Student Number: ____________________ Class Teacher: ____________________

St George Girls High School

Trial Higher School Certificate Examination

2016

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen.
- Write your **student number** on each booklet.
- Board-approved calculators may be used.
- Marks may be deducted for careless or poorly presented work.
- A Reference Sheet is provided.
- In Question 11-16, show relevant mathematical reasoning and /or calculations

Total Marks – 100

Section I Pages 2 – 5

10 marks

- Attempt Questions 1 – 10.
- Allow about 15 minutes for this section.
- Answer on the multiple-choice sheet provided at the back of this paper.

Section II Pages 6 – 11

90 marks

- Attempt Questions 11 – 16.
- Allow about 2 hours 45 minutes for this section.
- Begin each question in a new booklet.

Section I	/10
Section II	
Question 11	/15
Question 12	/15
Question 13	/15
Question 14	/15
Question 15	/15
Question 16	/15
Total	/100

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1. Let $z = 2 - 3i$. What is the value of z^{-1}?

(A) $-\frac{1}{5}(2+3i)$

(B) $\frac{1}{13}(2+3i)$

(C) $\frac{1}{5}(2-3i)$

(D) $\frac{1}{13}(2-3i)$

2. What is the value of $\int_0^2 \sqrt{\frac{x}{4-x}}\,dx$ using the substitution $x = 4\sin^2\theta$?

(A) 0.75π

(B) $\pi - 2$

(C) $\pi + 6$

(D) $3\pi - 8$

3. The polynomial $x^3 + 3x^2 + 2x - 1 = 0$ has roots α, β and γ.
Which polynomial has roots $\frac{2}{\alpha}, \frac{2}{\beta}$ and $\frac{2}{\gamma}$?

(A) $x^3 - 4x^2 - 12x - 8 = 0$

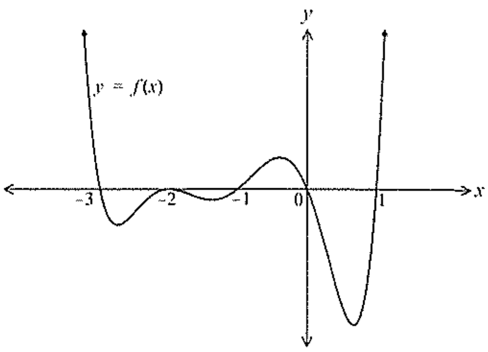
(B) $x^3 + 4x^2 - 12x + 8 = 0$

(C) $8x^3 - 12x^2 - 4x + 1 = 0$

(D) $8x^3 + 12x^2 + 4x - 1 = 0$

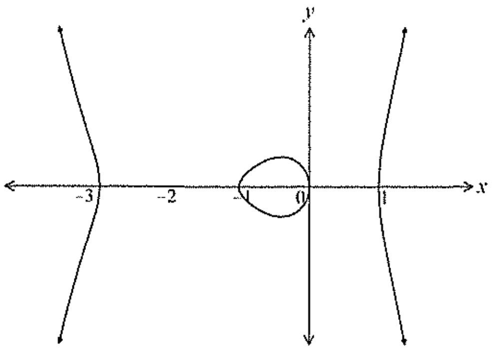
Section I (cont'd)

4. The diagram below shows the graph of the function $y = f(x)$.

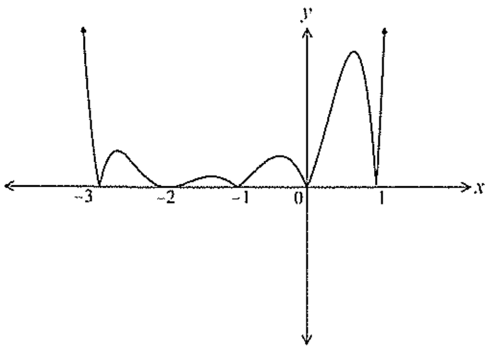


Which of the graphs below could represent the graph of $y = \dfrac{1}{f(x)}$?

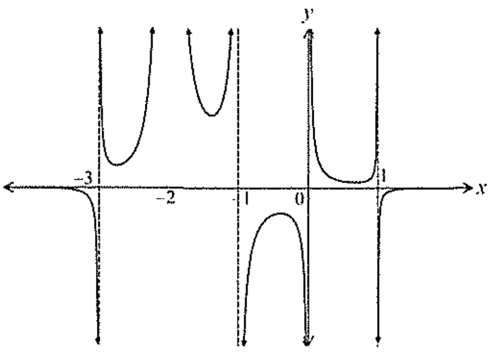
(A)



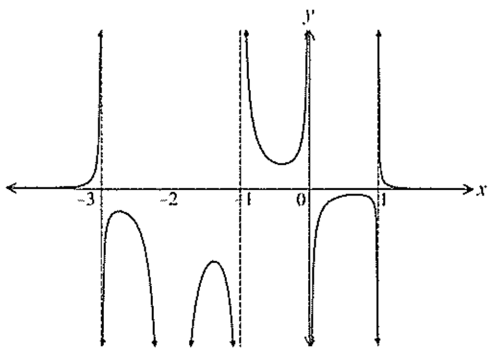
(B)



(C)



(D)



Section I (cont'd)

5. The area enclosed by the curve $y = 3x^2 - x^3$, the x-axis and the lines $x = 0$ and $x = 3$ is rotated about the y-axis.

What is the volume of the solid generated using the method of cylindrical shells?

(A) $\frac{27\pi}{4}$

(B) 12π

(C) $\frac{243\pi}{10}$

(D) $\frac{116\pi}{5}$

6. Given that $z = 3\left(cos\frac{\pi}{6} + isin\ \frac{\pi}{6}\right)$, what is the value of $(\bar{z})^3$?

(A) $9\left(cos\frac{\pi}{2} - isin\ \frac{\pi}{2}\right)$

(B) $9\left(cos\frac{\pi}{2} + isin\ \frac{\pi}{2}\right)$

(C) $27\left(cos\frac{\pi}{2} + isin\ \frac{\pi}{2}\right)$

(D) $27\left(cos\frac{\pi}{2} - isin\ \frac{\pi}{2}\right)$

7. What is the value of $\int_2^3 \frac{1}{\sqrt{4x-x^2}}dx$?

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{6}$

(D) $\frac{\pi}{4}$

Section I (cont'd)

8. Which expression gives the gradient of the normal to the curve $x^3 + xy + y^2 = 7$ at any point on the curve?

(A) $\dfrac{-3x^2 - y}{x + 2y}$

(B) $\dfrac{x + 2y}{3x^2 + y}$

(C) $\dfrac{3x^2 + y}{x + 2y}$

(D) $\dfrac{-x - 2y}{3x^2 + y}$

9. The hyperbola $16x^2 - 9y^2 = 144$ has foci $S\,(5, 0)$ and $S'(-5, 0)$.

What are the equation of its' directrices?

