Name: ______________________________

St George Girls High School

Trial Higher School Certificate Examination

2015

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen.
- Write your student number on each booklet.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- The mark allocated for each question is listed at the side of the question.
- Marks may be deducted for careless or poorly presented work.

Total Marks – 100

Section I – Pages 2 – 5
10 marks

- Attempt Questions 1 – 10.
- Allow about 15 minutes for this section.
- Answer on the sheet provided.

Section II – Pages 6 – 11
90 marks

- Attempt Questions 11 – 16.
- Allow about 2 hours 45 minutes for this section.
- Begin each question in a new booklet.
- Show all necessary working in Questions 11 – 16.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1. Find $$\int \frac{dx}{x^2 - 4x + 13}$$

(A) $\frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right) + C$

(B) $\frac{2}{3}\tan^{-1}\left(\frac{x-2}{3}\right) + C$

(C) $\frac{1}{3}\tan^{-1}\left(\frac{2x-4}{3}\right) + C$

(D) $\frac{2}{3}\tan^{-1}\left(\frac{2x-4}{3}\right) + C$

2. The foci of the hyperbola $\frac{y^2}{8} - \frac{x^2}{12} = 1$ are

(A) $(\pm 2\sqrt{5}, 0))$

(B) $(\pm\sqrt{30}, 0))$

(C) $(0, \pm 2\sqrt{5}))$

(D) $(0, \pm\sqrt{30}))$

3. The region bounded by the curves $y = x^2$ and $y = x^3$ in the first quadrant is rotated about the y-axis. The volume of the solid of revolution formed can be found using:

(A) $V = \pi\int_0^1 \left(y^{\frac{1}{3}} - y^{\frac{1}{2}}\right) dy$

(B) $V = \pi\int_0^1 \left(y^{\frac{1}{2}} - y^{\frac{1}{3}}\right) dy$

(C) $V = \pi\int_0^1 \left(y^{\frac{2}{3}} - y\right) dy$

(D) $V = \pi\int_0^1 (x^4 - x^6)\ dx$

Section I (cont'd)

4. The five fifth roots of $1 + \sqrt{3}i$ are:

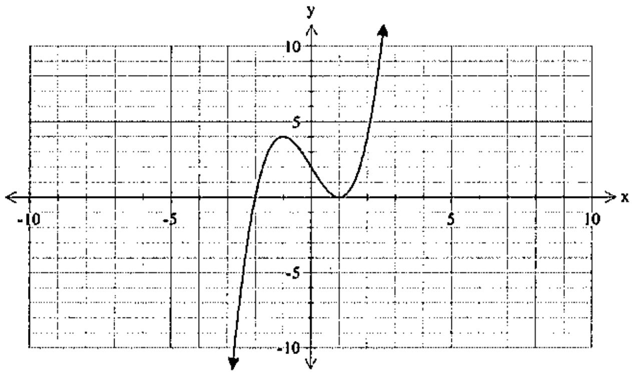
(A) $2^{\frac{1}{5}} \operatorname{cis}\left(\frac{2k\pi}{5} + \frac{\pi}{15}\right), k = 0, 1, 2, 3, 4$

(B) $2^{5} \operatorname{cis}\left(\frac{2k\pi}{5} + \frac{\pi}{15}\right), k = 0, 1, 2, 3, 4$

(C) $2^{\frac{1}{5}} \operatorname{cis}\left(\frac{2k\pi}{5} + \frac{\pi}{30}\right), k = 0, 1, 2, 3, 4$

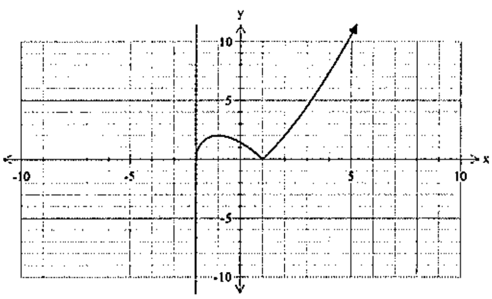
(D) $2^{5} \operatorname{cis}\left(\frac{2k\pi}{5} + \frac{\pi}{30}\right), k = 0, 1, 2, 3, 4$

5. The diagram of $y = f(x)$ is drawn below.

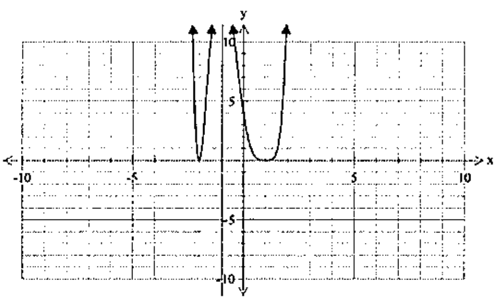


Which of the diagrams below best represents $y = \sqrt{f(x)}$

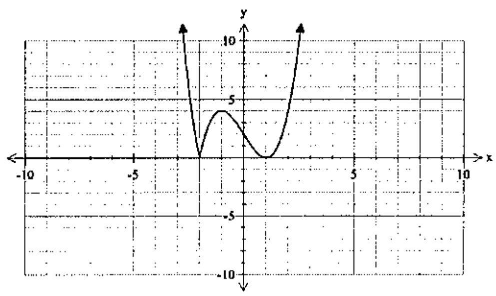
(A)



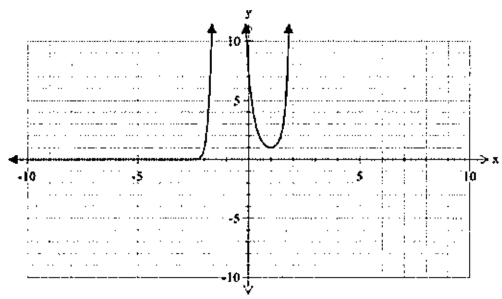
(B)



(C)



(D)



Section I (cont'd)

6. What is the remainder when $P(x) = x^3 + x^2 - x + 1$ is divided by $(x - 1 - i)$?

   (A) $-3i - 2$

   (B) $3i - 2$

   (C) $3i + 2$

   (D) $2 - 3i$

7. $P(x)$ is a polynomial of degree 5 with real coefficients. $P(x)$ has $x = -3$ as a root of multiplicity 3 and $x = i$ as a root. Which of the following expressions is a factorised form of $P(x)$ over the complex numbers?

   (A) $P(x) = (x+3)^3(x-1)(x+1)$

   (B) $P(x) = (x+3)^3(x-1)^2$

   (C) $P(x) = (x+3)^3(x-i)(x+i)$

   (D) $P(x) = (x+3)^3(x-i)^2$

8. Let the point A represent the complex number z on an Argand diagram. Which of the following describes the locus of A specified by $|z+3| = |z|$?

   (A) Perpendicular bisector of the interval joining (0,0) and (3,0)

   (B) Perpendicular bisector of the interval joining (0,0) and (−3,0)

   (C) Circle with a centre (0,0) and radius of 1.5 units

   (D) Circle with a centre (0,0) and radius of 3 units

9. A particle of mass m is moving in a straight line under the action of a force.
$$F = \frac{m(5-7x)}{x^3}$$
   Which of the following equations is the representation of its velocity, if the particle starts from rest at $x = 1$?

   (A) $v = \pm\frac{3}{x}\sqrt{x^2 - 7x + 5}$

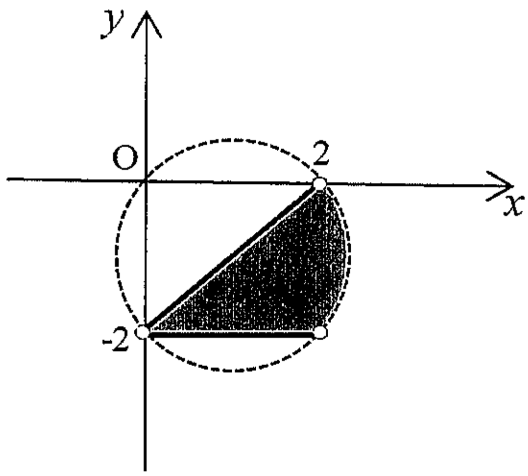
   (B) $v = \pm\frac{1}{x}\sqrt{-9x^2 + 14x - 5}$

   (C) $v = \pm 3x\sqrt{x^2 - 7x + 5}$

   (D) $v = \pm x\sqrt{9x^2 + 14x - 5}$

Section I (cont'd)

10. A region on the Argand Diagram is part of a circle with centre $(1, -1)$, as shown below.



Which inequality could define the shaded area?

