \rightarrow \quad \frac{16e - 40}{e}$

(c) (i) If α is a double root of $P(x)$

Then $\quad P(\alpha) = 0$ and $P'(\alpha) = 0$

Now $\quad P'(x) = 4x^3 + p$

So $\quad P'(\alpha) = 4\alpha^3 + p$

$= 0$

$\alpha^3 = -\frac{p}{4}$

$\alpha = \left(-\frac{p}{4}\right)^{\frac{1}{3}}$

(ii) (I) Now on substitution

$\alpha^4 + p\alpha + q = 0$

then $\quad \alpha(\alpha^3 + p) = -q$

$\left(-\frac{p}{4}\right)^{\frac{1}{3}}\left[-\frac{p}{4} + p\right] = -q$

$\left(-\frac{p}{4}\right)^{\frac{1}{3}}\left(\frac{3p}{4}\right) = -q$

Cube both sides of equation

$\left(-\frac{p}{4}\right)\left(\frac{3p}{4}\right)^3 = -q^3$ $\quad (\times -1)$

$\frac{3^3 p^4}{4^4} = q^3$

$3^3 p^4 = 4^4 q^3$ $\quad \left[27p^4 = 256q^3\right]$

$3^3 p^4 = 2^8 q^3$

OR II $\quad$ Substitute:- $\left[\left(-\frac{p}{4}\right)^{\frac{1}{3}}\right]^4 + p\left[\left(-\frac{p}{4}\right)^{\frac{1}{3}}\right] + q = 0$

$\frac{p^{4/3}}{4^{4/3}} - \frac{p^{4/3}}{4^{1/3}} + q = 0$

$p^{4/3}\left(\frac{1}{4^{4/3}} - \frac{1}{4^{1/3}}\right) = -q$

$p^{4/3}\left(-\frac{3}{4^{4/3}}\right) = -q$

cube both sides

and $\times -1$ $\quad p^4 . \frac{3^3}{4^4} = q^3$

$3^3 p^4 = 4^4 q^3$

Question 6

(a) (i) $\quad y = \frac{-1}{x^2+1}$

$\frac{dy}{dx} = -(x^2+1)^{-2} \times 2x = -\frac{2x}{(x^2+1)^2}$

Stationary points $\frac{dy}{dx} = 0 \quad \rightarrow \quad x = 0,\ y = 1$

$(0,1)$ is a stationary point

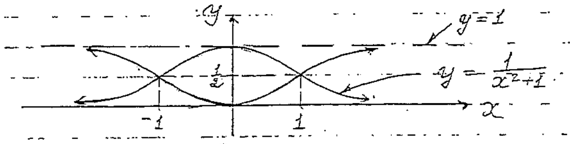
For $x > 0 \quad \frac{dy}{dx} < 0$

For $x < 0 \quad \frac{dy}{dx} > 0$

$\therefore (0,1)$ is a maximum turning point

as $x \to \infty$ $y \to 0^+$ and as $x \to -\infty$ $y \to 0^+$

$y = 0$ is a horizontal asymptote.



(ii) $\quad y = \frac{x^2}{x^2+1} = \frac{x^2+1-1}{x^2+1} = 1 - \frac{1}{x^2+1}$

which is the graph from part (i) reflected in x-axis and translated 1 unit up.

Points of intersection : $\frac{1}{x^2+1} = 1 - \frac{1}{x^2+1}$

$\frac{2}{x^2+1} = 1$

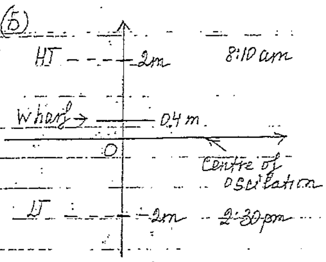
$x^2 + 1 = 2$

$x^2 = 1$

$x = -1 \quad y = \frac{1}{2}$

or $\quad x = 1 \quad y = \frac{1}{2}$

(5)