(A) $x = \frac{9}{5}$ and $x = -\frac{9}{5}$

(B) $y = \frac{9}{5}$ and $y = -\frac{9}{5}$

(C) $y = \frac{12}{5}$ and $y = -\frac{12}{5}$

(D) $x = \frac{12}{5}$ and $x = -\frac{12}{5}$

10. A particle of mass m is projected vertically upwards with an initial velocity of u ms^{-1} in a medium in which the resistance to the motion is proportional to the square of the velocity v ms^{-1} of the particle or mkv^2. Let x be the displacement in metres of the particle above the point of projection, O, so that the equation of motion is $\ddot{x} = -\left(g + kv^2\right)$ where g ms^{-2} is the acceleration due to gravity. Assume $k = 10$ and the acceleration due to gravity is 10 ms^{-2}.
Which of the following gives the correct expression for the time taken?

(A) $t = \dfrac{1}{10}\left(\tan^{-1} u + \tan^{-1} v\right)$

(B) $t = \dfrac{1}{10}\left(\tan^{-1} v - \tan^{-1} u\right)$

(C) $t = \dfrac{1}{10}\left(\tan^{-1} u - \tan^{-1} v\right)$

(D) $t = \dfrac{1}{10}\left(\tan^{-1} v + \tan^{-1} u\right)$

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

a) Consider the complex numbers $\omega = -1 + \sqrt{3}\,i$ and $Z = \sqrt{3} + 2i$.

(i) Evaluate $\omega\bar{z}$. 1

(ii) Evaluate $|\omega|$. 1

(iii) Find the value of $arg\ (\omega)$. 1

(iv) Find the value of ω^5. 1

(v) Evaluate $\frac{\omega}{Z}$. 2

b) Sketch the region in the Argand diagram where $-\frac{\pi}{6} \le arg\ z \le \frac{\pi}{3}$ and $z\bar{z} \le 4$. 3

c) Evaluate $\int_0^4 x\sqrt{x^2+9}\,dx$. 3

d) Find $\int \frac{\sqrt{x^2-25}}{x}\,dx$, using the trigonometric substitution $x = 5\,sec\,\theta$. 3

Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

a) A solid is formed by rotating about the y-axis the region bounded by the curve $y = \sin x$ and the x-axis between $0 \le x \le \pi$. Find the volume of this solid using the method of cylindrical shells. 4

b) (i) If $\frac{x}{x^2 - x - 6} \equiv \frac{A}{x-3} + \frac{B}{x+2}$, find the values of A and B. 2

(ii) Hence find $\int \frac{\sin\theta\cos\theta}{\sin^2\theta - \sin\theta - 6}\,d\theta$ 2

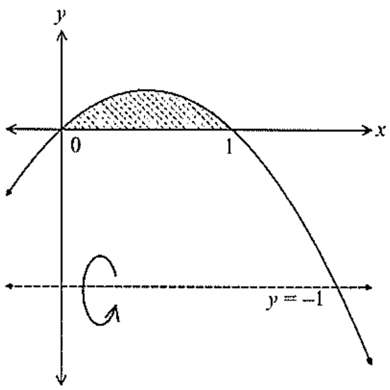
c) Let two complex numbers be $z_1 = 2(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$ and $z_2 = 2i$.

(i) On an Argand diagram sketch the vectors OA and OB to represent z_1 and z_2 respectively. 1

(ii) Draw the vectors $z_1 + z_2$ and $z_1 - z_2$ on the same Argand diagram. 1

(iii) What are the exact values of $\arg(z_1 + z_2)$ and $\arg(z_1 - z_2)$? 2

d) The area enclosed by the curve $y = x(1 - x)$ and the x-axis is rotated about the line $y = -1$. 3



Find the volume of the solid of revolution formed.

Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) A particle of mass m is moving in a straight line under the action of a force. 3

$$F = \frac{m}{x^3}(6 - 10x)$$

What is the velocity in any position, if the particle starts from rest at $x = 1$?

b) Consider the function $y = \cos^{-1}(e^x)$

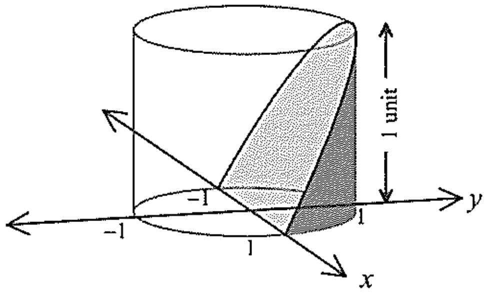
(i) Find the domain and the range. 2

(ii) Sketch the graph of $y = \cos^{-1}(e^x)$? 2

(iii) Hence or otherwise sketch the graph of $y = \left[\cos^{-1}(e^x)\right]^2$. 1

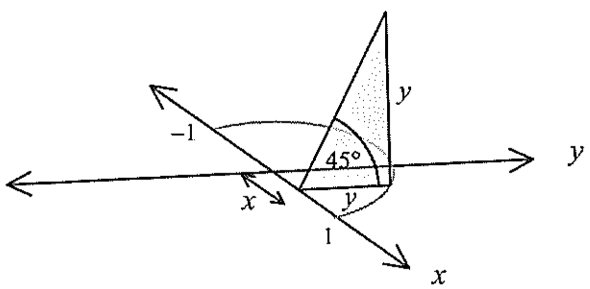
c) Use integration by parts to evaluate $\int_1^e \frac{\ln x}{x^2}\,dx$ 3

d) A cylinder has the circle $x^2 + y^2 = 1$ as its base and is 1 unit in height. 4
The shaded wedge is formed by a plane, which passes along the x-axis and is angled at 45° to the base of the cylinder.

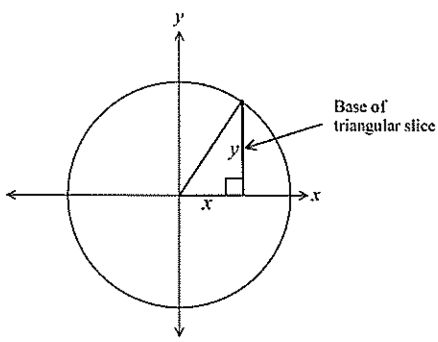


Slices are taken through this wedge at right angles to the x axis, and perpendicular to the base of the cylinder, through a point (x, y) on the circle

Triangular slice through the wedge



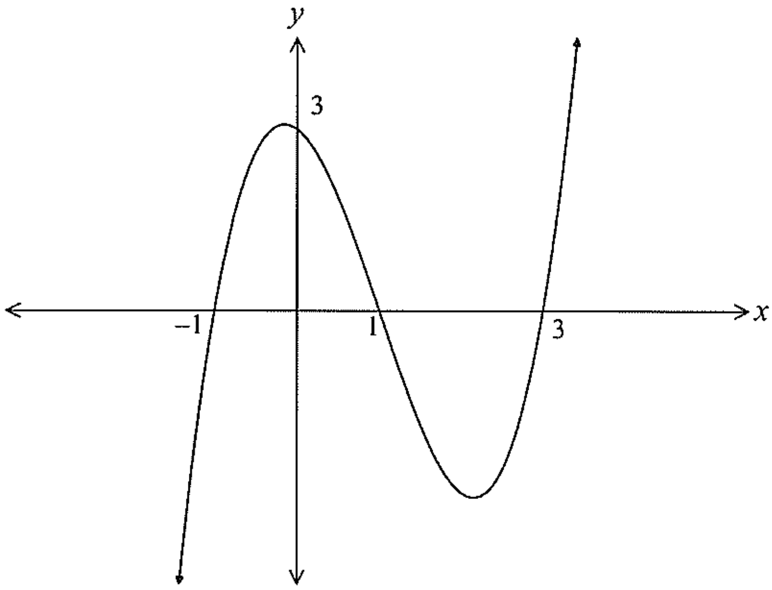
Base of the cylinder



Find the volume of the wedge.