(A) $|z - 1 + i| \leq 1$ and $0 < \arg(z + 2i) < \frac{\pi}{4}$

(B) $|z - 1 - i| < \sqrt{2}$ and $0 \leq \arg(z - 2i) \leq \frac{\pi}{4}$

(C) $|z - 1 + i| \leq 1$ and $0 < \arg(z + 2i) \leq \frac{\pi}{4}$

(D) $|z - 1 + i| < \sqrt{2}$ and $0 \leq \arg(z + 2i) \leq \frac{\pi}{4}$

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

a) Let $A = 3 + 3\sqrt{3}i$ and $B = -5 - 12i$. Express each of the following in the form $x + iy$:

(i) $\bar{B}$ 1

(ii) $\dfrac{A}{B}$ 2

(iii) $\sqrt{B}$ 2

b) i) Find the modulus and argument of A, where $A = 3 + 3\sqrt{3}i$ 2

ii) Hence find A^4 in the form of $x + iy$. 1

c) The roots of the polynomial equation $2x^3 - 3x^2 + 4x - 5 = 0$ are α, β and γ. Find the polynomial equation which has roots:

(i) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$. 2

(ii) $2\alpha, 2\beta$ and 2γ. 2

d) Find $\displaystyle\int \frac{dx}{\sqrt{9 + 16x - 4x^2}}$. 3

Question 12 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Evaluate $\displaystyle\int_0^{\frac{\sqrt{\pi}}{2}} 3x \ sin(x^2) \, dx$. 3

b) (i) Find the values of $A, B,$ and C such that: 4

$$\frac{4x^2 - 3x - 4}{x^3 + x^2 - 2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

(ii) Hence find $\displaystyle\int \frac{4x^2 - 3x - 4}{x^3 + x^2 - 2x} \, dx$

c) Solve the equation $x^4 - 7x^3 + 17x^2 - x - 26 = 0$, given that $x = (3 - 2i)$ is a root of the equation. 3

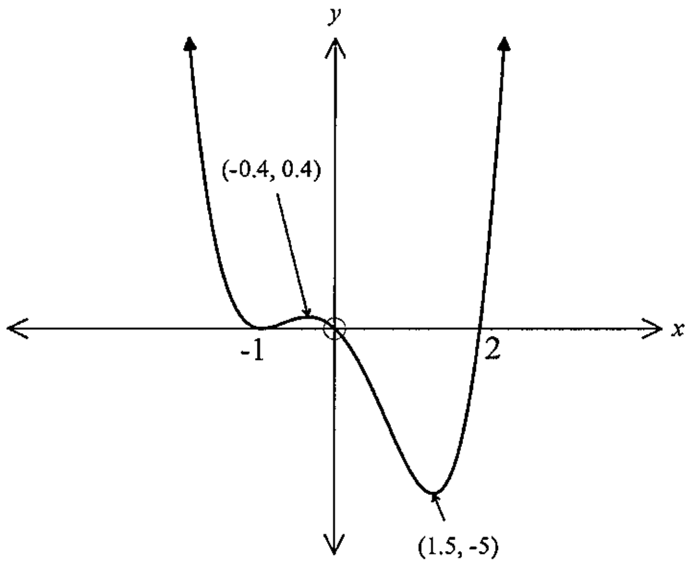
d) (i) Find the equation of the tangent at the point $P\left(ct, \frac{c}{t}\right)$ on the rectangular hyperbola $xy = c^2$. 2

(ii) Find the coordinates of A and B where this tangent cuts the x and y axis respectively. 2

(iii) Prove that the area of the triangle OAB is a constant. (Where O is the origin). 1

Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) The graph of $y = f(x)$ is shown below.



Draw separate sketches for each of the following:

(i) $y = |f(x)|$ 1

(ii) $y = \dfrac{1}{f(x)}$ 2

(iii) $y^2 = f(x)$ 2

(iv) $y = e^{f(x)}$ 2

b) At the start of the observation yesterday, the upper deck of a ship, anchored at Sydney Wharf was 1.2 metres above the wharf at 6:13am, when the tide was at its lowest level. At 12:03pm, at the following high tide, the last observation record shows that the upper deck was 2.6 metres above the wharf. Considering that the tide moves in simple harmonic motion, find:

(i) At what time, during the observation period, was the upper deck exactly 2 metres above the wharf? 2

(ii) What was the maximum rate at which the tide increased during this period of observation? 2

c) Use the method of cylindrical shells to find the volume of the solid generated by revolving the region enclosed by $y = 3x^2 - x^3$ and the x axis around the y-axis. 4

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

a) A particle of mass m kg is dropped from rest in a medium where the resistance to the motion has magnitude $\frac{1}{40}mv^2$ when the speed of the particle is v ms^{-1}. After t seconds the particle has fallen x metres. The acceleration due to gravity is 10 ms^{-2}.

(i) Explain why $\ddot{x} = \frac{1}{40}(400 - v^2)$. 1

(ii) Find an expression for t in terms of v. 2

(iii) Show that $v = 20\left(1 - \frac{2}{1+e^t}\right)$. 1

(iv) Show that $x = 20\left[t + 2ln\left(\frac{1+e^{-t}}{2}\right)\right]$ 2

b) Consider the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(i) Show that the equation of the tangent at the point $P(a\sec\theta, b\tan\theta)$ has the equation $bx\sec\theta - ay\tan\theta = ab$. 2

(ii) Find the equation of the normal at P. 2

(iii) Find the coordinates of the points A and B where the tangent and normal respectively cut the y-axis. 2

(iv) Show that AB is the diameter of the circle that passes through the foci of the hyperbola. 3

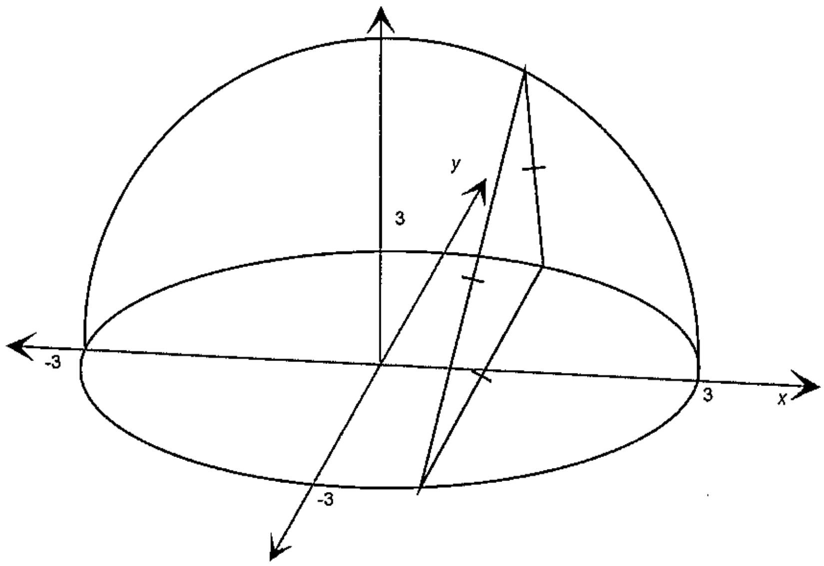
Question 15 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Derive the reduction formula: 4

$$\int x^n e^{-x^2} dx = -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{n-1}{2} \int x^{n-2} e^{-x^2} dx$$

and use this reduction formula to evaluate $\int_0^1 x^5 e^{-x^2} dx$

b) 4



The diagram above shows a solid which has the circle $x^2 + y^2 = 9$ as its base. All cross-sections perpendicular to the x axis are equilateral triangles. Calculate the volume of the solid.

c) Given that $x^4 - 6x^3 + 9x^2 + 4x - 12 = 0$, has a double root at $x = \alpha$, find the value of α. 3

d) If z represents the complex number $x + iy$, Sketch the regions:

(i) $|arg\, z| < \frac{\pi}{4}$ 2

(ii) $Im\,(z^2) = 4$ 2

Question 16 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Show that: $\dfrac{\cos A - \cos(A+2B)}{2\sin B} = \sin(A+B)$. 3

b) Consider the area enclosed between the graphs of the hyperbola $xy = 9$ and the line $x + y = 10$ in the first quadrant. This area is rotated about the x axis. By taking a cross-section perpendicular to the axis of rotation and sketching an appropriate diagram, find the volume of the generated solid. 4

c) Consider the function $f(x) = \sqrt{3 - \sqrt{x}}$

(i) Find the domain of $f(x)$. 1

(ii) Show that $f(x)$ is a decreasing function and deduce the range of $f(x)$ 2

(iii) Show that $f''(x) = \dfrac{6-3\sqrt{x}}{16\,[\sqrt{3x - x\sqrt{x}}\ ]^3}$ and find the coordinates of any inflection points . 3

(iv) Sketch the graph of $y = f(x)$ and show that $\displaystyle\int_0^9 \sqrt{3-\sqrt{x}}\,dx = \frac{24\sqrt{3}}{5}$ 2

End of Examination

TRIAL Paper 2015

Ext 2 Solutions

1. $\int \frac{dx}{x^2-4x+13} = \int \frac{dx}{x^2-4x+4+9}$

$= \int \frac{dx}{(x-2)^2+9}$ A

$= \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+c$

2. $\frac{y^2}{8} - \frac{x^2}{12} = 1$

$a = 2\sqrt{2} \quad b = 2\sqrt{3}$

$\therefore b^2 = a^2(e^2-1)$

$(2\sqrt{3})^2 = (2\sqrt{2})^2(e^2-1)$

$12 = 8(e^2-1)$

$\frac{12}{8} = e^2-1$

$e^2 = \frac{20}{8}$

$e^2 = \frac{10}{4}$

$e = \frac{\sqrt{10}}{2}$

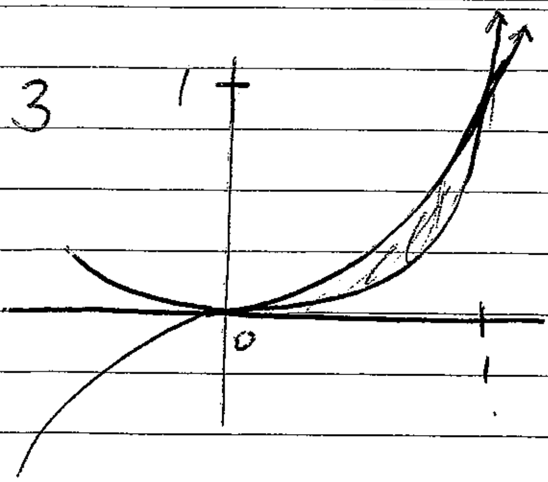
$\therefore$ Foci $= (0, \pm ae)$

$= \left(0, \pm \cancel{2}\sqrt{2}\frac{\sqrt{10}}{\cancel{2}}\right)$ C

$= (0, \pm\sqrt{20})$

$= (0, \pm 2\sqrt{5})$

3



if $y = x^3$, $x = y^{1/3}$

$y = x^2$, $x = y^{1/2}$

$$\therefore V = \pi \int_0^1 \left[\left(y^{1/3}\right)^2 - \left(y^{1/2}\right)^2 \right] dy$$

