St George Girls High School

Trial Higher School Certificate Examination

2013

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen.
- Write your student number on each booklet.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- The mark allocated for each question is listed at the side of the question.
- Marks may be deducted for careless or poorly presented work.

Total Marks – 100

Section I – Pages 2 – 5
10 marks

- Attempt Questions 1 – 10.
- Allow about 15 minutes for this section.
- Answer on the sheet provided.

Section II – Pages 6 – 13
90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.
- Begin each question in a new booklet.
- Show all necessary working in Questions 11 – 16.
- Templates for Q14(b) to be detached and placed in Q14 answer booklet.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Section I

Marks

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1. Let $z = 1 + 2i$ and $w = -2 + i$. What is the value of $\bar{z}\,.\,w$

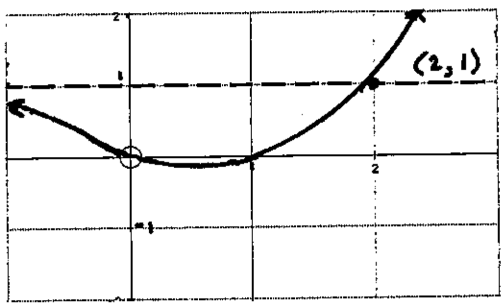
   (A) $5i$

   (B) $-4 + 5i$

   (C) $-3i$

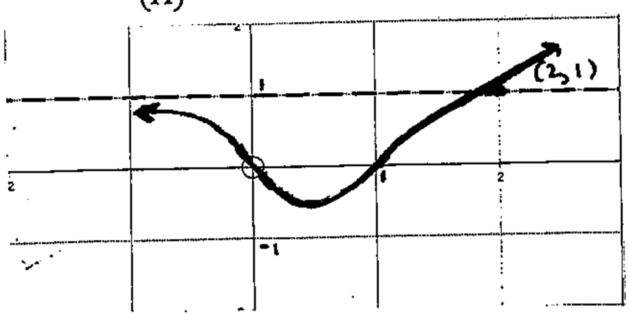
   (D) $-4 - 3i$

2. The graph of $y = f(x)$ is shown below

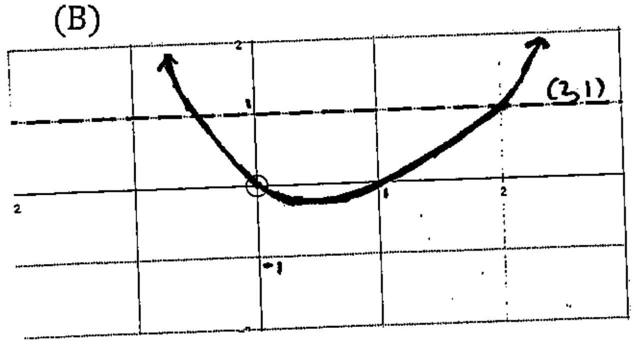


Which of the following is the graph of $y = \sqrt{f(x)}$?

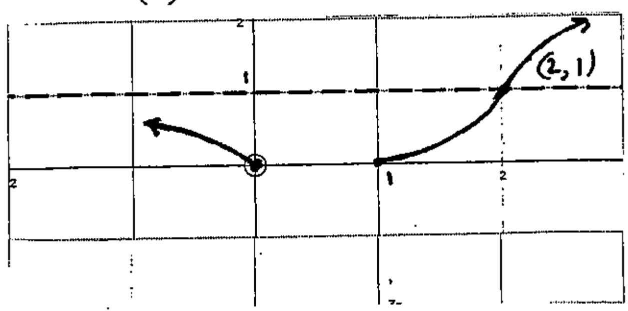
(A)



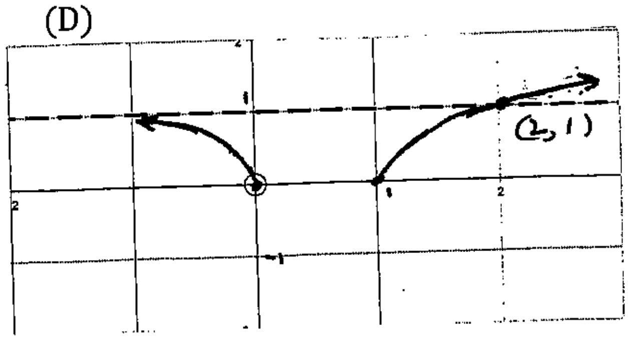
(B)



(C)

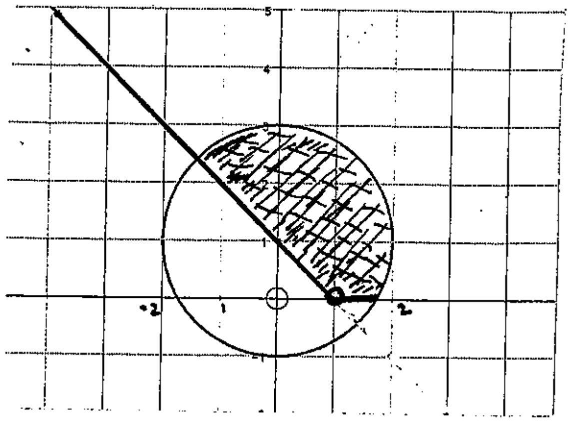


(D)



Section I (cont'd)

3. Consider the Argand diagram below.



Which inequality could define the shaded area?

(A) $|z-i| \leq 2$ and $0 \leq \arg(z-1) \leq \frac{3\pi}{4}$

(B) $|z+i| \leq 2$ and $0 \leq \arg(z-1) \leq \frac{3\pi}{4}$

(C) $|z-i| \leq 2$ and $0 \leq \arg(z-1) \leq \frac{\pi}{4}$

(D) $|z+i| \leq 2$ and $0 \leq \arg(z-1) \leq \frac{\pi}{4}$

4. The points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\emptyset, b\sin\emptyset)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ and the chord PQ subtends a right angle at $(0,0)$. Which of the following is the correct expression?

(A) $\tan\theta.\tan\emptyset = -\frac{b^2}{a^2}$

(B) $\tan\theta.\tan\emptyset = -\frac{a^2}{b^2}$

(C) $\tan\theta.\tan\emptyset = \frac{b^2}{a^2}$

(D) $\tan\theta.\tan\emptyset = \frac{a^2}{b^2}$

Section I (cont'd)

5. Which of the following is an expression for $\int \frac{\sin x . \cos x}{4 + \sin x}\, dx$

Use the substitution $u = 4 + \sin x$

(A) $-4 \ln |4 + \sin x| + C$
(B) $4 \ln |4 + \sin x| + C$
(C) $-\sin x - 4 \ln |4 + \sin x| + C$
(D) $4 + \sin x - 4 \ln |4 + \sin x| + C$

6. The polynomial $P(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$ has $x = 1$ as a root of multiplicity 3 and $x = i$ is a root. Which of the following expressions is a factorised form of $P(x)$ over the complex numbers?

(A) $P(x) = (x + 1)^3 (x - 1)(x + 1)$
(B) $P(x) = (x - 1)^3 (x - 1)(x + 1)$
(C) $P(x) = (x + 1)^3 (x - i)(x + i)$
(D) $P(x) = (x - 1)^3 (x - i)(x + i)$

7. What is the eccentricity of the hyperbola

$$\frac{(x-1)^2}{10} - \frac{(y+1)^2}{4} = 1$$

(A) $\frac{\sqrt{6}}{2}$ B. $\sqrt{\frac{7}{5}}$ C. $\frac{2}{\sqrt{6}}$ D. $\frac{\sqrt{14}}{2}$

8. The polynomial equation $x^3 - 3x^2 - x + 2 = 0$ has roots α, β and γ. Which of the following polynomial equations have roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$

(A) $2x^3 - 2x^2 - 3x + 1 = 0$
(B) $2x^3 - x^2 - 3x + 1 = 0$
(C) $x^3 - 2x^2 - 3x + 1 = 0$
(D) $x^3 - x^2 - 3x + 1 = 0$

Section I (cont'd)

9. A particle of mass m is moving in a straight line under the action of a force

$$F = \frac{m}{x^3}(6 - 10x)$$

What of the following is an expression for its velocity in any position, if the particle starts from rest at $x = 1$?

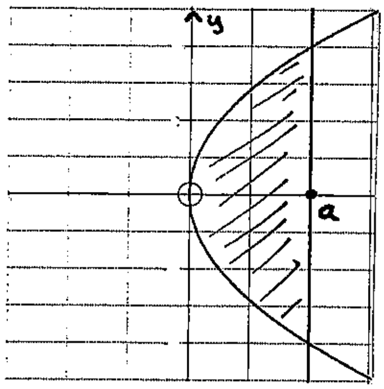
(A) $v = \pm\frac{1}{x}\sqrt{(-3 + 10x - 7x^2)}$

(B) $v = \pm x\sqrt{(-3 + 10x - 7x^2)}$

(C) $v = \pm\frac{1}{x}\sqrt{2(-3 + 10x - 7x^2)}$

(D) $v = \pm x\sqrt{2(-3 + 10x - 7x^2)}$

10. A solid is formed by rotating the region enclosed by the parabola $y^2 = 4ax$, its vertex $(0, 0)$ and the line $x = a$, about the y-axis.



What is the volume of this solid using the method of slicing.