Question 14 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) A sketch of the function $f(x)$ is shown below.



Draw a separate half-page graph for each of the following functions, showing all asymptotes and intercepts.

(i) $y = |f(x)|$ 2

(ii) $y^2 = f(x)$ 2

(iii) $y = f(|x|)$ 2

(iv) $y = e^{f(x)}$ 2

b) If one root of the equation $x^3 - px^2 + qx - r = 0$ is equal to the product of the other two, 3

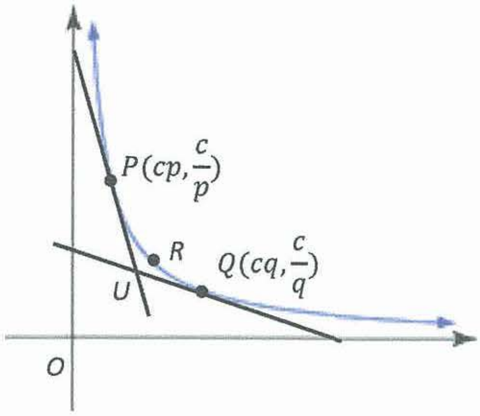
show that:

$$(q + r)^2 = r(p + 1)^2$$

c) Given that $P(x) = x^4 - 2x^3 + 2x - 1 = 0$ has a root of multiplicity 3, find the factors of $P(x)$. 4

Question 15 (15 marks) Use a SEPARATE writing booklet. **Marks**

a)



On the hyperbola $xy = c^2$, three points P, Q and R are on the same branch, with parameters p, q and r respectively. The tangents at P and Q intersect in U.

(i) If the equation of the tangent at P is $x + p^2y = 2cp$, show that the coordinates of U are: 1

$$\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$$

(ii) If O, U and R are collinear, prove that 3

$$r^2 = pq.$$

b) (i) Let $I_n = \int_0^1 (1 - x^r)^n dx$, where $r > 0$, for $n = 0, 1, 2, 3, \ldots$ 3

Show that $I_n = \dfrac{nr}{nr+1} I_{n-1}$.

(ii) Hence or otherwise, find the value of $I_n = \int_0^1 (1 - x^{\frac{3}{2}})^3 dx$. 2

c) (i) Show that the normal to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point $P(a \sec\theta, b\tan\theta)$ is given by: 3

$$ax\sin\theta + by = (a^2 + b^2)\tan\theta\,.$$

(ii) If the normal meets the $x-$axis in G and PN is the perpendicular from P onto the $x-$axis, prove that $OG = e^2 ON$. 3

Question 16 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) (i) Show that the condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is: 3

$$c^2 = 16m^2 + 9$$

(ii) Hence show that the pair of tangents drawn from (3, 4) to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ are at right angles to each other. 2

b) A polynomial $P(x)$ is divided by $x^2 - a^2$, where $a \neq 0$, and the remainder is $px + q$.

(i) Show that $p = \frac{1}{2a}\{P(a) - P(-a)\}$ and $q = \frac{1}{2}\{P(a) + P(-a)\}$. 2

(ii) Find the remainder when the polynomial $P(x) = x^n - a^n$ is divided by $x^2 - a^2$ for the cases:

(α) n even 1

(β) n odd. 1

c) A mass of 1 kg is moving along the x –axis under the influence of two forces: an accelerating force of $\frac{F}{v}$ and a resisting force of kv^2, where v is the velocity of the mass.

(i) Write down the equation of motion. 1

(ii) If the maximum velocity attained is V, show that $k = \frac{F}{V^3}$ 2

(iii) Show that the distance travelled from $v = \frac{V}{4}$ to $v = \frac{V}{2}$ is $\frac{V^3}{3F} ln\frac{9}{8}$. 3

End of Examination

St George Girls High School

Solutions - Ext 2 Trial Exam 2016

Q1 $z = 2 - 3i$

$z^{-1} = \frac{1}{2-3i} \times \frac{2+3i}{2+3i}$

$= \frac{2+3i}{4+9}$

$= \frac{1}{13}(2+3i)$

B.

Q2. $x = 4\sin^2\theta$ when $x=0,\ \theta=0$

$\frac{dx}{d\theta} = 8\sin\theta\cos\theta$ $x=2,\ \theta=\frac{\pi}{4}$

$dx = 8\sin\theta\cos\theta\, d\theta$

$\int_0^2 \sqrt{\frac{x}{4-x}}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}}\,.\,8\sin\theta\cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{4}} \sqrt{\frac{\cancel{4}\sin^2\theta}{4\cos^2\theta}}\,.\,8\sin\theta\cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cancel{\cos\theta}}\,.\,8\sin\theta.\cancel{\cos\theta}\,d\theta$

$= \int_0^{\frac{\pi}{4}} 8\sin^2\theta\,d\theta$

$= 8\int_0^{\frac{\pi}{4}} \frac{1}{2}(1-\cos 2\theta)\,d\theta$

$= \frac{8}{2}\left[x - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$

$= \pi - 2$

B.

Q3 For $x^3 + 3x^2 + 2x - 1 = 0$.

Let $y = \frac{2}{x}$ $\therefore x = \frac{2}{y}$ sub $\frac{2}{x}$ for x

$$\left(\frac{2}{x}\right)^3 + 3\left(\frac{2}{x}\right)^2 + 2\left(\frac{2}{x}\right) - 1 = 0$$

$$\frac{8}{x^3} + 3 \times \frac{4}{x^2} + \frac{4}{x} - 1 = 0$$

Mult by x^3

$$8 + 12x + 4x^2 - x^3 = 0$$

$$\therefore x^3 - 4x^2 - 12x - 8 = 0$$

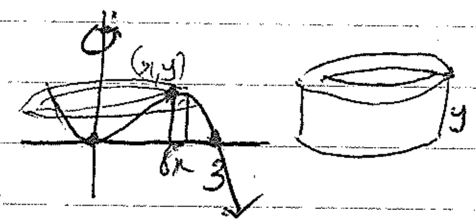
A.

Q4. D.