$1.6 + 2.4 = 4\,m$

Amplitude $= \frac{4}{2} = 2m$

Period = one cycle $= (2{:}30\,pm - 8{:}10\,am) \times 2$
$= 12\,h\ 40\,min$

$\therefore T = \frac{760}{60} = \frac{38}{3}$ hours

(or 760 minutes)

The motion is a SHM

$\therefore \ddot{x} = -n^2 x$, where $n = \frac{2\pi}{T} = \frac{6\pi}{38} = \frac{3\pi}{19}$

$\left(\text{or } n = \frac{2\pi}{760} = \frac{\pi}{380}\right)$

$x = 2\cos(nt + \alpha)$

Initial condition: when $t = 0$ $x = 2$ (highest point)

$\therefore 2 = 2\cos\alpha \Rightarrow \cos\alpha = 1 \Rightarrow \alpha = 0$

$\therefore x = 2\cos\left(\frac{3\pi}{19}t\right)$ or $x = 2\cos\left(\frac{\pi}{380}t\right)$

$v = \dot{x} = -\frac{3\pi}{19} \times 2\sin\left(\frac{3\pi}{19}t\right)$ | $v = \dot{x} = -\frac{\pi}{380} \times 2\sin\left(\frac{\pi}{380}t\right)$

$v = -\frac{6\pi}{19}\sin\left(\frac{3\pi}{19}t\right)$ | $v = -\frac{\pi}{190}\sin\left(\frac{\pi}{380}t\right)$

Deck is level with the wharf when $x = 0.4 = \frac{2}{5}$

$2\cos\left(\frac{3\pi}{19}t\right) = \frac{2}{5}$ or $2\cos\left(\frac{\pi}{380}t\right) = \frac{2}{5}$

$\cos\left(\frac{3\pi}{19}t\right) = \frac{1}{5}$ | $\cos\left(\frac{\pi}{380}t\right) = \frac{1}{5}$

$\frac{3\pi}{19}t = \cos^{-1}\left(\frac{1}{5}\right)$ | $\frac{\pi}{380}t = \cos^{-1}\left(\frac{1}{5}\right)$

$t = \frac{19\cos^{-1}\left(\frac{1}{5}\right)}{3\pi}$ | $t = \frac{380\cos^{-1}\left(\frac{1}{5}\right)}{\pi}$

$\doteqdot 2.761\,h \doteqdot 166$ minutes. | $\doteqdot 166$ minutes.

166 minutes = 2h 46 minutes

8:10 + 2h 46 min = 10:56 am

$\therefore$ The deck is level with the wharf at 10:56 am

(ii) max speed when $\sin\left(\frac{3\pi}{19}t\right) = \pm 1$ $\left(\text{or } \sin\frac{\pi}{380}t = \pm 1\right)$

$|v_{max}| = \frac{6\pi}{19}\,m\,h^{-1}$ or $|v_{max}| = \frac{\pi}{190}\,m/min$

(c) RTP $1 + \sin 2x = (\cos x + \sin x)^2$

(i) $RHS = \cos^2 x + 2\cos x\sin x + \sin^2 x$
$= 1 + 2\cos x \sin x$
$= 1 + \sin 2x$
$= LHS$

(ii) $\cos x + \sin x = 1 + \sin 2x$

$\cos x + \sin x = (\cos x + \sin x)^2$ (using part (i))

$\cos x + \sin x - (\cos x + \sin x)^2 = 0$

$(\cos x + \sin x)(1 - (\cos x + \sin x)) = 0$

$\cos x + \sin x = 0$ or $\cos x + \sin x = 1$ (since if $AB = 0$, then $A = 0$ or $B = 0$)

($\div \cos x$, since $\cos x \neq 0$)

$1 + \tan x = 0$

$\tan x = -1$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$LHS = \cos x + \sin x$
$= R\sin(x + \lambda)$
$= R\cos x \sin\lambda + R\cos\lambda \sin x$