(A) $8\pi a^3$

(B) $\dfrac{16\pi a^3}{5}$

(C) $\dfrac{8\pi a^3}{5}$

(D) $4\pi a^3$

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours 45 minutes for this section

Question 11 – Start A New Booklet – (15 marks) Marks

a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x}\, dx$ using the substitution $\tan\left(\frac{x}{2}\right)=t$ 3

b) Let $z=3-2i$ and $w=1+\sqrt{2}i$

(i) Find $|z|$ 1

(ii) Express $\frac{w}{z}$ in the form $a+bi$ where a and b are real numbers. 2

c) Find $\int \frac{dx}{20-4x+x^2}$ 2

d) (i) Write $z=1+i$ in modulus-argument form. 2

(ii) Hence express z^{-4} in the form $x+yi$, where x and y are real. 2

e) The area bounded by the curve $y=\frac{1}{x+1}$, the x-axis, the line $x=2$ and the line $x=8$, is rotated about the y-axis.

Find the volume of the solid generated using the method of cylindrical shells. 3

Question 12 – Start A New Booklet – (15 marks) Marks

a) Find $\displaystyle\int \frac{x}{(1-x)(1+x^2)}\,dx$ 3

b) Let $f(x) = \dfrac{1-x}{x}$

On <u>separate</u> diagrams sketch the graph of the following functions. For each graph label any asymptotes and critical points.

(i) $y = |f(x)|$ 2

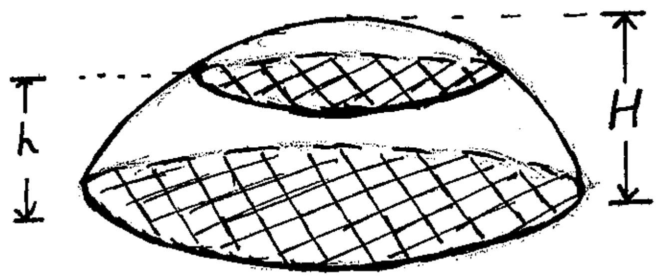
(ii) $y = e^{f(x)}$ 2

(iii) $y^2 = f(x)$ 2

c) (i) Verify that $\displaystyle\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{1}{4}\pi a^2$ 1

(ii) Deduce that the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab 2

(iii)



The diagram shows a mound of height H. At height h above the horizontal base, the horizontal cross-section of the mound is an ellipse with equation 3

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2 \text{ where } \lambda = 1 - \frac{h^2}{H^2}$$

(x, y are appropriate co-ordinates in the plane of the cross-section).

Show that the volume of the mound is $\dfrac{8\pi abH}{15}$

Question 13 – Start A New Booklet – (15 marks) Marks

a) On a school camp, one of the girls of mass M kg jumps vertically (feet first) from a rock ledge into a river below.

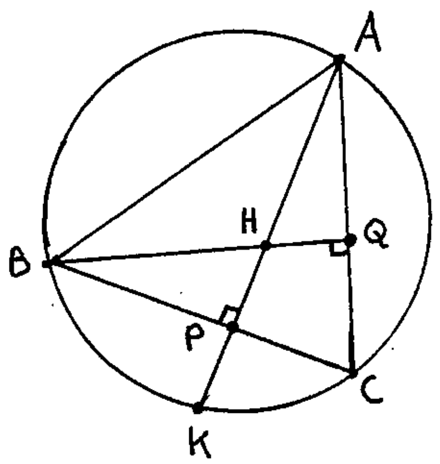
When she is falling at v m/s, she encounters air resistance equal to $\dfrac{Mv}{20}$ Newtons. She hits the water at a speed of V m/s. Let x be the displacement below the rock ledge at time t seconds after jumping.

(i) Show that $\ddot{x} = g - \dfrac{v}{20}$, where g is the acceleration due to gravity. 1

(ii) If it takes two seconds for her feet to hit the water, using $g = 10$ m/s^2 show that $V = 200\left(1 - e^{-\frac{1}{10}}\right)$ 3

(iii) Find the height of the rock ledge above the water (to the nearest 0.1 metre) 3

b)



The altitudes AP and BQ of the acute triangle ABC intersect at H. 3

AP produced cuts the circle at K.

Prove that $HP = PK$

Question 13 (cont'd) Marks

c) Show that the equation of the tangent to the ellipse 1

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \ \ (a>b>0)$$

at the point $P(a\cos\theta,\ b\sin\theta)$ is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

(i) The tangent at P meets the x-axis at S and the y-axis at T.

Find the area of ΔOST. 2

(ii) If A is the point $(a,0)$ and B is the point $(0,b)$, the area of ΔAPB is $\frac{1}{2}ab(\cos\theta+\sin\theta-1)$ [do not prove this]

Prove that, as θ varies in the interval $0<\theta<\frac{\pi}{2}$, the area of ΔAPB is a maximum when the tangent to the ellipse is parallel to AB. 2

Question 14 – Start A New Booklet – (15 marks) Marks

a) On an Argand diagram, sketch the locus of the points z such that 2

$$\arg\left\{\frac{(z-1)}{(z+1)}\right\} = \frac{\pi}{2}$$

b) $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are two points on the rectangular hyperbola $xy = c^2$.

Given that the equation of the tangent at P is

$$x + p^2 y = 2cp.$$

(i) The tangents at P and Q meet in T. Find the co-ordinates of T in terms of c, p and q. 2

(ii) If T lies on the hyperbola $xy = k^2$ for all positions of P and Q, prove that 1

$$\frac{pq}{(p+q)^2} = \frac{k^2}{4c^2}$$

c) (i) Given $z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1)$ let w be a solution to $z^5 + 1 = 0$ where $w \neq -1$. Prove that $1 + w^2 + w^4 = w + w^3$ 1

(ii) Hence show that 3

$$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$$

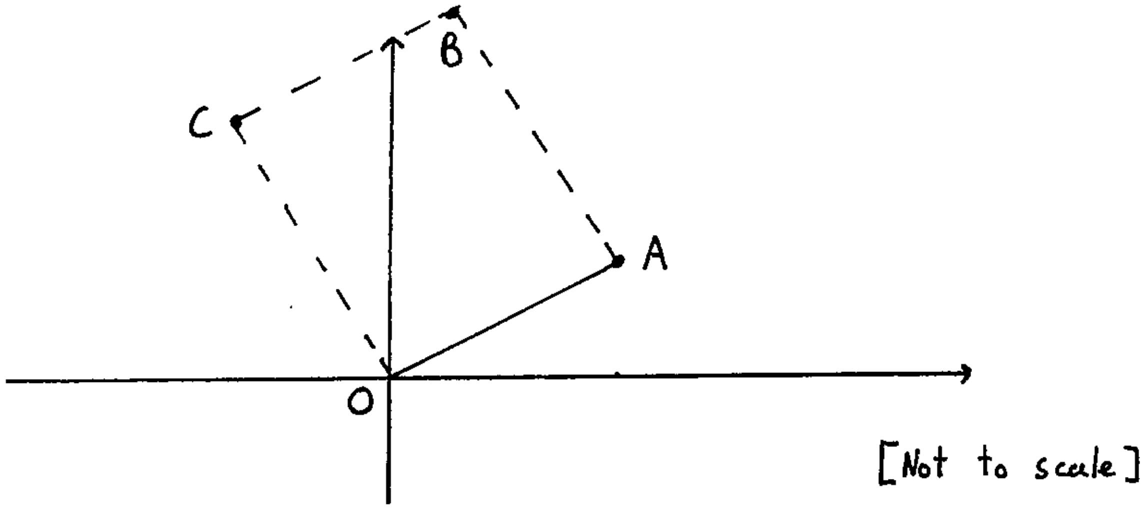
Question 14 (cont'd) Marks

d) (i) If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$ where $n \geq 1$, use integration by parts to prove that 3

$$I_n = \frac{n-1}{n}\, I_{n-2}$$

(ii) Hence show that $I_5 = \frac{8}{15}$ 1

e) $OABC$ is a rectangle in an Argand diagram where O is the Origin and point A corresponds to the complex number $2 + i$



Given that the length of the rectangle is twice its breadth and OA is one of the shorter sides, find the complex number representing C. 2

Question 15 – Start A New Booklet – (15 marks) Marks

a) Suppose α, β and γ are the roots of the polynomial equation

$$x^3 + x + 12 = 0$$

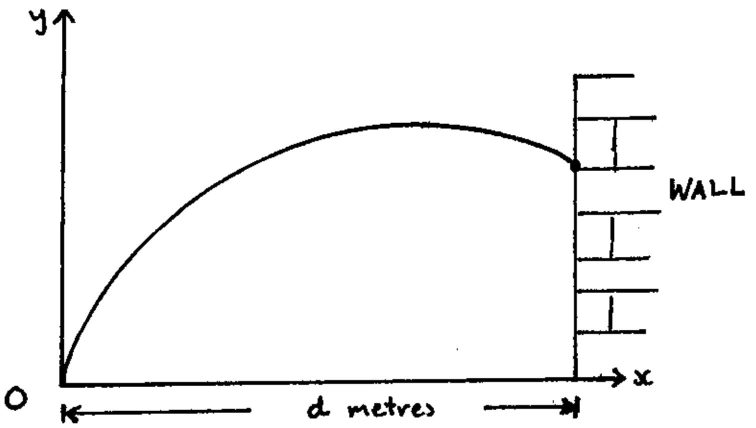
(i) Find $\alpha^2 + \beta^2 + \gamma^2$ 2