D

Q5. Use the method of cylindrical shells

$$V = \int_a^b 2\pi xy \, dx$$



$$= \int_0^3 2\pi x(3x^2 - x^3) \, dx$$

$$= 2\pi \int_0^3 (3x^3 - x^4) \, dx$$

$$= 2\pi \left[\frac{3}{4}x^4 - \frac{x^5}{5}\right]_0^3$$

$$= 2\pi \left\{\left[\frac{3}{4}\left(3^4 - \frac{3^5}{5}\right)\right] - \left[\frac{3}{4}0 - \frac{0}{5}\right]\right\}$$

$$= \frac{243\pi}{10}$$

C

Q6 $z = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$\bar{z} = 3\left(\cos\frac{\pi}{6} - i\sin\pi/6\right)$

$\bar{z}^3 = 3^3\left(\cos\frac{3\pi}{6} - i\sin\frac{3\pi}{6}\right)$

$= 27\left(\cos\frac{\pi}{2} - i\sin\pi/2\right)$ D

Q7 $\int_2^3 \frac{1}{\sqrt{4x-x^2}}\,dx$

$= \int_2^3 \frac{1}{\sqrt{-(x^2-4x+4)+4}}\,dx$

$= \int_2^3 \frac{1}{\sqrt{4-(x-2)^2}}\,dx$

$= \left[\sin^{-1}\frac{(x-2)}{2}\right]_2^3$ C.

$= \sin^{-1}\frac{1}{2}$

$= \pi/6$

Q8 $x^3 + xy + y^2 = 7$

By implicit differentation

$3x^2 + y + x\cdot\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(x+2y) = -3x^2 - y$

$\frac{dy}{dx} = \frac{-3x^2-y}{x+2y}$

Gradient of tangent $m_1 = \frac{-3x^2-y}{x+2y}$

Gradient of normal $m_2 = \frac{x+2y}{3x^2+y}$ B.

Q9 $16x^2 - 9y^2 = 144$

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

$\therefore a = 3, \; b = 4$

Foci are $(ae, 0)$ and $(-ae, 0)$

$= (5, 0)$ and $(-5, 0)$

$ae = 5$

$3e = 5$

$e = \frac{5}{3}$

Equation of directrices

$$x = \pm \frac{a}{e}$$

$$= \pm \frac{3}{5/3}$$

$$= \pm \frac{9}{5}$$

A.

Q10 $\ddot{x} = -(g + kv^2)$

$= -(10 + 10v^2)$

$$\frac{dv}{dt} = -10(1 + v^2)$$

$$\frac{dt}{dv} = \frac{-1}{10(1+v^2)}$$

$$t\Big]_0^t = \frac{-1}{10} \cdot \tan^{-1} v \Big]_u^v$$

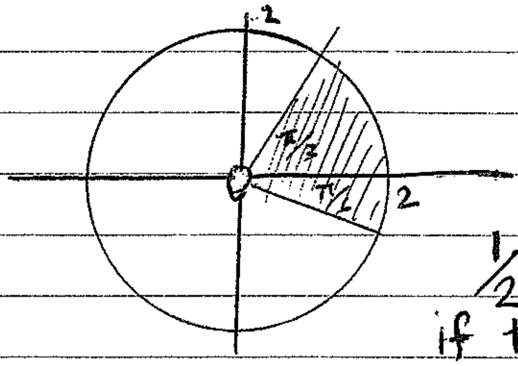
$$t = \frac{-1}{10}\left[\tan^{-1} v - \tan^{-1} u\right]$$

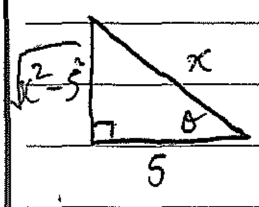
$$= \frac{1}{10}\left[\tan^{-1} u - \tan^{-1} v\right]$$

C

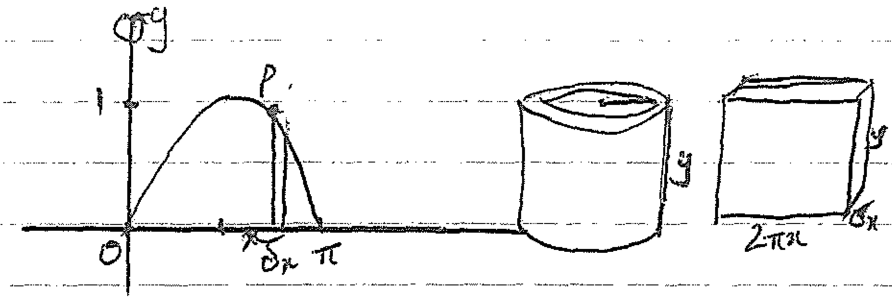
Ext 2 Trial Exam 2016 St George Girls HS — Solutions

Solution	Mark
Q11	
a) i) $\omega = -1+\sqrt{3}i$ $\quad z = \sqrt{3}+2i \quad \bar{z} = \sqrt{3}-2i$	
$\omega\bar{z} = (-1+\sqrt{3}i)(\sqrt{3}-2i)$	
$= -\sqrt{3}+2i+3i+2\sqrt{3}$	
$= \sqrt{3}+5i$	1 mark for correct answer (1)
ii) $\lvert\omega\rvert = \sqrt{(-1)^2+(\sqrt{3})^2}$	
$= \sqrt{4}$	
$= 2$	1 mark for correct answer (1)
iii) $\arg\omega = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$ $= \theta$	
related angle $= \tan^{-1}\sqrt{3}$	
$= \pi/3$	
but ω is in 2nd quadrant	
$\therefore \arg\omega = \pi - \pi/3$	
$= \frac{2\pi}{3}$	1 mark (1)
iv) $\omega = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$	
$\omega^5 = 2^5\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^5$	$\frac{1}{2}$
$= 32\left(\cos\frac{10\pi}{3}+i\sin\frac{10\pi}{3}\right)$	
$= 32\left(\cos\frac{-2\pi}{3}+i\sin\frac{-2\pi}{3}\right)$	$\frac{1}{2}$ (1)
$32\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -16 - 16\sqrt{3}i$	
v) $\frac{\omega}{z} = \frac{-1+\sqrt{3}i}{\sqrt{3}+2i} \times \frac{\sqrt{3}-2i}{\sqrt{3}-2i}$	
$= \frac{\sqrt{3}+5i}{3+4}$ from (i)	1
$= \frac{\sqrt{3}+5i}{7}$	1 (2)

Solutions	Mark
Q11	
b) $z\bar{z} = 4$	
$(x-iy)(x+iy) = 4$	
$x^2+y^2 = 4$	1 mark for drawing correct circle
$z\bar{z} \leq 4$ is interior of circle	
	1 mark for correct range of argument.
$\frac{1}{2}$ mark if the origin inclusive.	1 mark for correct region. (3)
c) $I = \int_0^4 x\sqrt{x^2+9}\,dx$	
Let $u = x^2+9$	1 mark for substitution and change of limits.
$du = 2x\,dx$	
when $x=0$, $u=9$	
$x=4$, $u=25$	
$\therefore I = \frac{1}{2}\int_0^4 2x\sqrt{x^2+9}\,dx$	
$= \frac{1}{2}\int_9^{25} \sqrt{u}\,du$	1 mark for integrand
$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_9^{25}$	
$= \frac{1}{3}\left[25^{3/2} - 9^{3/2}\right]$	
$= \frac{1}{3}\left[125 - 27\right]$	
$= \frac{98}{3}$ or $32\frac{2}{3}$	1 mark for answer. (3)