$$= \pi \int_0^1 \left(y^{2/3} - y \right) dy \qquad C$$

4. Let $z = r(\cos\theta + i\sin\theta)$

if $z^5 = 1 + \sqrt{3}\, i$

then $z^5 = r^5 \operatorname{cis} 5\theta$

$= 2\left(\cos\frac{\pi}{3} + 2n\pi\right)$

$z = 2^{1/5} \operatorname{cis}\left(\frac{\pi}{15} + \frac{2k\pi}{5}\right)$

for $k = 0, 1, 2, 3, 4$

Now $r^5 = \sqrt{1 + (\sqrt{3})^2}$

$= \sqrt{1+3}$

$= \sqrt{4}$

$= 2$

$r = 2^{1/5}$

$5\theta = \tan^{-1}\sqrt{3}$

$= \frac{\pi}{3} + 2k\pi$

$\theta = \frac{\pi}{15} + \frac{2k\pi}{5}$

5. Graph A

6. $P(x) = x^3 + x^2 - x + 1$ **B**

Let $x = 1+i$

$P(1+i) = (1+i)^3 + (1+i)^2 - (1+i) + 1$

$(1+i)^2 = 1 + 2i - 1 = 2i$

$(1+i)^3 = (1+i)^2(1+i)$

$= 2i(1+i)$

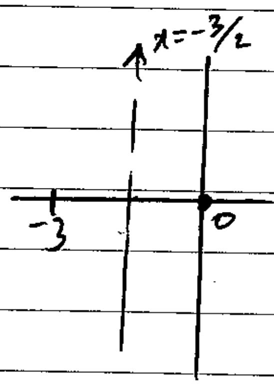
$= 2i(1+i) + 2i - 1 - i + 1$

$= 2i - 2 + 2i - i$

$= 3i - 2$

7. **C**

8. $|z+3| = |z|$ **B**



$|x+iy+3| = |x+iy|$

$(x+3)^2 + y^2 = x^2 + y^2$

$x^2 + 6x + 9 + y^2 = x^2 + y^2$

$6x + 9 = 0$

$2x + 3 = 0$

$x = \frac{-3}{2}$

$\therefore$ Perpendicular bisector of the line joining $(0,0)$ and $(-3,0)$

9. $F = m\dfrac{(5-7x)}{x^3}$

$\cancel{m}\ddot{x} = \cancel{m}\dfrac{(5-7x)}{x^3}$

$\dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = \dfrac{5-7x}{x^3}$

$\left.\dfrac{1}{2}v^2\right]_0^v = \displaystyle\int_1^x 5x^{-3} - 7x^{-2}\,dx$

$\dfrac{1}{2}v^2 = \left.\dfrac{5x^{-2}}{-2} + 7x^{-1}\right]_1^x$

$= \left[-\dfrac{5}{2x^2} + \dfrac{7}{x}\right]_1^x$

$= \dfrac{1}{2}\left[\dfrac{14}{x} - \dfrac{5}{x^2}\right]_1^x$

$= \dfrac{1}{2}\left[\left(\dfrac{14}{x} - \dfrac{5}{x^2}\right) - (14-5)\right]$

$= \dfrac{1}{2}\left[\dfrac{14}{x} - \dfrac{5}{x^2} - 9\right]$

$v^2 = \dfrac{14x - 5 - 9x^2}{x^2}$

$v = \pm\dfrac{1}{x}\sqrt{14x - 5 - 9x^2}$ B

10. D.

Question 11

a) i) $A = 3 + 3\sqrt{3}i \qquad B = -5 - 12i$

$\bar{B} = \overline{-5 - 12i}$
$= -5 + 12i$ — 1 mark — 1

ii) $\dfrac{A}{B} = \dfrac{3 + 3\sqrt{3}i}{-5 - 12i} \times \dfrac{-5 + 12i}{-5 + 12i}$ — 1 mark

$= \dfrac{-15 + 36i - 15\sqrt{3}i - 36\sqrt{3}}{25 - 144i^2}$

$= \dfrac{-15 - 36\sqrt{3} + i(36 - 15\sqrt{3})}{169}$ — 1 mark — 2

iii) $\sqrt{B} = \sqrt{-5 - 12i}$

Let $z = x + iy$ so $z^2 = -5 - 12i$

Let $(x + iy)^2 = -5 - 12i$

$x^2 + 2ixy - y^2 = -5 - 12i$

$x^2 - y^2 + 2ixy = -5 - 12i$

Equate real part

$x^2 - y^2 = -5$ --- ①

Equate imaginary part

$2xy = -12$ --- ②

From (2) $y = \dfrac{-6}{x}$ sub in ①

$x^2 - \left(\dfrac{-6}{x}\right)^2 = -5$ — 1 mark

$x^4 - 36 = -5x^2$

$x^4 + 5x^2 - 36 = 0$

$(x^2 + 9)(x^2 - 4) = 0$

$x^2 = -9$ or $x^2 = 4$

$x = \pm 2$ as x is real.

Sub this in ② $\quad x = 2, y = -3$

$\therefore \quad x = -2, y = 3$

$\therefore \sqrt{-5 - 12} = 2 - 3i$ or $-2 + 3i$ — 1 mark — 2

$= \pm(2 - 3i)$

bi) mod $r = \sqrt{(3)^2 + (3\sqrt{3})^2}$
$= \sqrt{9 + 27}$
$= \sqrt{36}$
$= 6$ 1

arg B : $\tan\theta = \frac{3\sqrt{3}}{3} = \sqrt{3}$
$\theta = \pi/3$ 1 (2)

ii) $A = 6 \text{ cis } \pi/3$
$A^4 = 6^4 \text{ cis } \frac{4\pi}{3}$

$= 1296 \text{ cis } -\frac{2\pi}{3}$

$= 1296\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$
$= 1296\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$= -648(1 - \sqrt{3}i)$ 1 mark (1)

c) i) $2x^3 - 3x^2 + 4x - 5 = 0$
Let $X = \frac{1}{x}$ $\therefore x = \frac{1}{X}$

$\therefore$ equation is $2\left(\frac{1}{X}\right)^3 - 3\left(\frac{1}{X}\right)^2 + 4\left(\frac{1}{X}\right) - 5 = 0$ 1 mark
$\frac{2}{X^3} - \frac{3}{X^2} + \frac{4}{X} - 5 = 0$

$2 - 3X + 4X^2 - 5X^3 = 0$
$\therefore 5x^3 - 4x^2 + 3x - 2 = 0$ 1 mark (2)

ii) Let $X = 2x$ $\therefore x = \frac{X}{2}$
equation is
$2\left(\frac{X}{3}\right)^3 - 3\left(\frac{X}{2}\right)^2 + 4\left(\frac{X}{2}\right) - 5 = 0$ 1 mark
$\frac{2x^2}{8} - \frac{3x^2}{4} + \frac{4x}{2} - 5 = 0$

$\frac{x^2}{4} - \frac{3x^2}{4} + 2x - 5 = 0$
$\therefore x^3 - 3x^2 + 8x - 20 = 0$ 1 mark (2)

d) $$\int \frac{dx}{\sqrt{9+16x-4x^2}}$$

$$= \int \frac{dx}{\sqrt{9+4(4x-x^2)}}$$

$$= \int \frac{dx}{\sqrt{9-4(x^2-4x)}}$$

$$= \int \frac{dx}{\sqrt{9-4(x^2-4x+4)+16}}$$

$$= \int \frac{dx}{\sqrt{25-4(x-2)^2}}$$

1 mark for completing the squares correctly

$$= \int \frac{dx}{\sqrt{4\left(\frac{25}{4}-(x-2)^2\right)}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{25/4-(x-2)^2}}$$

Let $u = x-2$

$du = dx$

1 mark

$$= \frac{1}{2}\int \frac{du}{\sqrt{25/4-u^2}}$$

$$= \frac{1}{2}\sin^{-1}\frac{2u}{5} + c$$

$$= \frac{1}{2}\sin^{-1}\frac{2(x-2)}{5} + c$$

1 mark

3

11 c) Comments

When writing the new equation it is important that it is written as an equation in x.

1 mark was taken off for an equation not written with respect to x

Preferrably equations should be written where the highest coefficient is positive.

11 d) Care needs to be taken when completing the squares, especially when the quadratic is non-monic. Many students lost 1 mark for not completing the squares correctly.