(ii) Hence explain why only one of the roots is real. 2

(iii) Let the real root be denoted by α. Prove that $-3 < \alpha < -2$ 1

(iv) Hence prove that the modulus of each of the other roots lies between 2 and $\sqrt{6}$ 3

b) 4



In the diagram above, the wall of a building stands on level ground, d metres from a fire hose located at O If water leaves the hose with velocity $V\text{ms}^{-1}$ at an angle θ to the ground:

Given $V > \sqrt{gd}$, (g is acceleration in the vertical plane due to gravity) show that the particle will strike the wall above ground level provided that $\beta < \theta < \frac{\pi}{2} - \beta$ where $\beta = \frac{1}{2}\sin^{-1}\left(\frac{gd}{V^2}\right)$

You may assume that the range on the horizontal plane from the point of projection is

$$\frac{V^2 \sin 2\theta}{g}$$

c) Find $\int \frac{dx}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$ 3

[Hint: • choose an appropriate substitution]
• $\frac{x^3}{x+1} = x^2 - x + 1 - \frac{1}{x+1}$]

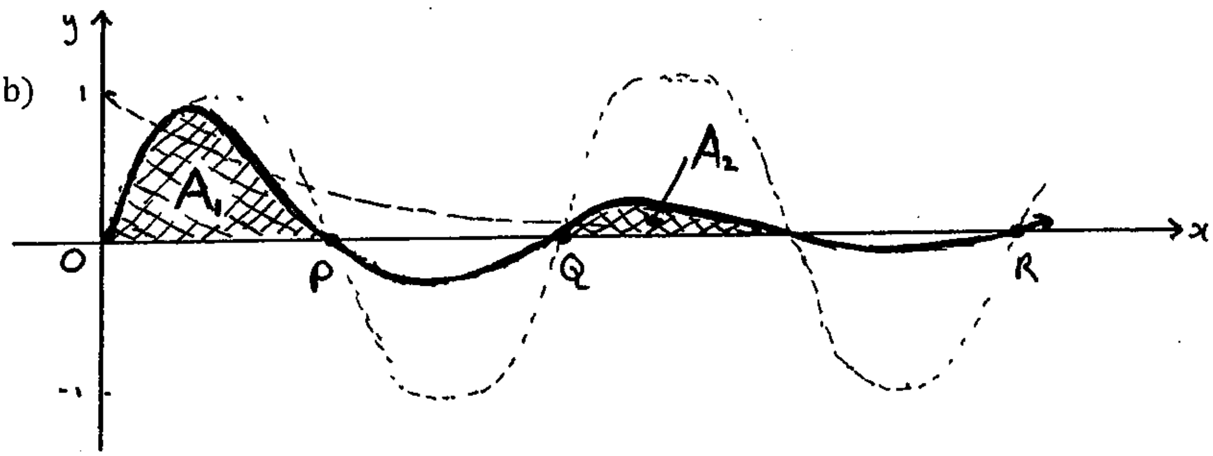
Question 16 – Start A New Booklet – (15 marks) Marks

a) A polynomial of degree n is given by $P(x) = x^n + ax - b$. It is given that the polynomial has a double root at $x = \alpha$.

(i) Find the derived polynomial $P'(x)$ and show that $\alpha^{n-1} = -\frac{a}{n}$ 2

(ii) Show that $\left(\frac{a}{n}\right)^n + \left(\frac{b}{n-1}\right)^{n-1} = 0$ 4

b)



The diagram shows a sketch of part of the curve $y = f(x)$ with equation

$$y = e^{-x}.\sin x, \qquad x \geq 0$$

(i) Find the coordinates of the points P, Q and R where $y = f(x)$ cuts the x-axis. 1

(ii) Integrating by parts gives 3

$$\int e^{-x}.\sin x\, dx = -\frac{1}{2}e^{-x}(\sin x + \cos x) + C$$

[you do not have to prove this]

If the terms $A_1, A_2, \ldots, A_n$ represent areas between $y = f(x)$ and the x-axis for successive portions of $y = f(x)$ where y is positive. (The areas represented by A_1 and A_2 are shown as the shaded regions in the diagram).

Show that $A_n = \frac{1}{2}\left(e^{(1-2n)\pi} + e^{(2-2n)\pi}\right)$

(iii) Show that $A_1 + A_2 + A_3 + \cdots$ is a geometric series and that 3

$$S_\infty = \frac{e^\pi}{2(e^\pi - 1)}$$

(iv) Given that $\int_0^\infty e^{-x}.\sin x\, dx = \frac{1}{2}$, find the exact value of $\int_0^\infty |e^{-x}.\sin x|\, dx$ 2

End of Paper

Student Number: ____________________________ Teacher: __________________

Year 12 Mathematics Extension 2 Trial HSC Examination 2013

Section I

Multiple-choice Answer Sheet – Questions 1 – 10

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $\quad 2+4 =$ (A) 2 $\quad$ (B) 6 $\quad$ (C) 8 $\quad$ (D) 9

A ○ $\quad$ B ● $\quad$ C ○ $\quad$ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● $\quad$ B ~~●~~ $\quad$ C ○ $\quad$ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

correct

A ~~●~~ $\quad$ B ~~●~~ $\quad$ C ○ $\quad$ D ○

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

SECTION I

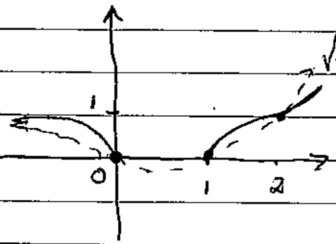
1/ $z = 1+2i \qquad w = -2+i$

$\bar{z} = 1-2i \qquad \bar{z}.w = (1-2i)(-2+i)$

$= -2+5i-2i^2$

$= 5i$ (A)

2/



$\sqrt{f(x)}$ defined only for $f(x) \geq 0$.

where $f(x) > 1$, $\sqrt{f(x)} < f(x)$

where $f(x) = 1$, $\sqrt{f(x)} = f(x)$

where $f(x) < 1$, $\sqrt{f(x)} > f(x)$

(D)

3/ $x^2 + (y-1)^2 \leq 4$

$|z - i| \leq 2$ and $0 \leq \arg(z-1) \leq \frac{3\pi}{4}$ (A)

4/ $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$

lie on $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ chord PQ subtends a right angle at $(0,0)$

$m_1 \times m_2 = -1$

$\frac{b\sin\theta}{a\cos\theta} \times \frac{b\sin\phi}{a\cos\phi} = -1$

$\frac{b^2}{a^2}\tan\theta \times \tan\phi = -1 \qquad \therefore \tan\theta\tan\phi = -\frac{a^2}{b^2}$ (B)

5/ $\int \frac{\sin x.\cos x}{4+\sin x}\,dx. \qquad u = 4+\sin x \quad \therefore \sin x = u-4$

$\frac{du}{dx} = \cos x. \qquad \therefore du = \cos x\,dx$

$\int \frac{(u-4)}{u}\,du$

$= \int 1 - \frac{4}{u}\,du$

$= u - 4\ln u + c$

$= 4+\sin x - 4\ln|4+\sin x| + c$ (D)

A
D
A
B
D
D
B
B
C
B

6. $P(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$

$= (x-1)^3(x-i)(x+i)$

$(-1)^3 \times (-i) \times (i) = -1 \times -i^2$

$= -1$ (D)

7. $\frac{(x-1)^2}{10} - \frac{(y+1)^2}{4} = 1$ hyperbola $\qquad b^2 = 4 \quad a^2 = 10$

$e = \sqrt{1 + \frac{b^2}{a^2}}$

$= \sqrt{\frac{14}{10}}$

$= \sqrt{7/5}$ (B)

8. $x^3 - 3x^2 - x + 2 = 0$

$x = \frac{1}{\alpha}, \quad \alpha = \frac{1}{x} \qquad (1/x)^3 - 3(1/x)^2 - (1/x) + 2 = 0$

$\frac{1}{x^3} - \frac{3}{x^2} - \frac{1}{x} + 2 = 0$

$1 - 3x - x^2 + 2x^3 = 0$ (B)

9. $F = \frac{m}{x^3}(6-10x) \qquad F = ma$

$a = 6x^{-3} - 10x^{-2}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6x^{-3} - 10x^{-2}$

$\frac{1}{2}v^2 = \int 6x^{-3} - 10x^{-2}\,dx$

$\frac{1}{2}v^2 = -3x^{-2} + 10x^{-1} + c$

when $x=1$, $v=0 \quad \therefore c = -7$

$\frac{1}{2}v^2 = -3x^{-2} + 10x^{-1} - 7$

$v^2 = 2(-3x^{-2} + 10x^{-1} - 7)$

$= \frac{2}{x^2}(-3 + 10x - 7x^2)$

$v = \pm\frac{1}{x}\sqrt{2(-3+10x-7x^2)}$ (C)

QUESTION 11

(a) Let $t = \tan\left(\frac{x}{2}\right)$

then $\tan^{-1}(t) = \frac{x}{2}$

$\therefore \frac{2\,dt}{1+t^2} = dx$

When $x = 0$, $t = 0$

$x = \frac{\pi}{2}$, $t = 1$

Gives on substitution

$$\int_0^1 \frac{1}{3 + 5\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2\,dt}{(1+t^2)}$$

$$= \int_0^1 \frac{2\,dt}{3(1+t^2) + 5(1-t^2)}$$

$$* = \int_0^1 \frac{dt}{4-t^2}$$

$*$ Now: $\frac{1}{4-t^2} = \frac{A}{2-t} + \frac{B}{2+t}$

So $1 = 2(A+B) + t(A-B)$

$A - B = 0$ ... (I)