$\therefore R\sin\lambda = 1$

$R\cos\lambda = 1$

$R^2(\sin^2\lambda + \cos^2\lambda) = 2$

$R = \sqrt{2}$

$\sin\lambda = \frac{1}{\sqrt{2}}$, $\cos\lambda = \frac{1}{\sqrt{2}}$

$\lambda = \frac{\pi}{4}$

$\therefore \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = 1$

$\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\left(x + \frac{\pi}{4}\right) = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4} \therefore x = 0, \frac{2\pi}{4}, \frac{8\pi}{4}$

$\boxed{\begin{array}{l}\text{Solution:}\\ x = 0, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{4}, 2\pi\end{array}}$

(c)(ii) Alternative solution

$\cos x + \sin x = 1 + \sin 2x$

$(1 + \sin 2x)^{\frac{1}{2}} = 1 + \sin 2x$ (using part (i))

Squaring both sides:

$\cancel{1} + \sin 2x = \cancel{1} + 2\sin 2x + (\sin 2x)^2$

$(\sin 2x)^2 + \sin 2x = 0$

$\sin 2x(\sin 2x + 1) = 0$

$\sin 2x = 0$ or $\sin 2x = -1$

$2x = 0, \pi, 2\pi, 3\pi, 4\pi$ | $2x = \frac{3\pi}{2}, \frac{7\pi}{2}$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ | $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

Now, since we squared both sides we have increased the number of solutions

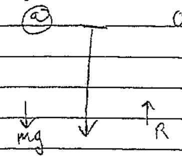
$\therefore$ we need to check the obtained solutions to satisfy the original equation.

when $x = 0$	$LHS = 1$	$RHS = 1$	✓
$x = \frac{\pi}{2}$	$LHS = 1$	$RHS = 1$	✓
$x = \pi$	$LHS = -1$	$RHS = 1$	✗
$x = \frac{3\pi}{2}$	$LHS = -1$	$RHS = 1$	✗
$x = 2\pi$	$LHS = 1$	$RHS = 1$	✓
$x = \frac{3\pi}{4}$	$LHS = 0$	$RHS = 0$	✓
$x = \frac{7\pi}{4}$	$LHS = 0$	$RHS = 0$	✓

Set of solutions: $x = 0, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{4}, 2\pi$

Question 7:

(a) at $t = 0$, $x = 0$ and $v = 0$



(i) $F = mg - R$

$m\ddot{x} = mg - \frac{1}{10}v$

$\ddot{x} = g - \frac{v}{10m}$

(ii) Terminal velocity when $\ddot{x} = 0$

let V_t be terminal velocity

then $0 = g - \frac{V_t}{10m}$

$V_t = 10mg$

(iii) Using $\ddot{x} = \frac{dv}{dt}$

then $\frac{dv}{dt} = g - \frac{v}{10m}$

$= \frac{10mg - v}{10m}$

so $\frac{dt}{dv} = \frac{10m}{V_t - v}$ $[V_t = 10mg]$

$dt = \frac{10m}{V_t - v}\,dv$

$t = \int_0^{V_t/2} \frac{10m}{V_t - v}\,dv$

$= 10m\left[-\ln(V_t - v)\right]_0^{V_t/2}$

$= -10m\left[\ln\left(\frac{V_t}{2}\right) - \ln(V_t)\right]$

$= -10m\left[\ln\left(\frac{1}{2}\right)\right]$

$= 10m\ln 2$

or $\frac{dv}{dt} = \frac{10mg - v}{10m}$

$dt = \frac{10m}{10mg - v}\, dv \quad [10mg = V_t]$

$= \frac{10m}{V_t - v}$

So $t = -10m \ln(V_t - v) + C$

at $t=0$, $v=0$ $\therefore$ $0 = -10m\ln(V_t) + C$

$C = 10m\ln(V_t)$

$t = -10m \ln\left[\frac{V_t - v}{V_t}\right]$

If $v = \frac{V_t}{2}$, then $t = -10m \ln\left(\frac{\frac{V_t}{2}}{V_t}\right)$