Solution	Mark
Q11d) $I = \int \frac{\sqrt{x^2-25}}{x}\,dx$	
Let $x = 5\sec\theta$	
$dx = 5\sec\theta\tan\theta\,d\theta$	
$I = \int \frac{\sqrt{(5\sec\theta)^2 - 25}}{5\sec\theta}.5\sec\theta\tan\theta\,d\theta$	1 mark
$= \int \frac{\sqrt{25\sec^2\theta - 25}}{5\cancel{\sec\theta}} . 5\cancel{\sec\theta}\tan\theta\,d\theta$	
$= \int \sqrt{25(\sec^2\theta - 1)}\ \tan\theta\,d\theta$	
$= \int 5\sqrt{(\tan^2\theta)}\ .\tan\theta\,d\theta$	
$= 5\int \tan\theta . \tan\theta\,d\theta$	
$= 5\int \tan^2\theta\,d\theta$	
$= 5\int \sec^2\theta - 1\,d\theta$	
$= 5\tan\theta - 5\theta + c$	1 mark
 	A few students did not change the variable back to x.
$\sec\theta = \frac{x}{5}$	
$\cos\theta = \frac{5}{x}$	
$\therefore I = 5\left(\frac{\sqrt{x^2-25}}{5}\right) - 5\sec^{-1}\left(\frac{x}{5}\right) + c$	
$= \sqrt{x^2 - 25} - 5\sec^{-1}\left(\frac{x}{5}\right) + c$	1 mark for correct answer

3

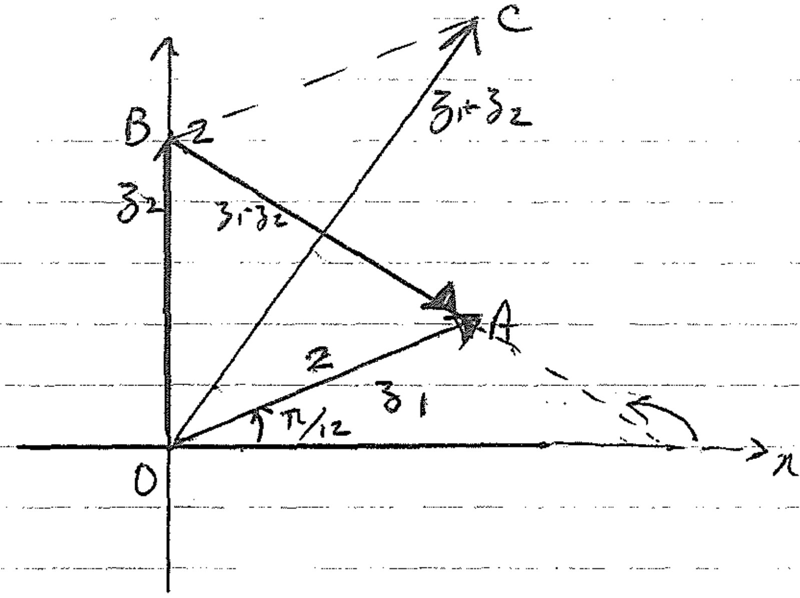
Solution	Mark
Q12 a)	
	1 mark
$A = 2\pi x y$	1 mark
$= 2\pi x \sin x$	Done very well by most students
$\delta V = 2\pi x \sin x \, \delta x$	
$V = \lim_{\delta x \to 0} \sum_{x=0}^{\pi} 2\pi x \sin x \, \delta x$	
$= 2\pi \int_0^{\pi} x \sin x \, dx$	1 mark
Using Integration by parts	
$= 2\pi$	
$u = x \quad v' = \sin x$	
$u' = 1 \quad v = -\cos x$	
$V = 2\pi\left(\left[-x\cos x\right]_0^{\pi} - \int_0^{\pi} -\cos x \, dx\right.$	1 mark
$= 2\pi\left((\pi - 0) + \left[\sin x\right]_0^{\pi}\right.$	
$= 2\pi\left[\pi + (0-0)\right]$	
$= 2\pi^2$	

Solution	Mark
Q12	
b) i) $x = A(x+2) + B(x-3)$	
when $x = -2$	
$-2 = A(0) + -5B$	
$B = \frac{2}{5}$	1 mark
when $x = 3$	
$3 = A(5) + B(0)$	
$A = \frac{3}{5}$	1 mark
$\therefore \dfrac{x}{x^2-x-6} = \dfrac{3}{5(x-3)} + \dfrac{2}{5(x+2)}$ --- (1)	
ii) $\displaystyle\int \frac{\sin\theta\cos\theta}{\sin^2\theta - \sin\theta - 6}\,d\theta$ Let $u = \sin\theta$, $du = \cos\theta\, d\theta$	Majority of students did well on this question
$= \displaystyle\int \frac{u\,du}{u^2-u-6}$	1 mark
$= \displaystyle\int \left(\frac{3}{5(u-3)} + \frac{2}{5(u+2)}\right) du$ from (1)	
$= \displaystyle\frac{3}{5}\int \frac{1}{u-3}\,du + \frac{2}{5}\int \frac{1}{u+2}\,du$	
$= \frac{3}{5}\ln(u-3) + \frac{2}{5}\ln(u+2) + C$	1 mark
$= \frac{3}{5}\ln(\sin\theta - 3) + \frac{2}{5}\ln(\sin\theta + 2) + C$	

Solution	Mark

Q12

c) i)



1 mark for z_1 at A and z_2 at B.

ii) On diagram

½ mark for vector z_1+z_2

½ mark for vector z_1-z_2

iii) Let vector z_1+z_2 be OC

Now OC and AB form a parallelogram.

Since $OA = OB = 2$

$\therefore$ OBCA is a rhombus.

Students had problems with finding the angle

$\angle BOA = \frac{\pi}{2} - \pi/12$

$\therefore \angle AOC = \frac{1}{2} \times \left(\frac{\pi}{2} - \frac{\pi}{12}\right)$ diagonals bisect the angles through which they pass

$= \frac{5\pi}{24}$

$\therefore \arg(z_1+z_2) = \pi/12 + \frac{5\pi}{24}$

$= \frac{7\pi}{24}$

1 mark.

Now $OC \perp AB$ (diagonals of a rhombus intersect at right angles

$\arg(z_1-z_2) = -\frac{\pi}{2} + \frac{7\pi}{24}$ (exterior angles)