$A + B = \frac{1}{2}$ ... (II)

(I) + (II) $2A = \frac{1}{2}$

$A = \frac{1}{4}$, $B = \frac{1}{4}$

$$= \int_0^1 \frac{\frac{1}{4}}{(2-t)} + \frac{\frac{1}{4}}{(2+t)}\,dt$$

$$= \frac{1}{4}\Big[-\ln(2-t) + \ln(2+t)\Big]_0^1$$

$$= \frac{1}{4}\left[\ln\left[\frac{2+t}{2-t}\right]\right]_0^1$$

$$= \frac{1}{4}(\ln 3 - \ln 1)$$

$$= \frac{1}{4}\ln 3$$

(b) (i) $|z| = \sqrt{3^2 + (-2)^2}$

$= \sqrt{13}$

(ii) $\frac{w}{z} = \frac{1+\sqrt{2}\,i}{3-2i} \times \left(\frac{3+2i}{3+2i}\right)$

$$= \frac{3 + 2i + 3\sqrt{2}\,i - 2\sqrt{2}}{3^2 + 4}$$

$$= \frac{3 - 2\sqrt{2}}{13} + \frac{2 + 3\sqrt{2}}{13} i$$

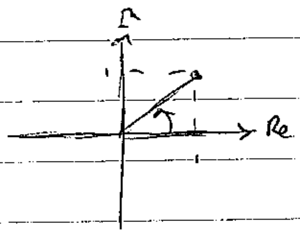
(c) $$\int \frac{dx}{20 - 4x + x^2} = \int \frac{dx}{x^2 - 4x + 4 + 16}$$

$$= \int \frac{dx}{(x-2)^2 + 4^2}$$

$$= \frac{1}{4} \tan^{-1}\left(\frac{x-2}{4}\right) + C$$

(d) (i) <u>Mod</u> $|z| = \sqrt{1^2 + 1^2}$ and <u>Arg</u> $\arg(z) = \tan^{-1}\left(\frac{1}{1}\right)$

$$= \frac{\pi}{4}$$



$$= \sqrt{2}$$

$$\therefore \quad z = r \operatorname{cis} \theta$$

$$= \sqrt{2} \operatorname{cis} \frac{\pi}{4}$$

$$= \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$$

(ii) Then $$z^{-4} = \left(\sqrt{2} . \operatorname{cis} \frac{\pi}{4}\right)^{-4}$$

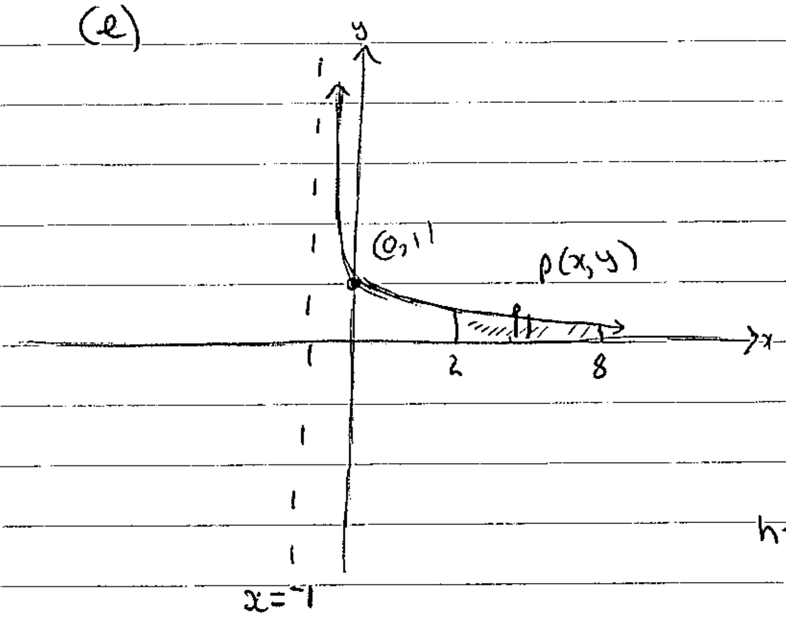
$$= 2^{-2} . \operatorname{cis}\left(-4 \times \frac{\pi}{4}\right) \quad \text{(by De Moivre's Theorem)}$$

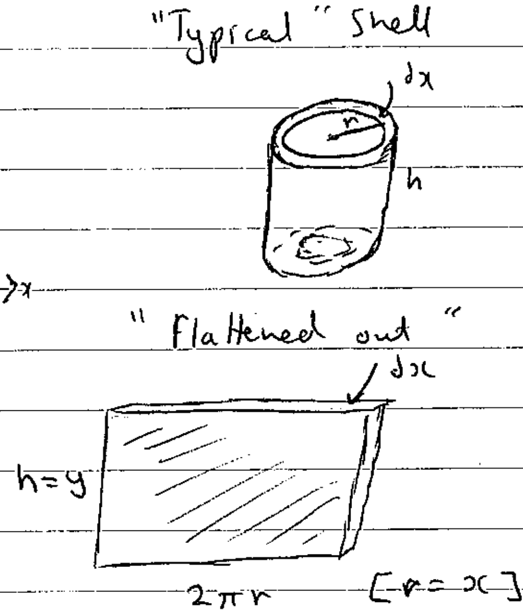
$$= \frac{1}{4} \left(\cos(-\pi) + i \sin(-\pi)\right)$$

$$= \frac{1}{4} (-1 - 0.i)$$

$$= -\frac{1}{4} \quad \left(\text{or } -\frac{1}{4} + 0.i\right)$$

(e)





So $\delta V = A \,.\, dx$

$= 2\pi x \,.\, y \,.\, dx$

$= 2\pi x \,.\, \dfrac{1}{x+1} \, dx$

Now $V = \lim\limits_{dx \to 0} \sum\limits_{x=2}^{8} 2\pi \,.\, \dfrac{x}{x+1} \, dx$

$= 2\pi \displaystyle\int_2^8 \frac{x}{x+1} \, dx$

$= 2\pi \displaystyle\int_2^8 \frac{x+1}{x+1} - \frac{1}{x+1} \, dx$

$= 2\pi \left[x - \ln|(x+1)| \right]_2^8$

$= 2\pi \left[(8 - \ln(9)) - (2 - \ln(3)) \right]$

$= 2\pi \left(6 - \ln\left(\frac{9}{3}\right) \right)$

$= 2\pi (6 - \ln 3) \; u^2$

QUESTION 12

(a) Now, partial fractions gives

$$\frac{x}{(1-x)(1+x^2)} \equiv \frac{A}{(1-x)} + \frac{Bx+C}{(1+x^2)}$$

$$x = A + Ax^2 + Bx - Bx^2 + C - Cx$$

$$= (A-B)x^2 + (B-C)x + (A+C)$$

Gives: $A - B = 0$ -- (I)

$B - C = 1$ -- (II)

$A + C = 0$ --- III

$\therefore$ $A = B$

so $A - C = 1$

$+\ A + C = 0$

$2A = 1$

$A = \frac{1}{2}$

Have $A = \frac{1}{2}$, $B = \frac{1}{2}$, $C = -\frac{1}{2}$

Integral

$$* \int \frac{x}{(1-x)(1+x^2)}\,dx$$

$$= \frac{1}{2}\int\left(\frac{1}{1-x} + \frac{x-1}{1+x^2}\right)dx$$

$$= \frac{1}{2}\int\left(\frac{1}{1-x} + \frac{x}{1+x^2} - \frac{1}{1+x^2}\right)dx$$

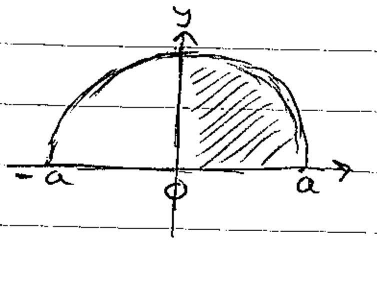
$$= \frac{1}{2}\left[-\ln|(1-x)| + \frac{1}{2}\ln(1+x^2) - \tan^{-1}x\right] + C$$

$*$ $x < 1$

$$= \frac{1}{2}\ln\left[\frac{(1+x^2)^{\frac{1}{2}}}{|1-x|}\right] - \frac{1}{2}\tan^{-1}x + C$$

(b) See attached Sheet.