$= -10m \ln(\frac{1}{2})$

$= 10m \ln 2$

(IV) (I) Let $v.\frac{dv}{dx} = \frac{10gm - v}{10m}$

$\frac{dv}{dx} = \frac{10gm - v}{10mv}$

$\frac{dx}{dv} = \frac{10mv}{V_t - v} \quad (10gm = V_t$

$= \frac{-10m(V_t - v) + 10mV_t}{V_t - v}$

$dx = -10m + \frac{10mV_t}{V_t - v}\, dv$

So

$x = -10mv - 10m \cdot V_t \ln(V_t - v) + C$

at $x=0$, $v=0$

gives $C = 10mV_t \ln(V_t)$

$\therefore x = -10mv - 10mV_t \ln\left[\frac{V_t - v}{V_t}\right]$

take $v = \frac{V_t}{2}$; $x = -10m\frac{V_t}{2} - 10mV_t \ln(\frac{1}{2})$

$= 10mV_t\left(\ln 2 - \frac{1}{2}\right)$

Distance fallen $= 100m^2g\left(\ln 2 - \frac{1}{2}\right)$

or (II) $t = -10m \ln\left(\frac{V_t - v}{V_t}\right)$

$-\frac{t}{10m} = \ln\left[\frac{V_t - v}{V_t}\right]$

$e^{-\frac{t}{10m}} = \frac{V_t - v}{V_t}$

$V_t - v = V_t e^{-\frac{t}{10m}}$

$v = V_t - V_t e^{-\frac{t}{10m}}$

$\frac{dx}{dt} = V_t\left(1 - e^{-\frac{t}{10m}}\right)$

$x = V_t t + 10mV_t e^{-\frac{t}{10m}} + C$

When $t=0$, $x=0$

$\therefore C = -10m.V_t$

So $x = V_t . t + 10mV_t e^{-\frac{t}{10m}} - 10mV_t$

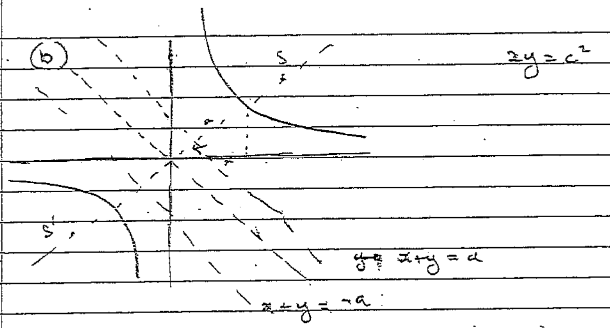
When $t = 10m\ln 2$

$x = 10mg . 10m\ln 2 + 10mV_t e^{-\ln 2} - 10mV_t$

$= 100m^2g\ln 2 + 10mV_t\frac{1}{2} - 10mV_t$

$*V_t = 10mg$ $\quad = 100m^2g\ln 2 - \frac{1}{2}100m^2g$

(b)



(i) $xy = 25$ at $P(5p, \frac{5}{p})$

$y + x.\frac{dy}{dx} = 0$ $\quad m = -\frac{1}{p^2}$

$\frac{dy}{dx} = \frac{-y}{x}$

and equation of tangent

$y - \frac{5}{p} = -\frac{1}{p^2}(x - 5p)$

$p^2y - 5p = -x + 5p$

$x + p^2y = 10p$ —— (A)

Similarly for $Q(5q, \frac{5}{q})$

tangent has eq. $x + q^2y = 10q$ —— (B)

Solve (A)−(B) $(p^2 - q^2)y = 10(p-q)$

$y = \frac{10}{p+q}$

So $q^2(A) - p^2(B) \Rightarrow q^2x + p^2q^2y = 10pq^2$

$p^2x + p^2q^2y = 10q^2p^2$

Gives $(q^2 - p^2)x = 10pq(q-p)$

$x = \frac{10pq}{p+q}$

Co-ords of $T\left(\frac{10pq}{p+q}, \frac{10}{p+q}\right)$

(ii) Chord PQ $\rightarrow$ gradient $m = \frac{\frac{5}{p} - \frac{5}{q}}{5p - 5q}$