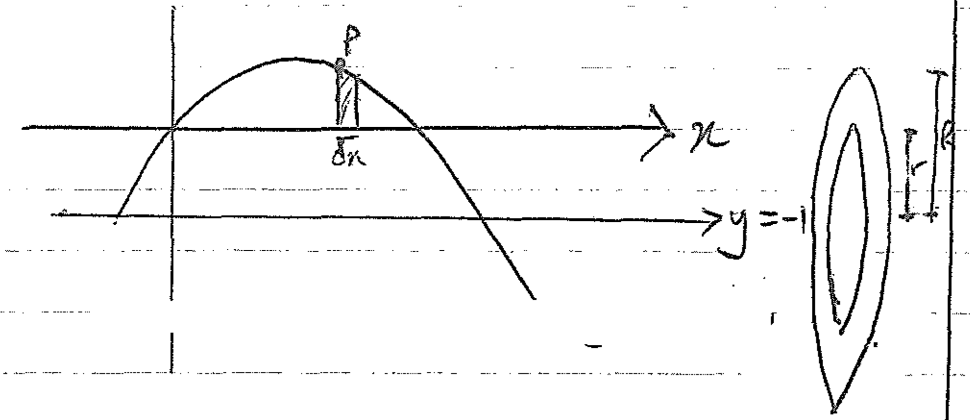
$= \frac{-5\pi}{24}$

1 mark

Solution	Mark

Q12

d) i)



$$A = \pi(R^2 - r^2)$$ $\quad R = y+1,\ r = 1,\ y = x(1-x)$

$$= \pi\left((y+1)^2 - 1^2\right)$$

$$= \pi\left[(x - x^2 + 1)^2 - 1\right]$$ 1

Students did not interpret the line about which the solid is rotated properly and so got the radii incorrect

$$= \pi\left[(x-x^2)^2 + 2(x-x^2) + 1 - 1\right]$$

$$= \pi\left[x^2 - 2x^3 + x^4 + 2x - 2x^2\right]$$

$$\delta V = \pi\left[x^4 - 2x^3 - x^2 + 2x\right]\delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{1} \pi\left[x^4 - 2x^3 - x^2 + 2x\right]\delta x$$ 1

$$= \pi\int_0^1 (x^4 - 2x^3 - x^2 + 2x)\,dx$$

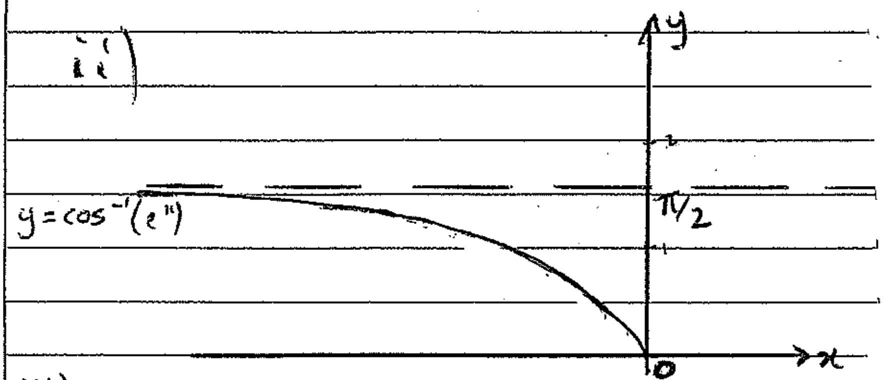
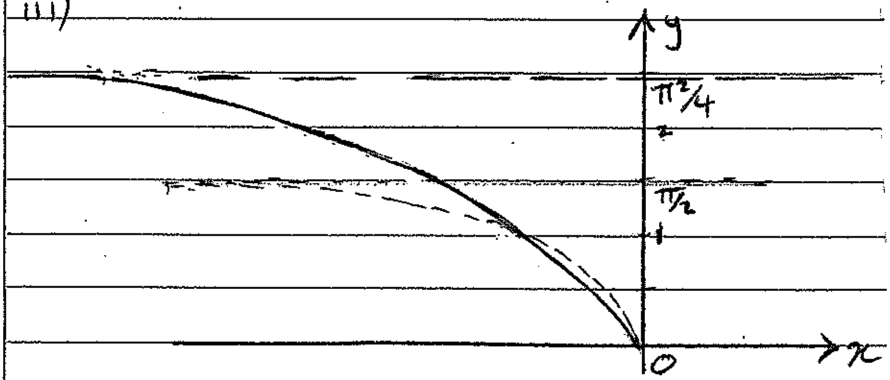
$$= \pi\left[\frac{x^5}{5} - \frac{x^4}{2} - \frac{x^3}{3} + x^2\right]_0^1$$

$$= \pi\left[\left(\frac{1}{5} - \frac{1}{2} - \frac{1}{3} + 1\right) - 0\right]$$

$$V = \frac{11\pi}{30} \text{ cu.}$$ 1

MATHEMATICS EXTENSION 2– QUESTION 13		
SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
13 a) $F = \frac{m}{x^3}(6-10x)$		
If $F = ma$		
$ma = \frac{m}{x^3}(6-10x)$		
$a = \frac{6}{x^3} - \frac{10}{x^2}$		Students who did
$\frac{d\frac{1}{2}v^2}{dx} = 6x^{-3} - 10x^{-2}$		not use $\ddot{x} = \frac{d(\frac{1}{2}v^2)}{dx}$
$\frac{1}{2}v^2 = \int 6x^{-3} - 10x^{-2}\,dx$	1 mark	found the question
$= -3x^{-2} + 10x^{-1} + c$		more difficult.
when $v = 0$, $x = 1$		
$\frac{1}{2}(0) = -3 + 10 + c$		
$c = -7$	1 mark	
$\therefore \frac{1}{2}v^2 = \frac{-3}{x^2} + \frac{10}{x} - 7$		
$v^2 = \frac{-6}{x^2} + \frac{20}{x} - 14$		
$v = \pm\sqrt{\frac{-6}{x^2} + \frac{20}{x} - 14}$	1 mark	
$= \pm\sqrt{\frac{-6 + 20x - 14x^2}{x^2}}$		½ mark off if
$v = \pm\frac{1}{x}\sqrt{2(-3 + 10x - 7x^2}$		v was not $\pm$.
	3	