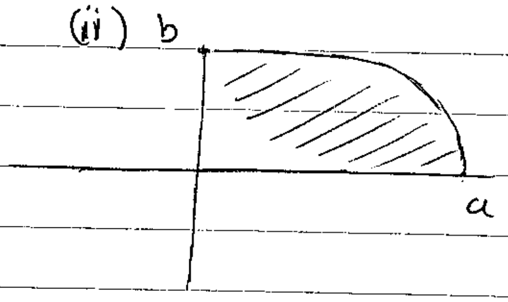
(c) (i) $\int_0^a \sqrt{a^2 - x^2}\,dx$ describes the area of a quadrant of the circle radius 'a' units, centre (0,0)



So, $A = \frac{1}{4}\pi r^2$

Gives $\int_0^a \sqrt{a^2-x^2}\,dx = \frac{1}{4}\pi a^2$

(ii)



Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

gives $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$

Area of one quadrant is

$$= \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

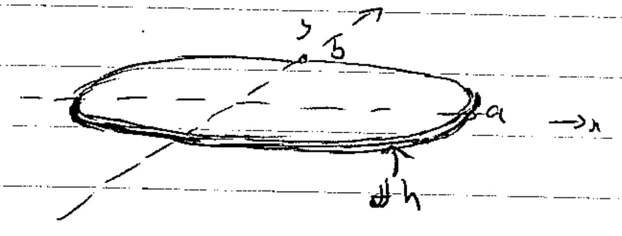
$$= \frac{b}{a}\cdot\frac{1}{4}\pi a^2 \quad [\text{from (i)}]$$

$$= \frac{1}{4}\pi a b.$$

$\therefore$ Area of four quadrants $4 \times \frac{1}{4}\pi a b$

$$= \pi a b.$$

(iii) "Typical" cross section is an ellipse in the $x-y$ plane



Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2$

then $\frac{x^2}{\lambda^2 a^2} + \frac{y^2}{\lambda^2 b^2} = 1$

and from part (ii) Area $= \pi\,(a\lambda).(b\lambda)$

$$= \pi a b \lambda^2$$

<u>So</u> $\delta V = A.\delta h$

$$V = \lim_{\delta h \to 0} \sum_0^H \pi . ab\lambda^2 . \delta h$$

$$= \int_0^H \pi ab \, \lambda^2 \, dh$$

$$= \pi ab \int_0^H \left[1 - \frac{h^2}{H^2} \right]^2 dh$$

$$= \pi ab \int_0^H \left(1 - 2\frac{h^2}{H^2} + \frac{h^4}{H^4} \right) dh$$

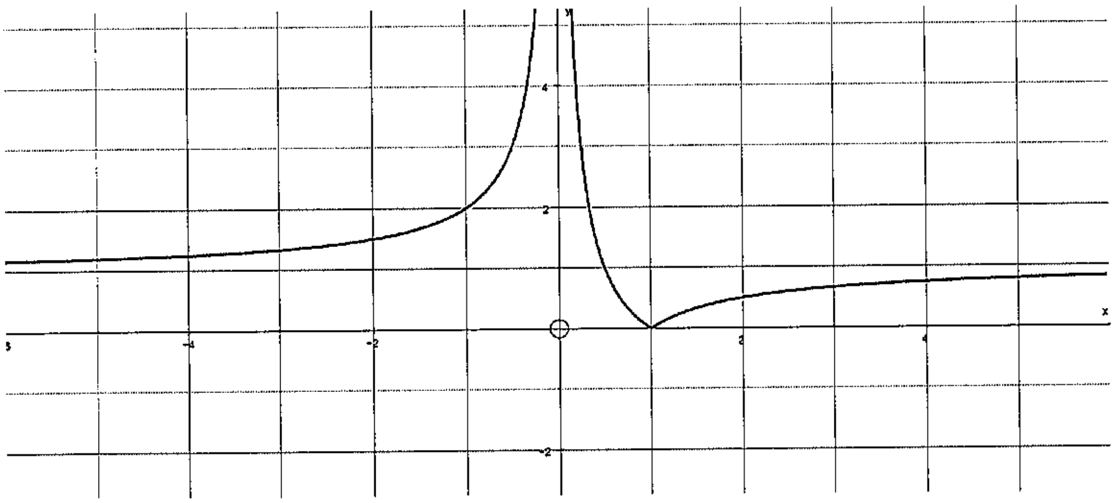
$$= \pi ab \left[h - \frac{2}{3} \cdot \frac{h^3}{H^2} + \frac{1}{5} \cdot \frac{h^5}{H^4} \right]_0^H$$

$$= \pi ab \left(H - \frac{2}{3} \cdot H + \frac{1}{5} \cdot H \right)$$

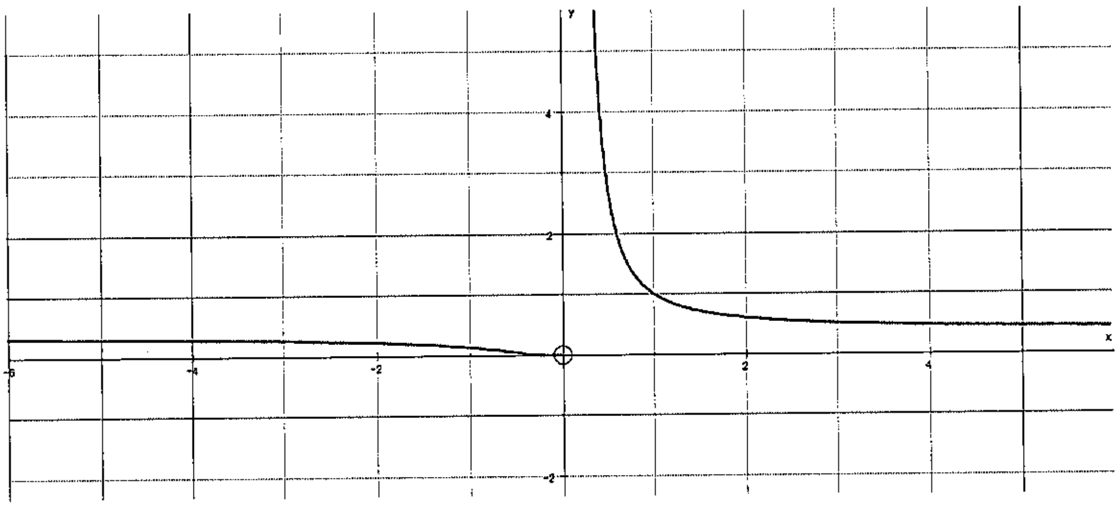
$$V = \frac{8 \pi ab H}{15} \; u^3$$

Graphs

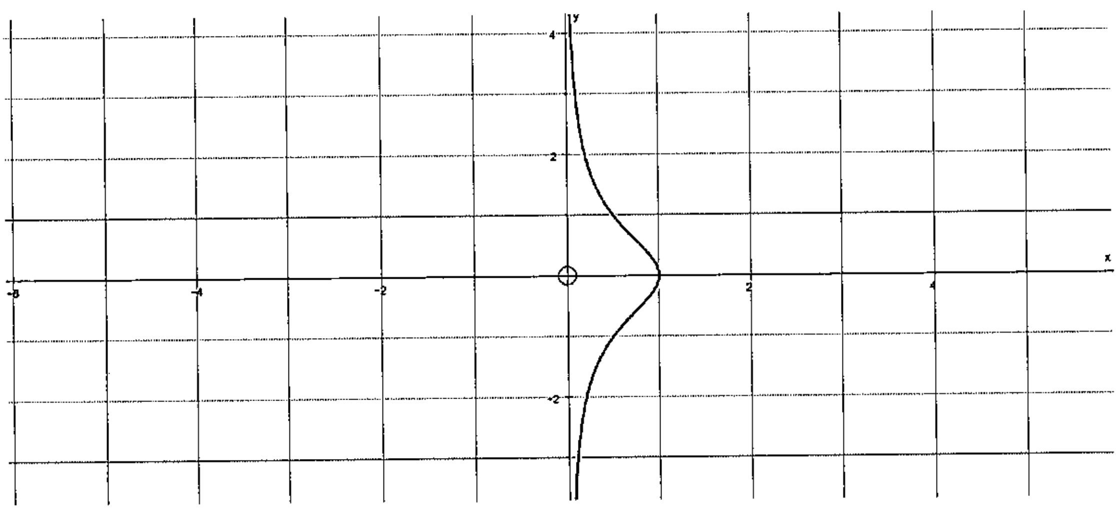
(i). $y = \left|\frac{1-x}{x}\right|$



(ii). $y = e^{\frac{1-x}{x}}$

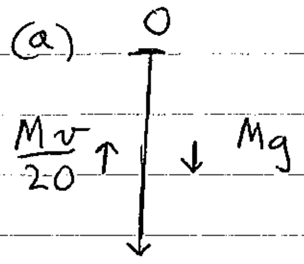


(iii). $y^2 = \frac{1-x}{x}$



QUESTION 13

(a)



Direction diagram

(i) $F = Mg - \frac{Mv}{20}$

$\therefore M\ddot{x} = Mg - \frac{Mv}{20}$

$\ddot{x} = g - \frac{v}{20}$

(ii) $\ddot{x} = \frac{dv}{dt} = \frac{20g - v}{20}$

$\frac{dt}{dv} = \frac{20}{20g - v}$

Gives $\frac{dt}{20} = \frac{dv}{20g - v}$

$\frac{t}{20} = -\ln(20g - v) + C$

at $t = 0$, $v = 0$ so $C = \ln(20g)$

Then $\frac{t}{20} = -\ln\left[\frac{20g - v}{20g}\right]$

$e^{-\frac{t}{20}} = \frac{20g - v}{20g}$

$20g\,e^{-\frac{t}{20}} = 20g - v$

★ $v = 20g\left(1 - e^{-\frac{t}{20}}\right)$

at $t=2$ gives $V = 20g(1-e^{-\frac{1}{10}})$
$v=V$

(iii) Also $\ddot{x} = v \cdot \frac{dv}{dx} = \frac{20g - v}{20}$

$$\frac{dv}{dx} = \cancel{20} \cdot \frac{g}{v} - \frac{1}{20}$$

$$= \frac{20g - v}{20v}$$

so $$\frac{dx}{dv} = \frac{20v}{20g-v}$$

$$dx = \frac{20v}{20g-v} \cdot dv$$

$$\therefore \quad x = -20 \int \frac{20g - v - 20g}{20g-v} \, dv$$

$$= -20 \int \frac{20g-v}{20g-v} - \frac{20g}{20g-v} \, dv$$

Gives $$x = -20 \int 1 - \frac{20g}{20g-v} \, dv$$

$$= -20 \left[v + 20g \ln(20g - v) \right] + C$$

at $x=0$, $v=0$ so $C = 20 [20g \ln 20g]$

$$\therefore \quad x = -20 \left[v + 20g \ln \left[\frac{20g-v}{20g} \right] \right]$$