$= \frac{5(q-p)}{pq} \Big/ 5(p-q)$

$= -\frac{1}{pq}$

Equation of chord $y - \frac{5}{p} = -\frac{1}{pq}(x - 5p)$

$pq.y - 5q = -x + 5p$

$x + pq.y = 5(p+q)$

Sub $R(0,5)$ $\quad 0 + 5pq = 5(p+q)$

$pq = p+q$

(iii) Now $X = \frac{10pq}{p+q}$, $Y = \frac{10}{p+q}$

then $X = \frac{10(p+q)}{p+q}$ and $Y = \frac{10}{pq}$ $\quad p>0, q>0$ $\therefore Y>0$

$X = 10$

Locus is vertical line, $x = 10$.

and, when $x = 10$ $\quad y = \frac{5}{2}$

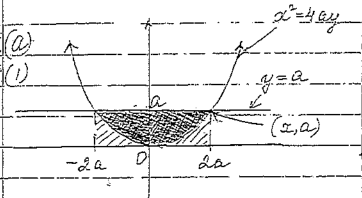
Line intersects hyperbola at $(10, \frac{5}{2})$

$\therefore$ locus is vertical line $x = 10$

with range $0 < y < \frac{5}{2}$

Question 8

(a) (i)



Method 1

$$Area = 4a\times a - 2\int_0^{2a}\frac{x^2}{4a}\,dx$$
$$= 4a^2 - 2\times\frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{2a}$$
$$= 4a^2 - \frac{1}{2a}\times\left(\frac{8a^3}{3} - 0\right)$$
$$= 4a^2 - \frac{4a^2}{3}$$
$$= \frac{12a^2 - 4a^2}{3} = \frac{8a^2}{3},$$

as required

Method 2

$x^2 = 4ay \Rightarrow x = 2\sqrt{ay}$

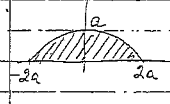
$$Area = 2\int_0^a 2\sqrt{ay}\,dy$$
$$= 4\int_0^a (ay)^{\frac{1}{2}}dy$$
$$= 4\left[\frac{(ay)^{\frac{3}{2}}}{\frac{3}{2}a}\right]_0^a$$
$$= 4\times\frac{2}{3}\left(\frac{(a^2)^{3/2}}{a} - 0\right)$$
$$= \frac{8}{3}\times\frac{a^3}{a} = \frac{8a^2}{3},$$

as required.

Method 3

Using the Simpson's Rule:

$$\int_a^b f(x)dx \doteq \frac{1}{6}(b-a)\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

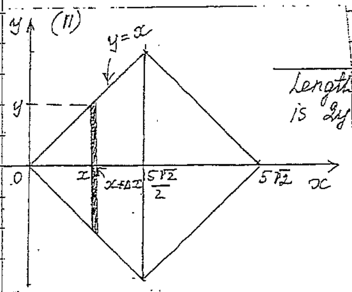


(note that this rule gives exact value for quadratics)

$\therefore$ Area $=$

$$\frac{1}{6}(2a+2a)(0+4\times a+0)$$
$$= \frac{4a}{6}\times 4a = \frac{16a^2}{6}$$
$$= \frac{8a^2}{3}, \text{ as required}$$

(ii)



Length of latus rectum of a cross-section is $2y$ or $2x$, since $y = x$.

From part (i): $4a = 2x \Rightarrow a = \frac{x}{2}$

$$A_{crosssection} = \frac{8a^2}{3} = \frac{8}{3}\left(\frac{x}{2}\right)^2 = \frac{2x^2}{3}$$