MATHEMATICS EXTENSION 2 – QUESTION 13

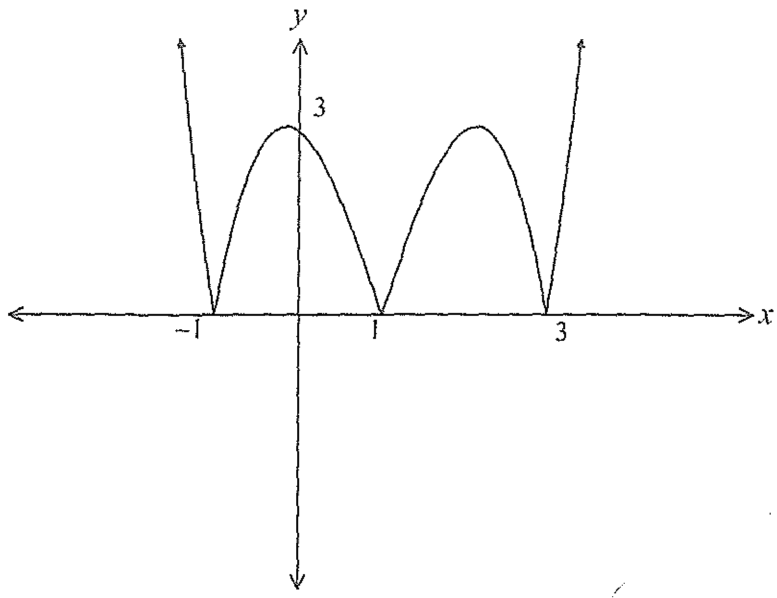
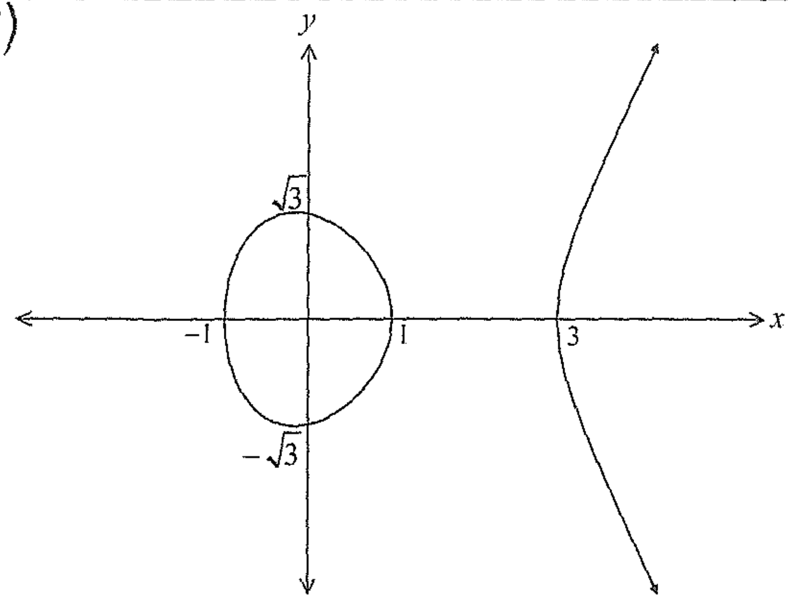
SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
13b) i) For $y=\cos^{-1}x$ $D_f: -1 < x < 1$ $\therefore D_f: -1 \leq e^{x} \leq 1$ but $e^{x} > 0$ for all x $\therefore 0 < e^{x} \leq 1$ For $y=\cos^{-1}(e^{x})$ $D: x \leq 0$ $R: 0 \leq y < \frac{\pi}{2}$	 1 1	Many students included $\pi/2$ in the range but in the diagram had it as an asymptote.
ii) 	1 mark asymptote 1 mark graph	This graph was mostly drawn correctly, however the graphs are still ~~not~~ drawn ⅓ page
iii) 	1 mark	This question was poorly done. Many students did not draw the original graph correctly which therefore produced a change in concavity at the point where the graph met $y=1$.
Note: $\frac{1}{2}$ mark off if the graph showed any change of concavity		

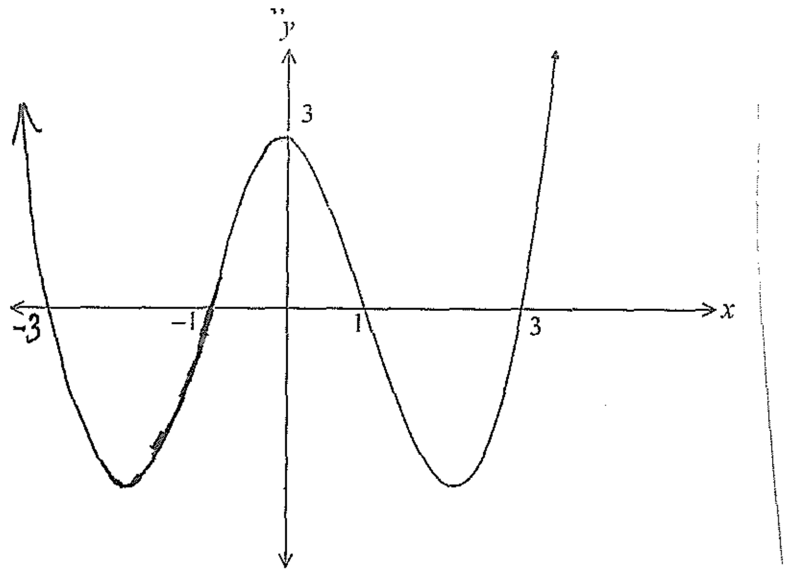
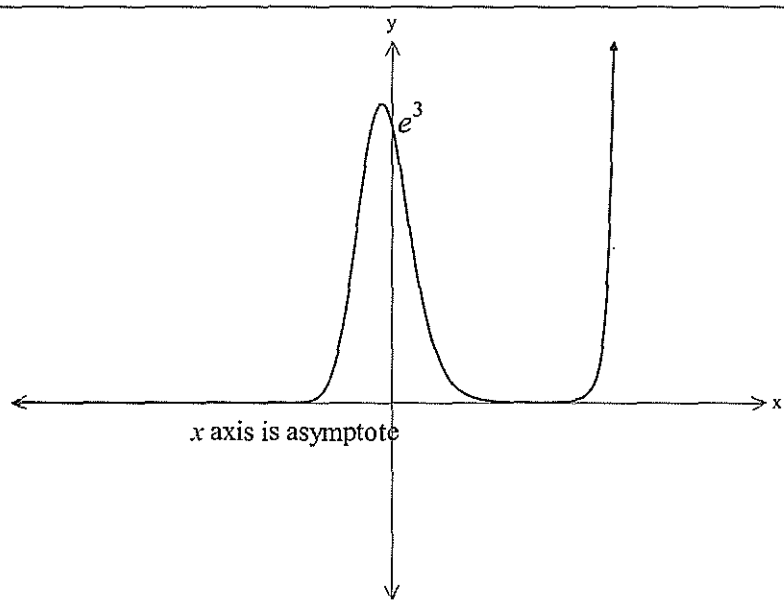
MATHEMATICS EXTENSION 2 – QUESTION 13

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) Let $I = \int_1^e \frac{\ln x}{x^2}\,dx$		
Let $u = \ln x \quad v' = x^{-2}$ $u' = \frac{1}{x} \quad v = \frac{-1}{x}$		This question was done well overall.
$\therefore I = \left[\frac{-\ln x}{x}\right]_1^e - \int_1^e -\frac{1}{x^2}\,dx$	1	
$= \left[\left(\frac{-\ln e}{e} - 0\right) + \int_1^e x^{-2}\,dx\right.$	1	
$= -\frac{1}{e} - \left[x^{-1}\right]_1^e$		
$= -\frac{1}{e} - \left(\frac{1}{e} - 1\right)$		
$= 1 - \frac{2}{e}$	1	

Solution	Mark
Q13 d)	
If $x^2+y^2=1$	
Semi-circle : $y=\sqrt{1-x^2}$	
Area of triangle :	
$A(x)=\frac{1}{2}bh$	
$=\frac{1}{2}y^2$	
$=\frac{1}{2}(1-x^2)$	1
$V=\int_0^b A(x)\,dx$	
$=\int_{-1}^{1}\frac{1}{2}(1-x^2)\,dx$	1
$=2\int_0^1\frac{1}{2}(1-x^2)\,dx$	
$=\int_0^1 1-x^2\,dx$	
$=\left[x-\frac{x^3}{3}\right]_0^1$	1
$=1-\frac{1}{3}$	
$=\frac{2}{3}u^2$	1

4

Question 14		2016	
	Solution	Marks	Allocation of marks
a)	(i) 	2	2 marks for a correct graph with intercepts shown. 1 mark for graph with wrong intercepts or wrong orientation, or other minor error. Very well done
	(ii) 	2	2 marks for a correct graph with intercepts shown. 1 mark for graph with wrong intercepts or wrong orientation, or other minor error. Well done except for as $x \to \infty$