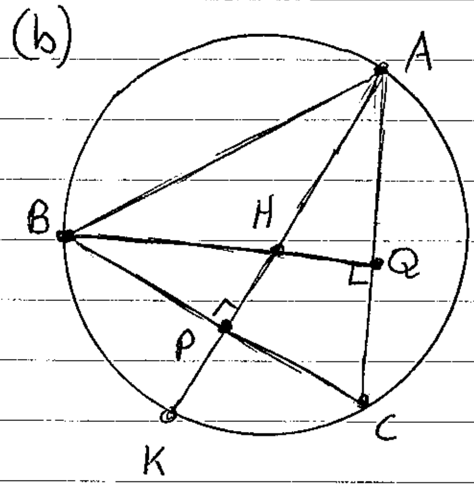
at $t=2$, $x=h$ and $v = V = 20g(1-e^{-\frac{1}{10}})$

We have $h = -20 \left[20g(1-e^{-\frac{1}{10}}) + 20g \ln(e^{-\frac{1}{10}}) \right]$

($g=10$) $$= -20 \left[200(1-e^{-\frac{1}{10}}) - 20 \right]$$

$\simeq$ 19.3 metres is height rock ledge.

(b)



Let $B\hat{K}A = \theta$

then $A\hat{C}B = \theta$
[angles at the circumference standing on the same arc are equal]

and $P\hat{A}C = 90° - \theta$
[angle sum of ΔAPC]

$\therefore$ $K\hat{B}C = 90° - \theta$ [angles at circumference standing on the same arc are equal]

gives $Q\hat{B}C = 90 - \theta$ [angle sum of ΔQBC]

Then, $B\hat{P}K = 90° = B\hat{P}H$ [given in data]
$K\hat{B}P = 90 - \theta = H\hat{B}C$ [from above]
PB is common

$\Delta KBP \equiv \Delta HBP$ by congruency test AAS

In congruent Δ's corresponding sides are equal, $HP = PK$

(c) Gradient function $\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$

$$\therefore \quad \frac{dy}{dx} = \frac{-b^2x}{a^2y}$$

at $P(a.\cos\theta, b\sin\theta)$

$$m = \frac{-b^2 a\cos\theta}{a^2 b\sin\theta} = \frac{-b\cos\theta}{a\sin\theta}$$

So Equation of tangent

$$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$\frac{y\sin\theta}{b} - \sin^2\theta = \frac{-x\cos\theta}{a} + \cos^2\theta$$

$$\therefore \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = \sin^2\theta + \cos^2\theta = 1$$

(i) Let $y = 0$, then $\frac{x\cos\theta}{a} = 1$

$$x = \frac{a}{\cos\theta}$$

$S\left(\frac{a}{\cos\theta}, 0\right)$

and $x = 0$, then $\frac{y\sin\theta}{b} = 1$

$$y = \frac{b}{\sin\theta}$$

$T\left(0, \frac{b}{\sin\theta}\right)$

$\therefore$ Area of $\triangle OST = \frac{1}{2} \cdot \frac{a}{\cos\theta} \cdot \frac{b}{\sin\theta}$

$$= \frac{ab}{\sin 2\theta}.$$

(ii) Let area be L for $\triangle APB$

then $\frac{dL}{d\theta} = \frac{1}{2}ab\left[-\sin\theta + \cos\theta\right]$

and $\frac{d^2L}{d\theta^2} = \frac{1}{2}ab\left[-\cos\theta - \sin\theta\right]$

$$= -\frac{1}{2}ab\left[\cos\theta + \sin\theta\right]$$

if $ab > 0$ and $0 \leq \theta \leq \frac{\pi}{2}$

$\cos\theta + \sin\theta \geq 0$

$\therefore$ concave down.

Let $\frac{dh}{d\theta} = 0$, $\Rightarrow$ $-\sin\theta + \cos\theta = 0$ $[ab \neq 0]$

$\sin\theta = \cos\theta$

$\tan\theta = 1$ $[\theta \neq \frac{\pi}{2}]$

$\theta = \frac{\pi}{4}$

Maximum Value at $\theta = \frac{\pi}{4}$

So $P\left(a\cos\frac{\pi}{4}, b\sin\frac{\pi}{4}\right) = P\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$

Gradient of tangent ; Gradient of AB

$m = -\dfrac{b^2 \frac{a}{\sqrt{2}}}{a^2 \frac{b}{\sqrt{2}}}$ $\quad m_{AB} = \dfrac{b-0}{0-a}$

$= -\frac{b}{a}$

$= -\frac{b}{a}$

Gradient of tangent equals gradient of AB, lines parallel for maximum area.

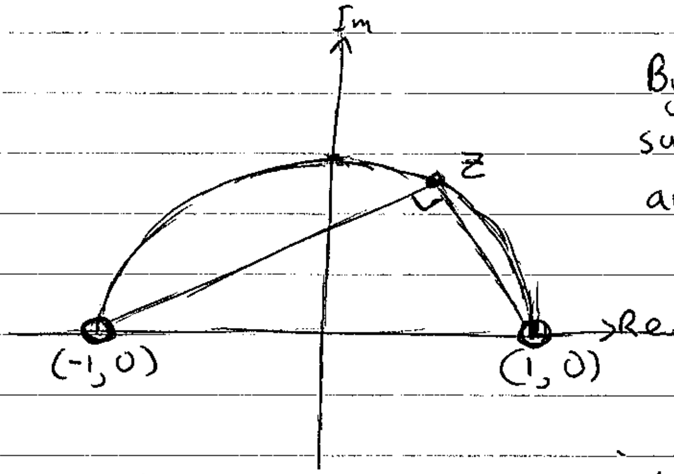
QUESTION 14

(a) $\arg\left(\frac{z-1}{z+1}\right) = \arg(z-1) - \arg(z+1)$

So $\arg(z-1) - \arg(z+1) = \frac{\pi}{2}$ (data)

$\therefore \arg(z-1) = \frac{\pi}{2} + \arg(z-(-1))$

Let $\arg(z-1) = \theta$
$\arg(z+1) = \beta$ $\Rightarrow$ $\theta = \frac{\pi}{2} + \beta$



By exterior angle equals sum of opposite interior angles $\Rightarrow$ z moves on a semi-circle [radius 1] above the x-axis.

[* angle in a semi-circle is a right angle].

(b) (i) Data: tangent at P $\Rightarrow$ $x + p^2 y = 2cp$ ... (I)
by symmetry at Q $\Rightarrow$ $x + q^2 y = 2cq$ ... (II)

(I) - (II) $\Rightarrow$ $(p^2 - q^2)y = 2c(p-q)$

$$y = \frac{2c}{p+q}$$

Substitute $x + p^2 \cdot \frac{2c}{p+q} = 2cp$

$$x = 2cp\left(1 - \frac{p}{p+q}\right)$$

$$x = \frac{2cp}{p+q}(p+q-p)$$

$$x = \frac{2cpq}{p+q}$$

$$\therefore \quad T \left(\frac{2cpq}{p+q}, \frac{2c}{p+q} \right)$$

(ii) By substitution in $xy = k^2$

$$\cancel{\frac{pq}{(p+q)^2}} \implies \frac{2cpq}{p+q} \times \frac{2c}{p+q} = k^2$$

$$\frac{4c^2pq}{(p+q)^2} = k^2$$

$$\frac{pq}{(p+q)^2} = \frac{k^2}{4c^2}$$

(c) (i) If ω is a solution and $\omega \neq 1$
then $\omega + 1 \neq 0$

Given $\omega^5 + 1 = 0$
then $(\omega+1)(\omega^4 - \omega^3 + \omega^2 - \omega + 1) = 0$

$\omega + 1 \neq 0 \quad \therefore \quad \omega^4 - \omega^3 + \omega^2 - \omega + 1 = 0$

$$\omega^4 + \omega^2 + 1 = \omega^3 + \omega^2$$

(ii) Let $z = r \operatorname{cis} \theta$

now $|z| = r$, $|z^5| = |z|^5 = 1$

So $r = 1$

gives $z = \cos\theta + i \sin\theta$

$$z^5 = \cos 5\theta + i \sin 5\theta \quad [\text{De Moivre}]$$

Equating $\cos 5\theta + i\sin 5\theta = -1 + 0.i$

then $\cos 5\theta = -1$ and $\sin 5\theta = 0$

$$\therefore \quad 5\theta = \pm\pi,\ \pm 3\pi,\ \pm 5\pi \ldots$$

$$\theta = \pm\frac{\pi}{5},\ \pm\frac{3\pi}{5},\ \pm\pi,\ \pm\frac{7\pi}{5},\ \pm\frac{9\pi}{5} \ldots$$

(i) $\theta = \pi$, $\cos\pi + i\sin\pi = -1$

(ii) $\theta = \frac{\pi}{5}$, $\cos\frac{\pi}{5} + i\sin\frac{\pi}{5} = \omega$

(iii) $\theta = -\frac{\pi}{5}$, $\cos -\frac{\pi}{5} + i\sin -\frac{\pi}{5} = \bar{\omega} = \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$

(iv) $\theta = \frac{3\pi}{5}$, $\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} = \omega^3$