Q12

a) $\int_0^{\sqrt{\pi/2}} 3x \sin(x^2)\, dx$

Let $u = x^2$

$du = 2x\,dx$

when $x = 0$ $u = 0$

$x = \sqrt{\pi/2}$, $u = \left(\sqrt{\frac{\pi}{2}}\right)^2 = \pi/4$

$= \frac{3}{2}\int_0^{\sqrt{\pi/2}} \sin x^2 . 2x\, dx$

$= \frac{3}{2}\int_0^{\pi/4} \sin u . du$ 1 changing limits + variable

$= \frac{3}{2}\left[-\cos u\right]_0^{\pi/4}$ 1 Integral

$= \frac{-3}{2}\left[\cos \pi/4 - \cos 0\right]$

$= \frac{-3}{2}\left(\frac{1}{\sqrt{2}} - 1\right)$

$= \frac{-3}{2}\left(\frac{1-\sqrt{2}}{\sqrt{2}}\right)$

$= \frac{3\sqrt{2}-3}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{6 - 3\sqrt{2}}{4}$ 1 Answer

Q12 b)

i) $$\frac{4x^2-3x-4}{x^3+x^2-2x} = \frac{4x^2-3x-4}{x(x^2+x-2)}$$

$$= \frac{4x^2-3x-4}{x(x-1)(x+2)}$$

$$\frac{4x^2-3x-4}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$4x^2-3x-4 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

when $x=0$

$$-4 = A(-1)(2)$$

$$A = 2$$

$x = 1$

$$-3 = B(3)$$

$$B = -1$$

when $x = -2$

$$18 = C(-2)(-3)$$

$$18 = 6C$$

$$C = 3$$

ii) $$\int \frac{4x^2-3x-4}{x(x-1)(x+2)} = \int \frac{2}{x} + \frac{-1}{x-1} + \frac{3}{x+2}\,dx$$

$$= 2\ln x - \ln|x-1| + 3\ln|x+2| + c$$

c) As there are real coefficients since $(3-2i)$ is a factor then $(3+2i)$ is also a factor

$$\therefore (x-(3-2i))(x-(3+2i)) = x^2 - x(3+2i) - x(3-2i) + (3-2i)(3+2i)$$

$$= x^2 - 3x - 2ix - 3x + 2ix + (9+4)$$

$$= x^2 - 6x + 13$$ is also a factor

$$\begin{array}{r}
x^2 - x - 2 \\
x^2 - 6x + 13 \;\big)\; x^4 - 7x^3 + 17x^2 - x - 26 \\
x^2 - 6x^3 + 13x^2 \\
\hline
-x^3 + 4x^2 - x \\
-x^3 + 6x^2 - 13x \\
\hline
-2x^2 + 12x - 26 \\
-2x^2 + 12x - 26 \\
\hline
0
\end{array}$$

$$\therefore\ P(x) = (x^2 - 6x + 13)(x^2 - x - 2)$$
$$= (x^2 - 6x + 13)(x - 2)(x + 1)$$
$$=$$

$\therefore$ Solution to $x^4 - 7x^3 + 17x^2 - x - 26 = 0$

is $3 \pm 2i$, 2 and -1.

d) i) $xy = c^2$

Using implicit differentiation

$$y + x.\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

when $x = ct$ and $y = c/t$

$$\frac{dy}{dx} = -c/t \div ct$$
$$= \frac{-1}{t^2}$$

$$\therefore\ y - \frac{c}{t} = \frac{-1}{t^2}(x - ct)$$

$$t^2 y - ct = -x + ct$$

$$* \quad x + t^2 y - 2ct = 0.$$

(12 ii) when $y=0$, $x+0-2ct=0$

$$x=2ct$$

$\therefore A\ (2ct, 0)$ 1

when $x=0$, $0+t^2y=0$

$$y=\frac{2c}{t}$$ 1

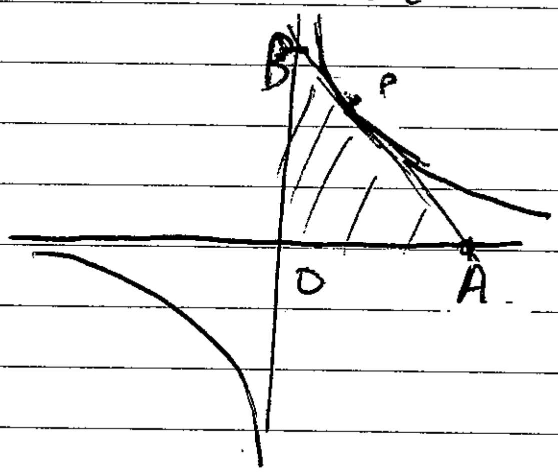
$\therefore B$ is $\left(0, \frac{2c}{t}\right)$

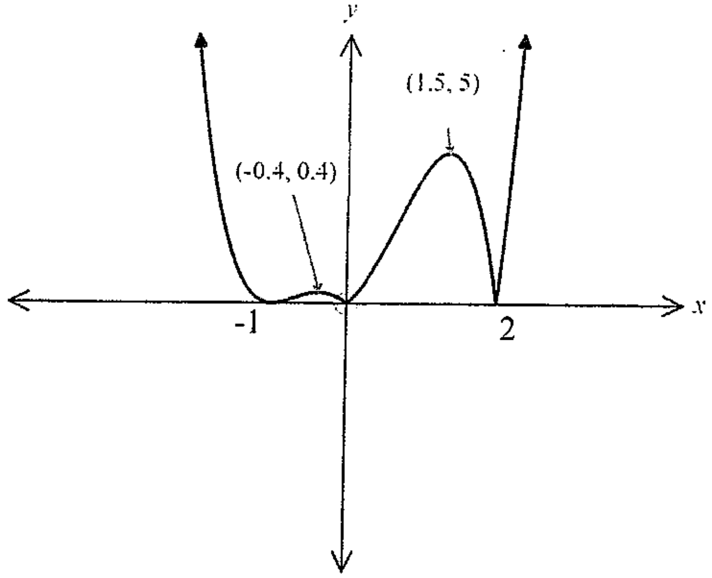
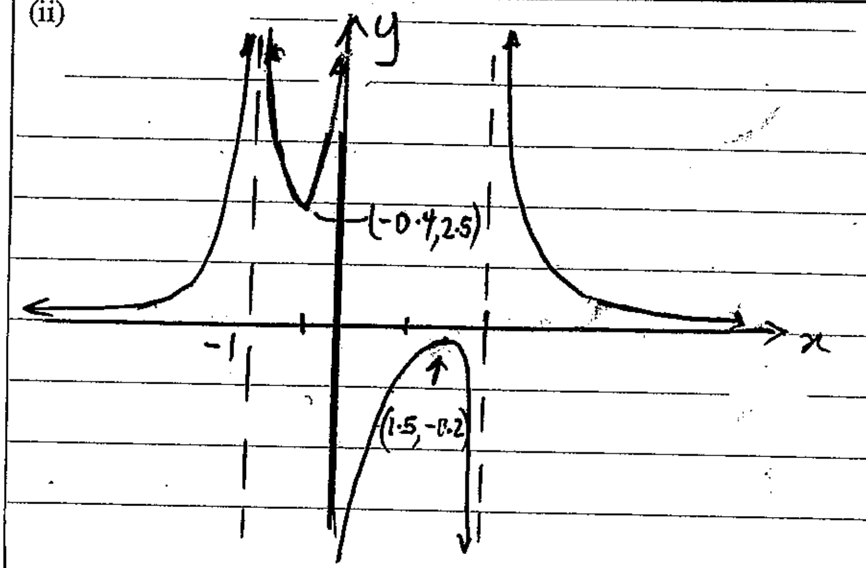
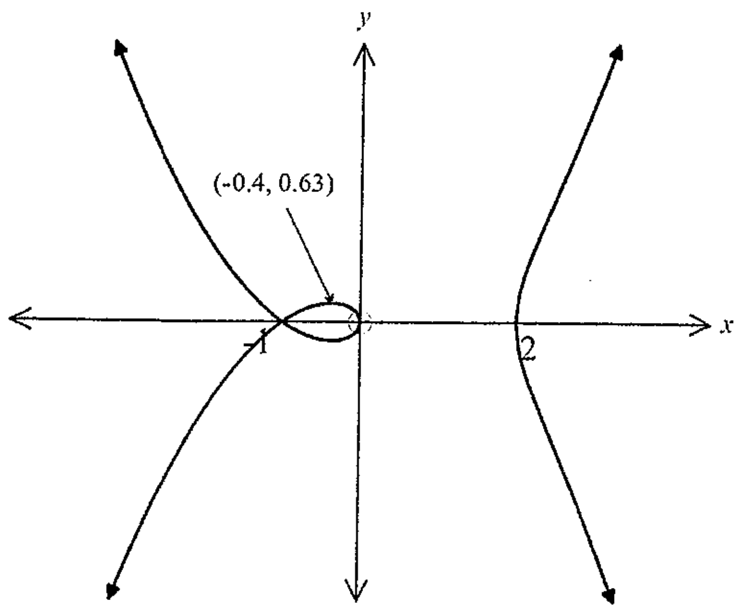
iii) Now $OA = 2ct$