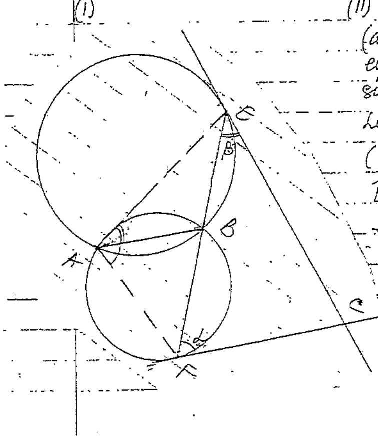
$$\Delta V = \frac{2x^2}{3}\Delta x$$

The solid is symmetrical in its diagonals and the latus rectum of the crossection equals to $2x$ ($2y$) only for $0 < x \le \frac{5\sqrt{2}}{2}$, for $x > \frac{5\sqrt{2}}{x}$ the relationship is

$$\therefore\ V \doteq 2\lim_{\Delta x\to 0}\sum_{x=0}^{\frac{5\sqrt{2}}{2}}\frac{2x^2}{3}\Delta x$$

$$V = 2\int_0^{\frac{5\sqrt{2}}{2}}\frac{2x^2}{3}dx$$
$$= \frac{4}{3}\int_0^{\frac{5\sqrt{2}}{2}}x^2\,dx$$
$$= \frac{4}{3}\left[\frac{x^3}{3}\right]_0^{\frac{5\sqrt{2}}{2}}$$
$$= \frac{4}{3}\times\left(\frac{125\times 2\sqrt{2}}{3\times 8} - 0\right)$$
$$= \frac{1}{3}\times\frac{125\sqrt{2}}{3}$$
$$= \frac{125\sqrt{2}}{9}\text{ units}^3$$

(b) (i)



(ii) Let $\widehat{CFE} = \alpha$, then $\widehat{BAF} = \alpha$

(angle between a tangent and a chord equals to the angle in alternate segment subtended by the chord)

Let $\widehat{CEF} = \beta$, then $\widehat{BAE} = \beta$

(reason as above)

$\widehat{EAF} = \alpha + \beta$ (sum of adjacent angles)

In ΔECF $\widehat{ECF} = 180^\circ - (\alpha+\beta)$

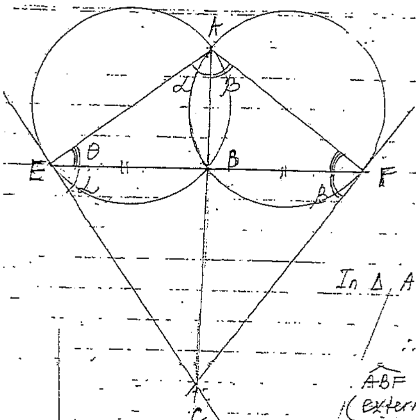
(sum of angles in a triangle is 180°)

In quadrilateral $AECF$

$\widehat{ECF} = 180^\circ - \widehat{EAF}$

$\therefore$ $AECF$ is cyclic

(opposite angles are supplementary)



(iii) In equal circles equal chords subtend equal angles

$\therefore\ \alpha = \beta$

AB subtends same angle in each circle.

Let this angle be θ

In ΔAEF: $2\alpha + 2\theta = 180^\circ$

(angle sum of a triangl. is 180°)

$\therefore\ \alpha + \theta = 90^\circ$

$\widehat{ABF} = \alpha + \theta = 90^\circ$

(exterior angle of a triangle is the sum of two opposite interior angles)

Join BC

in Δ EFC:

$EC = FC$ (sides opposite equal angles are equal, $\alpha = \beta$)

$\therefore\ \Delta BEC \equiv \Delta BFC$ (SAS: $EB = FB$ (given)

$\alpha = \beta$ (proven above)

$EC = FC$ (proven above))

$\widehat{EBC} = \widehat{FBC}$ (corresponding angles in congruent triangles are equal)

But $\widehat{EBC} + \widehat{FBC} = \widehat{EBF}$ (adjacent angles)

and $\widehat{EBF} = 180^\circ$ (given: EBF is a straight line)

$\therefore\ \widehat{FBC} = \frac{180^\circ}{2} = 90^\circ$

$\widehat{ABC} = \widehat{ABF} + \widehat{FBC} = 180^\circ$ (adjacent angles)

$\therefore$ A, B and C are collinear.