(v) $\theta = -\frac{3\pi}{5}$, $\cos -\frac{3\pi}{5} + i\sin \frac{3\pi}{5} = \bar{\omega}^3 = \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}$

Sum of roots for $z^4 - z^3 + z^2 - z + 1 = 0$

is
$$\omega + \bar{\omega} + \omega^3 + \bar{\omega}^3 = 1$$
$$2\left(\cos\frac{\pi}{5}\right) + 2\cos\left(\frac{3\pi}{5}\right) = 1$$

$$\cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right) = \frac{1}{2}$$

(d) (i) Given $I_n = \int_0^{\frac{\pi}{2}} \sin^n x . dx \qquad n \geq 1$

$$= \int_0^{\frac{\pi}{2}} \sin^{n-2} x . \sin^2 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^{n-2} x \left(1 - \cos^2 x\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx - \int_0^{\frac{\pi}{2}} \cos^2 x . \sin^{n-2} x \, dx$$

$$I_n = I_{n-2} - \int_0^{\frac{\pi}{2}} \underbrace{\cos x}_{u} \underbrace{(\cos x . \sin^{n-2} x)\, dx}_{dv}$$

$$= I_{n-2} - \left[\left(\cos x . \frac{1}{n-1} \sin^{n-1} x\right)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\sin x . \frac{1}{n-1} \sin^{n-1} x\, dx \right]$$

$$= I_{n-2} - \left[0 + \frac{1}{n-1} \int_0^{\frac{\pi}{2}} \sin^n x\, dx \right]$$

<u>Now</u> $I_n = I_{n-2} - \frac{1}{n-1} I_n$

$$I_n \left(1 + \frac{1}{n-1}\right) = I_{n-2} \qquad \left[1 + \frac{1}{n-1} = \frac{n-1+1}{n-1} \right.$$

$$\frac{n}{n-1} I_n = I_{n-2} \qquad \left. = \frac{n}{n-1} \right]$$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

(ii) From (i) $I_5 = \frac{4}{5} I_3$

$$I_3 = \frac{2}{3} I_1$$

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x\, dx$$

$$= \left[-\cos x \right]_0^{\frac{\pi}{2}}$$

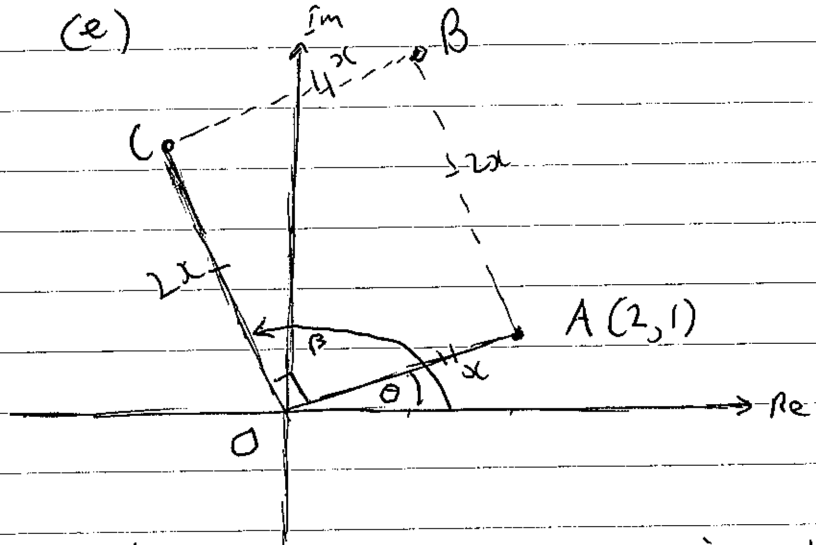
$$= -(0-1)$$

$$= 1$$

Then $I_5 = \frac{4}{5}\left(\frac{2}{3} \times 1\right)$

$$= \frac{8}{15}$$

(e)



For A, $|2+i| = \sqrt{5}$

$\therefore OA = \sqrt{5}$ and $OC = 2\sqrt{5}$

Now, OC is given by rotating OA through $\frac{\pi}{2}$ and doubling the length

So $\beta = \arg(OC) = \arg(2+i) + \frac{\pi}{2}$

$= \arg(2+i) + \arg(ki)$

$= \arg[k(-1+2i)]$

where $|k(-1+2i)| = \sqrt{k^2 \cdot 5}$

$= k\sqrt{5}$

$= 2\sqrt{5}$ $\quad \therefore k = 2$

Then C corresponds to the complex number $2(-1+2i)$

$= -2+4i$

Question 15

(a) (i) $(\alpha^2+\beta^2+\gamma^2) = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= 0^2 - 2\times 1$

$= -2$

* Sum roots $-\frac{b}{a}$

$\alpha+\beta+\gamma = -\frac{0}{1}$

$= 0$

* Sum two is $\frac{c}{a}$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$

$= \frac{1}{1}$

(ii) From (i), Sum of squares on roots < 0
Then at least one of the roots is complex, since $(i^2) = -1$.

Coefficients are real for x^3+x+12, so complex roots appear as conjugate pairs. Product of complex conjugates is real, so to get product of roots ($\alpha\beta\gamma = -12$) as real the other root must be real.

or let $P(x) = x^3+x+12$

$P'(x) = 3x^2+1$

Since $x^2 > 0$ $P'(x) \neq 0$, no turning points on this curve and since it is odd degree 3 it only cuts the x-axis at one point. Therefore, only one real root.

(iii) Let complex conjugate roots be $a+bi$ and $a-bi$ $[b \neq 0]$

then $x^3+x+12 = (x-\alpha)[x-(a+bi)][x-(a-bi)]$

$$x^3+x+12 = (x-\alpha)\left[x^2-2ax+(a^2+b^2)\right]$$

Gives $(a^2+b^2) \cdot \alpha = -12$

$(a^2+b^2)>0$ so $\alpha<0$

With α as the real root, the function will change sign as it passes through α.

Let $\alpha=-3$, $(-3)^3+-3+12=-18<0$

let $\alpha=-2$, $(-2)^3+-2+12=2>0$

Sign change gives $-3<\alpha<-2$ *

(b) Let point where the water strikes the wall be (d,h) [h is height above ground]

Given range is $\dfrac{V^2\sin 2\theta}{g}$

then $0<d<\dfrac{V^2\sin 2\theta}{g}$

* max distance at $\theta=\frac{\pi}{4}$ is $\frac{V^2}{g}$

Now $0<\dfrac{gd}{V^2}<\sin 2\theta$ $\quad$ $[0<\theta<\frac{\pi}{2}$ so $0<2\theta<\pi]$

(i) $0<2\theta<\frac{\pi}{2}$ ie $0<\theta<\frac{\pi}{4}$

then $\sin^{-1}\left(\dfrac{gd}{V^2}\right)<2\theta$

$\frac{1}{2}\sin^{-1}\left(\dfrac{gd}{V^2}\right)<\theta$ $\qquad$ let $\beta=\frac{1}{2}\sin^{-1}\left(\dfrac{gd}{V^2}\right)$

gives $\beta < \theta$ [Note: given $V > \sqrt{gd}$ then $0 < \frac{gd}{V^2} < 1$]

(ii) If $\frac{\pi}{2} < 2\theta < \pi$ then $\frac{\pi}{4} < \theta < \frac{\pi}{2}$

and angle $\sin 2\theta = \sin(\pi - 2\theta)$

So $0 < d < \frac{V^2 \sin(\pi - 2\theta)}{g}$

$$\frac{gd}{V^2} < \sin(\pi - 2\theta)$$

$$\sin^{-1}\left(\frac{gd}{V^2}\right) < \pi - 2\theta$$

Giv $2\theta < \pi - \sin^{-1}\left(\frac{gd}{V^2}\right)$

$$\theta < \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{gd}{V^2}\right)$$

$$\theta < \frac{\pi}{2} - \beta$$

From (i) & (ii) we have

$$\beta < \theta < \frac{\pi}{2} - \beta$$

(c) let $u^6 = x$

then

$(u^6)^{\frac{1}{2}} = x^{\frac{1}{2}}$

$u^3 = x^{\frac{1}{2}}$

and

$(u^6)^{\frac{1}{3}} = x^{\frac{1}{3}}$

$u^2 = x^{\frac{1}{3}}$

also $6u^5 . du = dx$

on Substitution

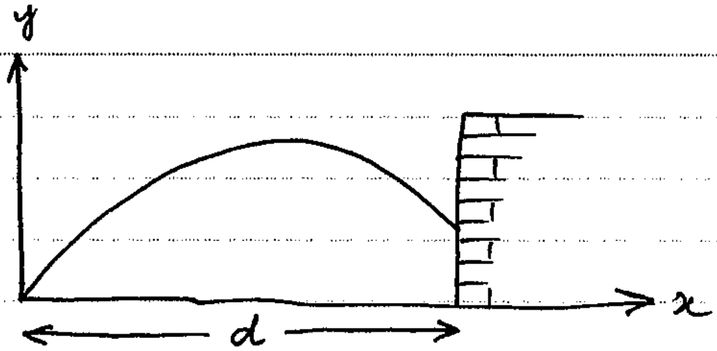
$$\int \frac{dx}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = 6\int \frac{u^5}{u^3 + u^2}\, du$$

$$= 6\int \frac{u^3}{u+1}\, du$$

$$= 6\int u^2 - u + 1 - \frac{1}{u+1}\, du$$

$$= 6\left[\frac{1}{4}u^4 - \frac{1}{2}u^2 + u - \ln(u+1)\right] + C$$

$$= 6\left(\frac{x^{\frac{2}{3}}}{4} - \frac{x^{\frac{1}{3}}}{2} + x^{\frac{1}{6}} - \ln(x^{\frac{1}{6}} + 1)\right) + C$$