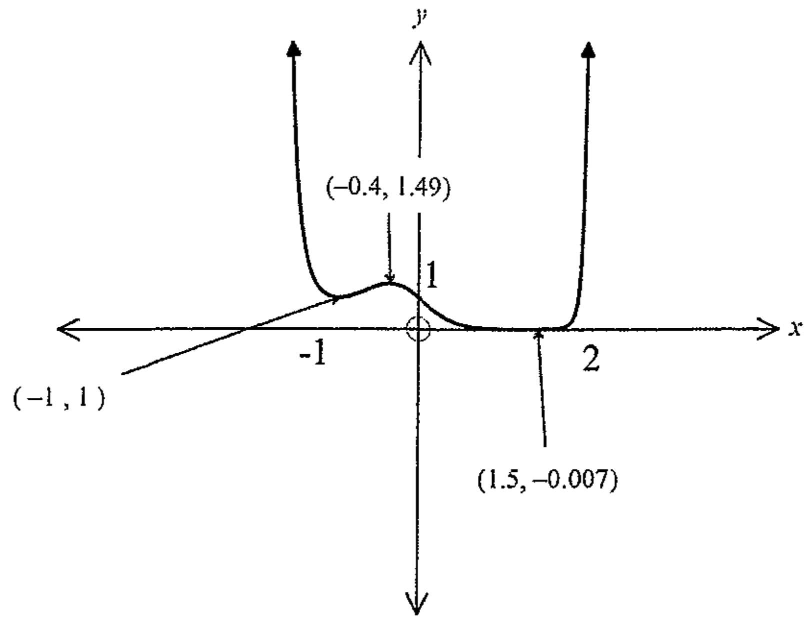
$$OB = \frac{2c}{t}$$

Area of $\Delta OAB = \frac{1}{2}\, 2ct \times \frac{2c}{t}$

$$= 2c^2$$

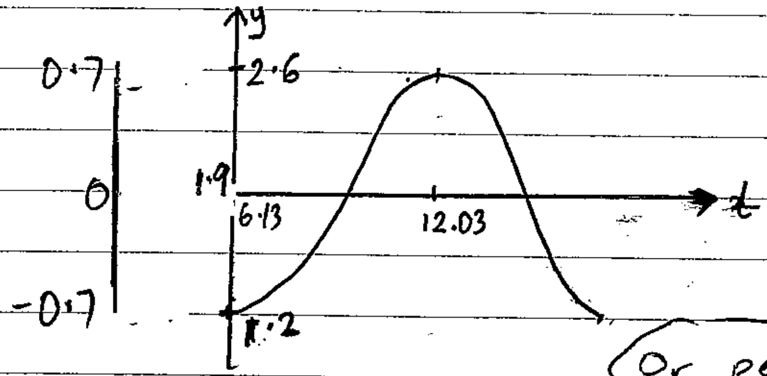


		2014	
	Solution	Marks	Allocation of marks
	(i) 	1	1 – graph
	(ii) 	2	1 – graph 1 - accuracy
	(iii) 	2	1 – graph 1 - accuracy Care needed to be taken at (-1,0) and (0,0) At (-1,0) the graph had to appear as two intersecting lines At (0,0) ⇒ vertical tangent

Question 13		2014	
	Solution	Marks	Allocation of marks
	(iv) 	**2**	1 – graph 1 - accuracy

<u>Note</u>: (0,0) was not a discontinuous point, The circle emphasised the origin. Marks were not deducted for this misconception, though.

Q13 b)



Period $= 5h\,50\,min \times 2$

$= \frac{35}{6} \times 2h$

$T = \frac{35}{3}\,h$

or

Or period can be found in min

period $= 350 \times 2$

$= 700$ min

$\therefore T = \frac{2\pi}{n}$

$= \frac{2\pi}{\frac{35}{3}}$

$n = \frac{6\pi}{35}$ (or $n = \frac{\pi}{350}$)

Centre of motion $= \frac{2.6 + 1.2}{2}$

$= 1.9$

Amplitude $= 0.7$ m

As the particle move in SHM
we use

$x = 1.9 - 0.7 \cos \frac{6\pi}{35} t$ 1 mark

or $x = \left(1.9 - 0.7 \cos \frac{\pi}{350} t\right)$

Using $x = 1{\cdot}9 - \cos\frac{6\pi t}{35}$

when $x = 2$

$2 = 1{\cdot}9 - 0{\cdot}7\cos\frac{6\pi t}{35}$

$0{\cdot}1 = -0{\cdot}7\cos\frac{6\pi t}{35}$

$-\frac{1}{7} = \cos\frac{6\pi t}{35}$

$\frac{6\pi t}{35} = \cos^{-1}\left(\frac{-1}{7}\right)$

$t = \frac{35\cos^{-1}\left(\frac{-1}{7}\right)}{6\pi}$

$t = 3h\ 11min$ after low tide

Using $x = 1{\cdot}9 - 0{\cdot}7\cos\frac{\pi}{350}t$

when $x = 2$

$2 = 1{\cdot}9 - 0{\cdot}7\cos\frac{\pi}{350}t$

$\frac{\pi}{350}t = \cos^{-1}\left(-\frac{1}{7}\right)$

$t = \frac{350\cos^{-1}\left(\frac{-1}{7}\right)}{\pi}$

$t =$ min

$t = 3h\ 11min.$

1

$\therefore$ The upper deck was exactly 2m above the wharf at 6.13 am + 3h 11min
ie 9.24 am.

1 mark

2

OR

ii) $\frac{dx}{dt} = -0{\cdot}7 \times \frac{6\pi}{35} \cdot -\sin\frac{6\pi t}{35}$ | $\frac{dx}{dt} = -0{\cdot}7 \times \frac{\pi}{350} \cdot -\sin\frac{\pi}{350}t$

1

The tide is moving fastest when:

$\sin\frac{6\pi t}{35} = 1$ **or** $\sin\frac{\pi}{350}t = 1$

$\max\frac{dx}{dt} = -0{\cdot}7 \times \frac{3\pi}{35}$

$= \frac{3\pi}{25}$ m/h

$\doteqdot 0{\cdot}377$ m/h

$\max\frac{dx}{dt} = -0{\cdot}7 \times \frac{\pi}{350}$

$= \frac{\pi}{500}$ m/min

1

$\doteqdot 0{\cdot}00628$ m/min

2

Q13 b) Alternative solution.

$$x = 0\cdot7\cos\left(\frac{\pi}{350}t + \alpha\right)$$

To find α when $t=0$, $x=-0\cdot7$

$$-0\cdot7 = 0\cdot7\cos\left(\frac{\pi}{350}(0) + \alpha\right)$$

$$-1 = \cos\alpha$$

$$\alpha = \pi$$

$$\therefore x = 0\cdot7\cos\left(\frac{\pi}{350}t + \pi\right)$$

when $x = 0\cdot1$

$$0\cdot1 = 0\cdot7\cos\left(\frac{\pi}{350}t + \pi\right)$$

$$\frac{1}{7} = \cos\left(\frac{\pi}{350}t + \pi\right)$$ 1

$$\frac{\pi}{350}t + \pi = \pm\cos^{-1}\left(\frac{1}{7}\right) + 2\pi k$$

$$\frac{\pi}{350}t = \pm\cos^{-1}\left(\frac{1}{7}\right) + 2\pi k - \pi$$

$$t = \pm\frac{350}{\pi}\left[\cos^{-1}\left(\frac{1}{7}\right) + 2\pi k - \pi\right]$$

when $k=0$

$t = -190.97\ldots$ or $t = 190.97\ldots$

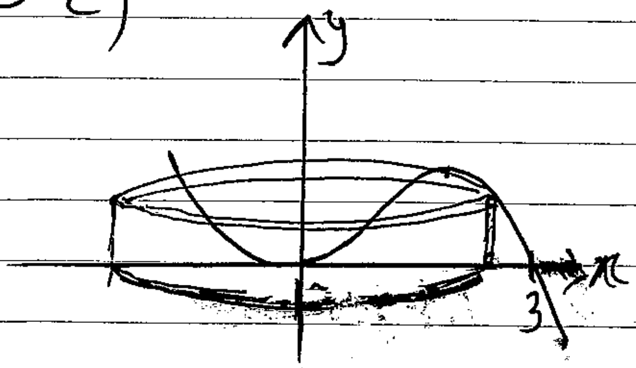
but $t > 0$

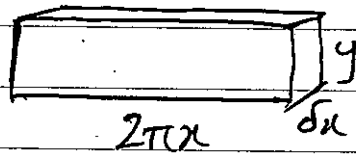
$\therefore t = 190.97$ min $\div 60$

$= 3$ h 11 min. 1

2

13 c)



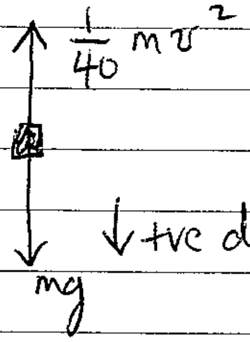


$$A = 2\pi x y$$
$$= 2\pi x(3x^2 - x^3)$$
$$\delta V = 2\pi x(3x^2 - x^3)\delta x \qquad 1$$

$$V \doteq \sum_{x=0}^{3} 2\pi x(3x^2 - x^3)\delta x$$
$$= 2\pi \int_0^3 3x^3 - x^4 \, dx \qquad 1$$
$$= 2\pi \left[\frac{3x^4}{4} - \frac{x^5}{5} \right]_0^3 \qquad 1$$
$$= \frac{243}{10}\pi \ u^3 \qquad 1 \qquad 4$$

This question was done relatively well.