$\ddot{y} = -g$ ——— ①

$\dot{y} = -gt + V\sin\theta$ ——— ②

$y = -\frac{1}{2}gt^2 + Vt\sin\theta$ ——— ③

$\ddot{x} = 0$ ——— ④

$\dot{x} = V\cos\theta$ ——— ⑤

$x = Vt\cos\theta$ ——— ⑥

When $x = d$: $Vt\cos\theta = d$

$$t = \frac{d}{V\cos\theta} \quad \text{sub in } ③$$

$$\therefore \; y = -\frac{g}{2}\cdot\frac{d^2}{V^2\cos^2\theta} + V\sin\theta\cdot\frac{d}{V\cos\theta}$$

$$= -\frac{gd^2}{2V^2\cos^2\theta} + d\tan\theta$$

when $x = d$ we want $y > 0$

$$\therefore \; -\frac{gd^2}{2V^2\cos^2\theta} + d\tan\theta > 0$$

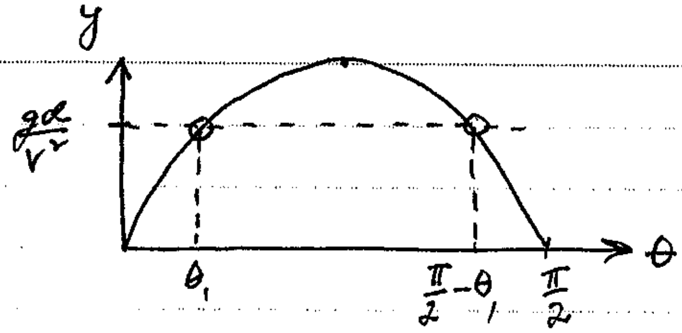
$$-d\left(\frac{gd}{2V^2\cos^2\theta} - \tan\theta\right) > 0$$

$$\text{ie} \quad \frac{gd}{2V^2\cos^2\theta} - \tan\theta < 0$$

$$\Rightarrow \quad \frac{gd}{2V^2\cos^2\theta} < \frac{\sin\theta}{\cos\theta}$$

$$\times\cos^2\theta] \Rightarrow \quad \frac{gd}{2V^2} < \sin\theta\cos\theta$$

$$\therefore \quad \frac{gd}{V^2} < \sin 2\theta$$



$y = \sin 2\theta$

when $\sin 2\theta = \frac{gd}{v^2}$

$$2\theta = \sin^{-1}\frac{gd}{v^2}, \quad (\underline{\text{ALSO}}\ \pi - \sin^{-1}\frac{gd}{v^2})$$

$$\Rightarrow \theta = \frac{1}{2}\sin^{-1}\frac{gd}{v^2}, \ \frac{\pi}{2} - \sin^{-1}\frac{gd}{v^2}.$$

Then $\sin 2\theta > \frac{gd}{v^2}$ from above graph

$$\Rightarrow \frac{1}{2}\sin^{-1}\frac{gd}{v^2} < \theta < \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\frac{gd}{v^2}$$

QUESTION 16

(a) (i) $P'(x) = n.x^{n-1} + a$

Given $x = \alpha$ is a double root

then $P(\alpha) = 0 = P'(\alpha)$

So $n.\alpha^{n-1} + a = 0$

$\therefore \alpha^{n-1} = -\frac{a}{n}$ --- (I)

(ii) From (i) $P(\alpha) = 0$

so $\alpha^n + a\alpha - b = 0$ --- (II)

From (i) $\alpha^n = \left(-\frac{a}{n}\right)\alpha$

So $\alpha\left(\frac{-a}{n}\right) + a\alpha - b = 0$

$$a\alpha\left(1 - \frac{1}{n}\right) = b$$

$$a\alpha = \frac{nb}{(n-1)}$$

$$\alpha = \frac{nb}{a(n-1)} \quad *$$

Sub into II, *

$$\left(\frac{nb}{a(n-1)}\right)^n + \left(\frac{nb}{a(n-1)}\right)a - b = 0$$

$$(nb)^n + nb\left[a^n(n-1)^{n-1}\right] - b\left[a^n(n-1)^n\right] = 0$$

$$(nb)^n = a^n b\left[(n-1)^n - n(n-1)^{n-1}\right]$$

$$= a^n b (n-1)^{n-1}\left[(n-1) - n\right]$$

$$n^n b^n = -a^n b (n-1)^{n-1}$$

gives $\dfrac{b^{n-1}}{(n-1)^{n-1}} = \dfrac{-a^n}{n^n}$

So $\left(\dfrac{a}{n}\right)^n + \left(\dfrac{b}{n-1}\right)^{n-1} = 0$

(b) (i) $y = e^{-x} . \sin x$

Now, $e^{-x} > 0$ for all x

Then $y = 0$ when $\sin x = 0$ $[x \geq 0]$

true at $x = \pi, 2\pi, 3\pi, 4\pi \ldots$

So points $P(\pi, 0)$; $Q(2\pi, 0)$; $R(4\pi, 0) \ldots$

(ii) Given $\int e^{-x} . \sin x \, dx = -\frac{1}{2} e^{-x}(\sin x + \cos x)$

we have $A_1 = -\frac{1}{2}\left[e^{-x}(\sin x + \cos x)\right]_0^{\pi}$

$= -\frac{1}{2}\left[\left(e^{-\pi}(0 + -1)\right) - \left(e^0(0+1)\right)\right]$

$= \frac{1}{2}\left(e^{-\pi} + 1\right)$

$A_2 = -\frac{1}{2}\left[e^{-x}(\sin x + \cos x)\right]_{2\pi}^{3\pi}$

$= -\frac{1}{2}\left[e^{-3\pi}(0 + -1) - e^{-2\pi}(0+1)\right]$

$= \frac{1}{2}\left(e^{-3\pi} + e^{-2\pi}\right)$

PTO

$$A_n = -\frac{1}{2}\left[e^{-x}(\sin x + \cos x)\right]_{(2n-2)\pi}^{(2n-1)\pi}$$

$$= -\frac{1}{2}\left[e^{-(2n-1)\pi}(0 + -1) - e^{-(2n-2)}(0+1)\right]$$

$$= \frac{1}{2}\left(e^{(1-2n)\pi} + e^{(2-2n)\pi}\right)$$

Since $\sin(2n-1)\pi = 0 = \sin(2n-2)\pi$

$\cos(2n-1)\pi = -1$; $\cos(2n-2)\pi = 1$

We can write $A_n = \frac{1}{2}\left(e^{(1-2n)\pi} + e^{(2-2n)\pi}\right)$

(iii) For G.P. $\frac{A_2}{A_1} = \frac{\frac{1}{2}(e^{-3\pi} + e^{-2\pi})}{\frac{1}{2}(e^{-\pi} + e^{0})}$

$$= \frac{e^{-2\pi}(e^{-\pi} + 1)}{(e^{-\pi} + 1)}$$

$$= e^{-2\pi}$$

and $\frac{A_3}{A_2} = \frac{\frac{1}{2}e^{-4\pi}(e^{-\pi}+1)}{\frac{1}{2}e^{-2\pi}(e^{-\pi}+1)}$

$$= e^{-2\pi}$$

Shown $\frac{A_2}{A_1} = \frac{A_3}{A_2}$ has common ratio $\frac{1}{e^{2\pi}}$

and so it represents a G.P., $a = \frac{1}{2}(e^{-\pi}+1)$

$r = e^{-2\pi}$

Now, $\left|\frac{1}{e^{2\pi}}\right| < 1$

With $|r| < 1$, $S_\infty = \frac{\frac{1}{2}(e^{-\pi}+1)}{1 - e^{-2\pi}}$

$$S_\infty = \frac{1+e^\pi}{2e^\pi} \Big/ \frac{e^{2\pi}-1}{e^{2\pi}}$$

$$= \frac{(1+e^\pi)\,e^\pi}{2(e^{2\pi}-1)}$$

$$= \frac{e^\pi\,(\cancel{1+e^\pi})}{2\,(\cancel{e^\pi+1})\,(e^\pi-1)}$$

$$S_\infty = \frac{e^\pi}{2\,(e^\pi-1)}$$

(iv) Let $B_1, B_2, B_3, \ldots$ be areas "below" the curve.

then $A_1+A_2+A_3+\ldots A_n + |B_1+B_2+B_3+\ldots B_n|$
gives total area bounded by the curve.

$$\int_0^\infty e^{-x}.\sin x\,dx = A_1+A_2+\ldots - (B_1+B_2+\ldots)$$

$$= \frac{e^\pi}{2(e^\pi-1)} - (B_1+B_2+\ldots)$$

From data $\frac{1}{2} = \frac{e^\pi}{2(e^\pi-1)} - (B_1+B_2+\ldots)$

Gvs $B_1+B_2+\ldots B_n = \frac{e^\pi-(e^\pi-1)}{2\,(e^\pi-1)}$

$$= \frac{1}{2(e^\pi-1)}$$

Exact value $\int_0^\infty |e^{-x}.\sin x|\,dx = \frac{e^\pi}{2(e^\pi-1)} + \frac{1}{(e^\pi-1)}$

$$= \frac{e^\pi+1}{2(e^\pi-1)}$$

QED