Q14a)



$\downarrow$ +ve direction

Resultant force $= mg - \frac{1}{40}mv^2$

$$m\ddot{x} = mg - \frac{1}{40}mv^2$$

$$\ddot{x} = g - \frac{1}{40}v^2$$

$$= \frac{40g - v^2}{40}$$

$$= \frac{400 - v^2}{40}$$

$$= \frac{1}{40}(400 - v^2)$$

ii) ($v-t$) rel'p

$$\frac{dv}{dt} = \frac{1}{40}(400 - v^2)$$

$$\frac{dt}{dv} = \frac{40}{400 - v^2}$$

$$= \frac{40}{(20-v)(20+v)}$$

Using partial fractions

$$\frac{40}{(20-v)(20+v)} = \frac{A}{20-v} + \frac{B}{20+v}$$

$$40 = A(20+v) + B(20-v)$$

when $v = -20$

$40 = 40B$

$B = 1$

when $v = 20$

$40 = A(40)$

$A = 1$

$$\therefore \int_0^t dt = \int_0^v \frac{1}{20-v} + \frac{1}{20+v}\, dv$$

$$t = \left[-\ln(20-v) + \ln(20+v)\right]_0^v$$

$$= \left[\ln\left(\frac{20+v}{20-v}\right)\right]_0^v$$

$$= \ln \frac{20+v}{20-v} - \ln \frac{20}{20}$$

$$t = \ln\left(\frac{20+v}{20-v}\right) \qquad \text{———} (1)$$

iii) From (1)

$$e^t = \frac{20+v}{20-v}$$

$$20e^t - ve^t = 20e^t - 20$$

$$v + ve^t = 20e^t - 20$$

$$v(1+e^t) = 20(e^t - 1)$$

$$v = \frac{20(e^t-1)}{1+e^t}$$

$$= \frac{20e^t - 20}{1+e^t}$$

$$= \frac{20(1+e^t-1-1)}{1+e^t}$$

$$= 20\left(\frac{1+e^{t}-2}{1+e^{t}}\right)$$

$$v = 20\left(1-\frac{2}{1+e^{t}}\right)$$

iv) $$\frac{dx}{dt} = 20\left(1-\frac{2}{1+e^{t}}\right)$$

$$= 20\left(1-\frac{2}{1+e^{t}} \times \frac{e^{-t}}{e^{-t}}\right)$$

$$= 20\left(1-\frac{2e^{-t}}{e^{-t}+1}\right)$$

$$\int_0^x dx = 20\int_0^t t + 2\ln|1+e^{-t}|$$

$$x = 20\left[t + 2\ln|1+e^{-t}|\right]_0^t$$

$$= 20\left[t + 2\ln(1+e^{-t}) - 2\ln 2\right]$$

$$x = 20\left[t + 2\ln\left[\frac{1+e^{-t}}{2}\right]\right.$$

#

14 b) i) $x = a\sec\theta$ $\quad y = b\tan\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$

and $\frac{dy}{d\theta} = b\sec^2\theta$

$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$= b\sec^2\theta \cdot \frac{1}{a\sec\theta\tan\theta}$

$= \frac{b\sec\theta}{a\tan\theta}$

Eqn of tangent

$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$

$bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta)$

$bx\sec\theta - ay\tan\theta = ab$

ii) $m_T = \frac{b\sec\theta}{a\tan\theta}$

$\therefore m_N = -\frac{a\tan\theta}{b\sec\theta}$ $\quad 1$

$\therefore y - b\tan\theta = \frac{-a\tan\theta}{b\sec\theta}(x - a\sec\theta)$

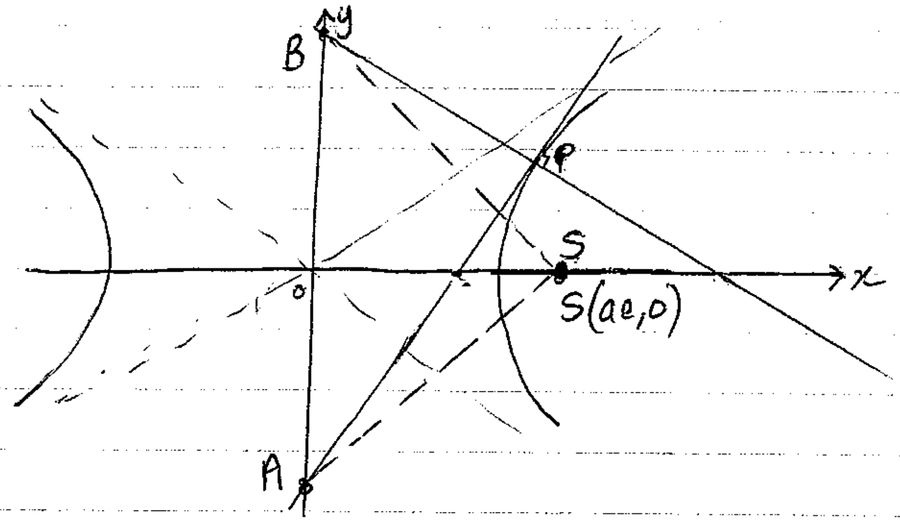
$by\sec\theta - b^2\tan\theta\sec\theta = -ax\tan\theta + a^2\tan\theta\sec\theta$

$\div \tan\theta\sec\theta$

$\frac{by}{\tan\theta} - b^2 = \frac{-ax}{\sec\theta} + a^2$

$\therefore \frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

iii)



Tangent cuts the y-axis at A when $x=0$

$$bx\sec\theta - ay\tan\theta = ab$$

when $x=0$, $\quad y = \dfrac{-b}{\tan\theta}$

$\therefore \quad A$ is $\left(0, -\dfrac{b}{\tan\theta}\right)$

For $\dfrac{ax}{\sec\theta} + \dfrac{by}{\tan\theta} = a^2+b^2$

when $x=0$

$$y = \frac{(a^2+b^2)\tan\theta}{b}$$

$\therefore \quad B$ is $\left(0, \dfrac{(a^2+b^2)\tan\theta}{b}\right)$

iv) Focus $= S(ae, 0)$

If AB is diameter of a circle
RTP, $\angle ASB = 90°$.
Gradient of AS

$$m_{AS} = \frac{0 - \frac{-b}{\tan\theta}}{ae - 0}$$

$$= \frac{b}{\tan\theta} \div ae$$

$$= \frac{b}{ae\tan\theta}$$

Gradient of BS

$$m_{BS} = \frac{0 - \frac{(a^2+b^2)\tan\theta}{b}}{ae - 0}$$

$$= \frac{-(a^2+b^2)\tan\theta}{b} \div ae$$

$$= \frac{-(a^2+b^2)\tan\theta}{abe}$$

Now

$$m_{AS} \times m_{BS} = \frac{\cancel{b}}{ae\cancel{\tan\theta}} \cdot \frac{-(a^2+b^2)\cancel{\tan\theta}}{a\cancel{b}e}$$

$$= \frac{-(a^2+b^2)}{a^2e^2} \quad \text{-- } (1)$$

From $e^2 - 1 = \frac{b^2}{a^2}$

$$a^2e^2 - a^2 = b^2$$

$$a^2e^2 = a^2 + b^2$$

sub in (1)

$$m_{AS} \times m_{BS} = -\frac{(a^2+b^2)}{a^2+b^2}$$

$$= -1$$

$\therefore \angle ASP = 90°$

$\therefore$ AB is a diameter of a circle passing through S.

Question 15

a) Let $I_n = \int x^n e^{-x^2} dx$

$$\int x^n e^{-x^2} dx = \int x^{n-1} x e^{-x^2} dx$$

$u = x^{n-1}$ $\quad v' = x e^{-x^2}$

$u' = (n-1) x^{n-2}$ $\quad v = -\frac{1}{2} e^{-x^2}$

$$\int x^{n-1} \cdot x e^{-x^2} dx = uv - \int vu'$$

$$= x^{n-1} \cdot -\frac{1}{2} e^{-x^2} - \int -\frac{1}{2} e^{-x^2} (n-1) x^{n-2}$$

$$= -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{n-1}{2} \int x^{n-2} e^{-x^2} dx$$

1 - mark was used to derive the reduction formula using integration by parts and by only rewriting the integrand as $\int x^{n-1} x e^{-x^2} dx$ where $u = x^{n-1}$ & $v' = x e^{-x^2}$

<u>Note</u>: No marks were awarded to students who took $u = x^n$ and $v' = e^{-x^2}$

We can't find the integral of e^{-x^2} to be $-\frac{1}{2x} e^{-x^2}$.

Method 1

Let $I_n = \int_0^1 x^n e^{-x^2} dx$

$$I_5 = \left[-\frac{1}{2} x^4 e^{-x^2}\right]_0^1 + 2\int_0^1 x^3 e^{-x^2} dx$$

1 - mark for use of reduction formula

$$= \frac{-1}{2e} + 2\left[\left[-\frac{1}{2} x^2 e^{-x^2}\right]_0^1 + \frac{3-1}{2}\int_0^1 x e^{-x^2} dx\right]$$

1 - mark for subsequent use of reduction formula

$$= \frac{-1}{2e} + 2\left[\frac{-1}{2e} + \int_0^1 x e^{-x^2} dx\right]$$

$$= \frac{-1}{2e} - \frac{1}{e} + 2\left[-\frac{e^{-x^2}}{2}\right]_0^1$$

$$= \frac{-1}{2e} - \frac{1}{e} - \frac{1}{e} + 1$$

$$= 1 - \frac{5}{2e}$$

1 - mark for answer 4

Method 2

Let $I_n = \int_0^1 x^n e^{-x^2}\,dx$

$$I_5 = \frac{-1}{2e} + \frac{5-1}{2} I_{5-2}$$

$$= \frac{-1}{2e} + 2\, I_3$$

$$I_3 = \frac{-1}{2e} + \frac{3-1}{2} I_1$$

$$= \frac{-1}{2e} + I_1$$

$$I_1 = \int_0^1 x\, e^{-x^2}\,dx$$

$$= \frac{-1}{2}\int_0^1 2x\, e^{-x^2}\,dx$$

$$= \frac{-1}{2}\left[e^{-x^2}\right]_0^1$$

$$= \frac{-1}{2}\left[\frac{1}{e} - 1\right]$$

$$\therefore I_3 = \frac{-1}{2e} + \frac{-1}{2}\left(\frac{1}{e} - 1\right)$$

$$= \frac{-1}{2e} - \frac{1}{2e} + \frac{1}{2}$$

$$I_5 = \frac{-1}{2e} + 2\left(\frac{-1}{2e} - \frac{1}{2e} + \frac{1}{2}\right)$$

$$= \frac{-1}{2e} - \frac{1}{e} - \frac{1}{e} + 1$$

$$= \frac{-5}{2e} + 1$$

Method 3

$$I_5 = \int x^5 e^{-x^2}\,dx$$

$$= \left[-\frac{1}{2} x^4 e^{-x^2}\right]_0^1 - \frac{4}{2}\int_0^1 x^3 e^{-x^2}\,dx$$

$$= \frac{-1}{2e} - 0 - 2\, I_3$$

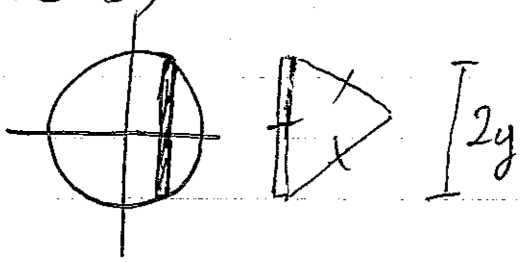
$$= \frac{-1}{2e} - 2\left[\frac{-1}{2e} - \frac{2}{2} I_1\right]$$

$$= \frac{-1}{2e} + \frac{1}{e} + 2\left[\frac{-1}{2e} + \frac{1}{2}\right]$$

$$= \frac{-1}{2e} + \frac{1}{e} - \frac{1}{e} + 1$$

$$= \frac{-5}{2e} + 1$$

15 b)



$x^2 + y^2 = 9$

$y^2 = 9 - x^2 \quad --- (1)$

Two methods of finding the area of the cross-section

Method 1

Using $A = \frac{1}{2} ab \sin C$

$= \frac{1}{2} \times 2y \times 2y \times \sin 60°$

$= \frac{1}{2} \times \quad \times \frac{\sqrt{3}}{2}$

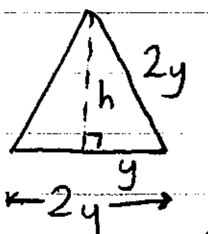
$A = \sqrt{3}\, y^2$

$A = \sqrt{3}\,(9 - x^2)$ from (1)

$\therefore \delta V = \sqrt{3}\,(9 - x^2)\,\delta x$

Method 2

Using $A = \frac{1}{2} b h$



$h^2 = 4y^2 - y^2$

$= 3y^2$

$h = \sqrt{3}\, y$

$\therefore A = \frac{1}{2} \times 2y \times \sqrt{3}\, y$

$= \sqrt{3}\, y^2$

$A = \sqrt{3}\,(9 - x^2)$

2 marks for finding the area of the cross-section

Care needs to be taken when finding the area of the triangle. Many students took the base to be y not $2y$.

Now $\delta V = \sqrt{3}\,(9 - x^2)\,\delta x$

$V \doteqdot \lim_{\delta x \to 0} \sum_{x=-3}^{3} \sqrt{3}\,(9 - x^2)\,\delta x$

$= \int_{-3}^{3} \sqrt{3}\,(9 - x^2)\, dx$

$$= \sqrt{3}\left[9x - \frac{x^3}{3}\right]_{-3}^{3}$$ 1 mark for integral

$$= \sqrt{3}\left[(27-9) - (-27+9)\right]$$

$$= \sqrt{3}\,(18+18)$$

$$= 36\sqrt{3}\ u^3$$ 1 - answer

4

Q15 c)

$f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

$f'(x) = 4x^3 - 18x^2 + 18x + 4$

Double root occurs when
$f'(x) = f(x) = 0$

1 mark for using the double root thm and finding the derivative.

Look at factors of 4
(ie $x = \pm 1$, $x = \pm 2$, $x = \pm 4$)
when $x = 2$

$f'(2) = 4(2^3) - 18(2^2) + 18(2) + 4$
$= 32 - 72 + 36 + 4$
$= 0$

$f(2) = 2^4 - 6(2^3) + 9(2^2) + 4(2) - 12$
$= 16 - 48 + 36 + 8 - 12$
$= 0$

1 mark for testing roots of $f'(x)$

Since $f'(2) = f(2) = 0$
then
$(x-2)$ is a repeated factor
$\therefore x = 2$ is a double root.

1 mark for testing in $f(x)$ and show $f'(2) = f(2) = 0$ and stating the value of α

3

Note: Care needs to be taken when differentiating and to test for a zero we use the factors of the constant term of $f'(x)$.

Question 15 d)

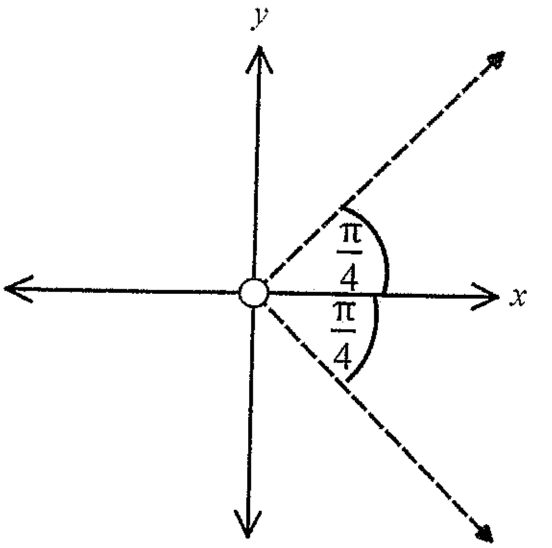
(d) (i)

$\arg z = \theta$

where $\tan\theta = \dfrac{y}{x}$

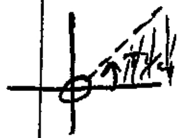
If $|\arg(z)| < \dfrac{\pi}{4}$

then $-\dfrac{\pi}{4} < \arg(z) < \dfrac{\pi}{4}$

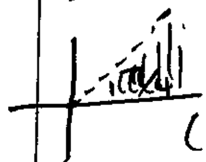


1 mark for the graph

1 mark for showing main features.



1mark



½ mark (if (0,0) was not removed)

2

(ii)

$z = x + iy$

$z^2 = (x+iy)^2 = x^2 + 2xyi - y^2$

$= x^2 - y^2 + 2xyi$

$Im(z^2) = 2xy$

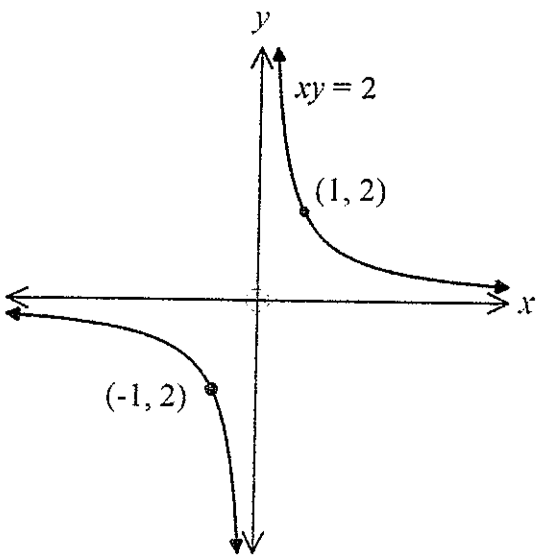
Graph required is $Im(z^2) = 4$

$2xy = 4$

ie $xy = 2$

or $y = \dfrac{2}{x}$

1 – determining equation



1 – Graph and points

Note: Generally well done, but always include a valued point on the graph

2

Q16

a) $$LHS = \frac{\cos A - (\cos A \cos 2B - \sin A \sin 2B)}{2\sin B}$$

$$= \frac{\cos A - \cos A \cos 2B + \sin A \sin 2B}{2\sin B}$$

$$= \frac{\cos A - \cos A(1 - 2\sin^2 B) + \sin A . 2\sin B \cos B}{2\sin B}$$

$$= \frac{\cancel{\cos A} - \cancel{\cos A} + 2\cos A \sin^2 B + 2\sin A \sin B \cos B}{2\sin B}$$

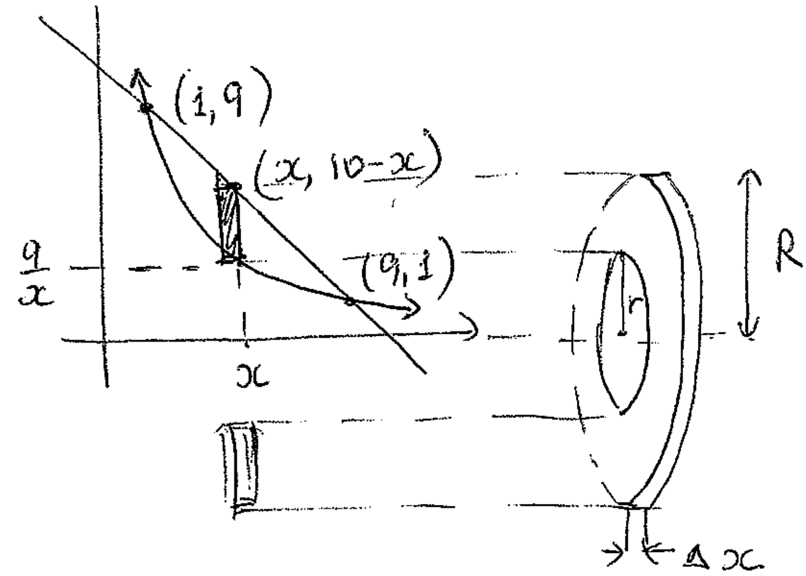
$$= \frac{2\sin^2 B \cos A + 2\sin A \sin B \cos B}{2\sin B}$$

$$= \frac{2\cancel{\sin B}(\sin B \cos A + \sin A \cos B)}{2\cancel{\sin B}}$$

$$= \sin(A+B)$$

$$= RHS$$

(b) Volume, using the annulus.



$$\Delta V \doteqdot \pi (R^2 - r^2)\,\Delta x$$

$$R = 10 - x$$
$$r = \frac{9}{x}$$

$$\Delta V \doteqdot \pi \left((10-x)^2 - \left(\frac{9}{x}\right)^2\right) \Delta x$$

$$V \doteqdot \pi \sum_{x=1}^{9} \left(100 - 20\cancel{0}x + x^2 - \frac{81}{x^2}\right) \Delta x$$

$$V = \lim_{\Delta x \to 0} \pi \sum_{x=1}^{9} \left(100 - 20\cancel{0}x + x^2 - \frac{81}{x^2}\right) \Delta x$$

$$= \pi \int_1^9 100 - 20\cancel{0}x + x^2 - 81x^{-2}\, dx$$

$$= \pi \left[100x - \frac{20\cancel{0}x^2}{2} + \frac{x^3}{3} - \frac{81x^{-1}}{-1}\right]_1^9$$

$$= \pi \left[100x - 10\cancel{0}x^2 + \frac{1}{3}x^3 + \frac{81}{x}\right]_1^9$$

$$= \pi \left[(900 - 810\cancel{0} + 243 + 9) - (100 - 10\cancel{0} + \tfrac{1}{3} + 81)\right]$$

$$= \pi \left(342 - 171 - \tfrac{1}{3}\right) = 170\tfrac{2}{3}\pi \ u^3$$

Graphs Question

16 c) $f(x) = \sqrt{3-\sqrt{x}} = \left(3 - x^{\frac{1}{2}}\right)^{\frac{1}{2}}$

(i) $3 - \sqrt{x} \geq 0$ and $x \geq 0$

$3 \geq \sqrt{x}$

$9 \geq x$ and $x \geq 0$

$\therefore$ Domain is $0 \leq x \leq 9$

(ii) $f'(x) = \frac{1}{2}\left(3-x^{\frac{1}{2}}\right)^{-\frac{1}{2}} \times -\frac{1}{2}x^{-\frac{1}{2}}$ (Chain rule)

$= -\frac{1}{4} \times \frac{1}{\sqrt{x}\sqrt{3-\sqrt{x}}}$

$= -\frac{1}{4} \times \frac{1}{\sqrt{3x - x\sqrt{x}}}$

Since $\sqrt{3x - x\sqrt{x}} \geq 0$ for all x in the domain
$f'(x) < 0$ for $0 < x < 9$ and
$f'(x)$ is undefined at $x = 0$ and $x = 9$.
as $x \to 0$ or $x \to 9$ $f'(x) \to -\infty$

$\therefore$ $f(x)$ is a decreasing function

$\therefore$ $f(x)_{max} = \sqrt{3}$ (when $x = 0$)

$f(x)_{min} = 0$ (when $x = 9$)

(iii) $f''(x) = -\frac{1}{4}\left(\left(3x - x^{\frac{3}{2}}\right)^{-\frac{1}{2}}\right)'$

$$= -\frac{1}{4} \times -\frac{1}{2}\left(3x - x^{3/2}\right)^{-\frac{3}{2}} \times \left(3 - \frac{3}{2}x^{\frac{1}{2}}\right)$$

$$= \frac{1}{8}\frac{3 - \frac{3}{2}\sqrt{x}}{\left(\sqrt{3x - x\sqrt{x}}\right)^3} = \frac{1}{16}\frac{6 - 3\sqrt{x}}{\left(\sqrt{3x - x\sqrt{x}}\right)^3}$$

Possible inflexion points:

$$3 - \frac{3}{2}\sqrt{x} = 0$$

$$\frac{3}{2}\sqrt{x} = +3$$

$$\sqrt{x} = \frac{6}{3}$$

$$\sqrt{x} = 2$$

$$x = 4, \text{ as } 0 \le x \le 9$$

Check the change of concavity around $x = 4$.

When $x = 2$

$$f''(x) = \frac{1}{8}\frac{3 - \frac{3}{2}\sqrt{2}}{\left(\sqrt{6 - 2\sqrt{2}}\right)^3} > 0, \text{ as}$$

$$\left(\sqrt{6 - 2\sqrt{2}}\right)^3 > 0$$

and $3 - \frac{3}{\sqrt{2}} > 0$.

When $x = 6$

$$f''(x) = \frac{1}{8}\frac{3 - \frac{3}{2}\sqrt{6}}{\left(\sqrt{6 - 2\sqrt{6}}\right)^3} < 0 \text{ as}$$

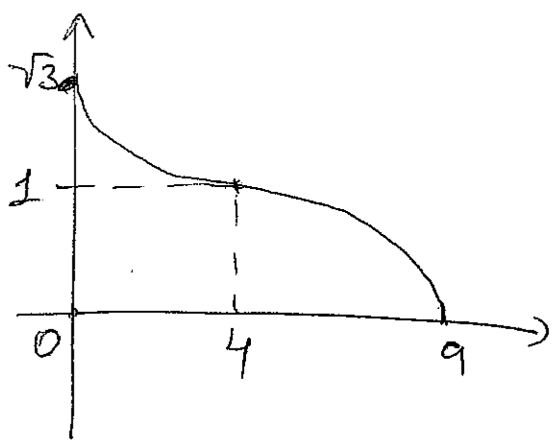
$3 - \frac{3}{2}\sqrt{6} < 0$ and $\left(\sqrt{6 - 2\sqrt{6}}\right)^3 > 0$

$\therefore$ There is a change in concavity,

When $x = 4$ $f(x) = \sqrt{3-2} = \sqrt{1} = 1$.

$\therefore$ The inflexion point is at $(4, 1)$

(iv) $A = \int_0^9 \sqrt{3-\sqrt{x}}\,dx = \int_0^{\sqrt{3}} x\,dy$



$$= \int_0^{\sqrt{3}} 9 - 6y^2 + y^4\,dy$$

$$= \left[9y - \frac{6y^3}{3} + \frac{y^5}{5}\right]_0^{\sqrt{3}}$$

$$= 9\sqrt{3} - 2\times 3\sqrt{3} + \frac{(\sqrt{3})^5}{5} - 0$$

$$= 9\sqrt{3} - 6\sqrt{3} + \frac{9\sqrt{3}}{5}$$

$$= 3\sqrt{3} + \frac{9\sqrt{3}}{5}$$

$$= \frac{15\sqrt{3} + 9\sqrt{3}}{5}$$

$$= \frac{24\sqrt{3}}{5}$$

$$y = \sqrt{3-\sqrt{x}}$$

$$y^2 = 3 - \sqrt{x}$$

$$y^2 - 3 = -\sqrt{x}$$

$$\sqrt{x} = 3 - y^2$$

$$x = (3-y^2)^2$$

$$x = 9 - 2\times 3\times y^2 + y^4$$

$$x = 9 - 6y^2 + y^4$$

Examiner's Comments.

SECTION 1 Q1—10

- Generally well done, however Q2, 3 and 9 caused some difficulty.

10% of candidates were incorrect on Q9
18% " " " " " Q2
21% " " " " " Q3

SECTION 2:

QUESTION 12:

(a) When making a substitution the integrand should contain one variable only.

(b) (i) 'Find the values' : requires students to set out their working and NOT use some short-cut method to simply write down the values of A, B and C.

(iii) Well done

(c) Well done

(d) Generally well done.

QUESTION 14:

(a) (i), (ii) and (iii) well done

(iv) Some students unable to integrate simply arrived at the answer magically.

(b) (i), (ii) and (iii) well done.

(iv) Many students failed to realise that the solution involved $m_1 \times m_2 = -1$

QUESTION 16:

(a) well done

(b) Generally well done although some students wrote $\left[(10-x) - \frac{x^2}{9}\right]^2$

rather than $(10-x)^2 - \left(\frac{x^2}{9}\right)^2$

(c) (i) Some students failed to realise there are TWO parts to this question

ie $\sqrt{x} \Rightarrow x \geq 0$ and $\sqrt{3-\sqrt{x}} \Rightarrow 3-\sqrt{x} \geq 0$

(ii) Too many students failed to realise that $y \geq 0$

(iii) Finding $f''(x)$ and solving $f''(x) = 0$ caused few problems. However, too many students failed to test for point of inflexion.

(iv) Sketches were generally quite poor. Students did not analyse $f'(x)$ at $x = 0$ and $x = 9$. Even those who did show that $f'(x)$ is undefined at $x = 0, 9$ then failed to interpret this correctly in their graphs.