Question 14		2016	
	Solution	Marks	Allocation of marks
	(iii) 	2	2 marks for a correct graph with intercepts shown. 1 mark for graph with wrong intercepts or wrong orientation, or other minor error. Well done
	(iv) 	2	2 marks for a correct graph with intercepts shown. 1 mark for graph with wrong intercepts or wrong orientation, or other minor error. Most had problems with the local maximum

Solution | Mark

Q 14

b) Let roots be α, β, $\alpha\beta$

$\Sigma\alpha$: $\alpha+\beta+\alpha\beta = p$ --- ①
$\Sigma\alpha\beta$: $\alpha\beta+\alpha^2\beta+\beta^2\alpha = q$ --- ②
$\Sigma\alpha\beta\gamma$: $\alpha^2\beta^2 = r$ --- ③
1

② + ③

$$(q+r) = \alpha^2\beta^2+\alpha\beta+\alpha^2\beta+\beta^2\alpha$$ 1
$$= \alpha\beta(1+\alpha\beta+\alpha+\beta)$$
$$= \alpha\beta(1+p) \quad \text{from ①}$$

Well done by most students

$$(q+r)^2 = \alpha^2\beta^2(1+p)^2$$ 1
$$(q+r)^2 = r(1+p)^2 \quad \#$$

3

c) Let $P(x) = x^4-2x^3+2x-1$
$$P'(x) = 4x^3-6x^2+2$$
$$P''(x) = 12x^2-12x$$ 1
$$= 12x(x-1)$$

when $x=0$ $\quad P(0) \neq P'(0) \neq P''(0) \neq 0$
when $x=1$ $\quad P(1) = 1-2+2-1$
$$= 0$$

Well done by most students

$$P'(1) = 4-6+2$$
$$= 0$$
$$P''(1) = 0$$ 1

Since $P(1) = P'(1) = P''(1) = 0$

$\therefore (x-1)$ is a factor of multiplicity 3

$\therefore P(x) = (x-1)^3\,Q(x)$ 1
$$= (x-1)^3(ax+b)$$

Some students did not write as factors

but $a=1$ monic polynomial
$b=1$

$\therefore P(x) = (x-1)^3(x+1)$ 1

4

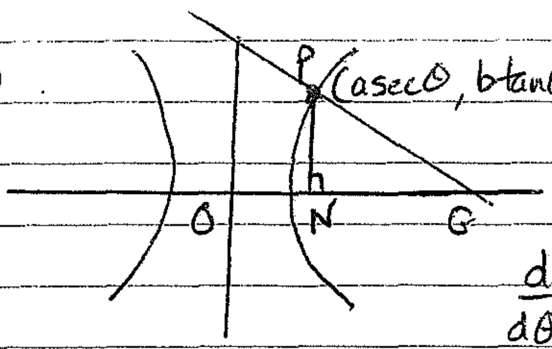
Solution	Mark
Q15	
a) i) The equation the tangent at P is	This question was done very well
$x+p^2y=2cp \quad \text{--} ①$	
The equation of the tangent at Q is	
$x+q^2y=2cq \quad \text{--} ②$	
Solving simultaneously	
① − ②	
$y(p^2-q^2)=2c(p-q)$	
$y\cancel{(p-q)}(p+q)=2c\cancel{(p-q)}$	
$y=\dfrac{2c}{p+q}$	½
Sub in ①	
$x+p^2\left(\dfrac{2c}{p+q}\right)=2cp$	
$x=2cp-\dfrac{2cp^2}{p+q}$	½
$=\dfrac{2cp(p+q)-2cp^2}{p+q}$	1
$=\dfrac{2cp^2+2cpq-2cp^2}{p+q}$	
$=\dfrac{2cpq}{p+q}$	
$\therefore U=\left(\dfrac{2cpq}{p+q}, \dfrac{2c}{p+q}\right)$	

MATHEMATICS EXTENSION 2– QUESTION 15

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
15 a) ii) Method 1:		This question was mostly done well.
Gradient of OU, $m_{OU} = \dfrac{\frac{2c}{p+q}}{\frac{2cpq}{p+q}}$		
$= \dfrac{2c}{p+q} \times \dfrac{p+q}{2cpq}$		If finding the equation of the line, OU, it is best to use the origin as the point not U.
$= \dfrac{1}{pq}$		
Equation of OU is:		
$y - 0 = \dfrac{1}{pq}(x-0)$		
$y = \dfrac{x}{pq}$ — (3)	1	
Now $xy = c^2$ — (4)		
Sub (3) in (4)		
$x\left(\dfrac{x}{pq}\right) = c^2$	1	
$\dfrac{x^2}{pq} = c^2$		
$x^2 = c^2pq$ — (5)		
but R is $\left(cr, \dfrac{c}{r}\right)$		
Sub $x = cr$ in (5)		
$(cr)^2 = c^2pq$	1	
$\cancel{c^2}r^2 = \cancel{c^2}pq$		
$r^2 = pq$	3	Some
Method 2: If OUR is collinear		Students who used this method, did not use the origin as the common point. Instead they used U. This made the question more difficult ie $M_{UR} = M_{UO}$
$m_{OU} = m_{OR}$	½	
$\dfrac{\frac{2c}{p+q} - 0}{\frac{2cpq}{p+q} - 0} = \dfrac{c/r - 0}{cr - 0}$	1	
$\dfrac{2}{2pq} = \dfrac{\frac{1}{r}}{r}$		
$\dfrac{1}{pq} = \dfrac{1}{r^2}$		
$r^2 = pq$		

Solution	Marks
Q15	
b) $I_n = \int_0^1 1(1-x^r)^n\,dx$	
Using Integration by parts $u=(1-x^r)^n \quad v'=1$	
$u' = n(1-x^r)^{n-1}\cdot -rx^{r-1} \quad v=x$	
$I_n = \left[x(1-x^r)^n\right]_0^1 - n\int_0^1 x(1-x^r)^{n-1}\cdot -rx^{r-1}dx$	1
$= 0 - nr\int_0^1 x(1-x^r)^{n-1}\cdot -x^{r-1}dx$	
$= -nr\int_0^1 -x^r(1-x^r)^{n-1}dx$	
$= -nr\int_0^1 [(1-x^r)-1](1-x^r)^{n-1}dx$	1
$= -nr\left(\int_0^1 (1-x^r)^n - (1-x^r)^{n-1}dx\right)$	
$I_n = -nr(I_n - I_{n-1}) = -nrI_n + nrI_{n-1}$	1
$= nrI_{n-1} - nrI_n$	
$I_n(nr+1) = nrI_{n-1}$	
$I_n = \frac{nr}{nr+1}\cdot I_{n-1}$	3
ii) $r=\frac{3}{2} \quad n=3$	
$I_3 = \frac{3\times\frac{3}{2}}{3\times\frac{3}{2}+1}\times I_2 = \frac{9}{11}I_2$	
$I_2 = \frac{2\times\frac{3}{2}}{2\times\frac{3}{2}+1}\times I_1 = \frac{3}{4}I_1$	
$I_1 = \frac{1\times\frac{3}{2}}{1\times\frac{3}{2}+1}\times I_0 = \frac{3}{5}I_0 \; ; \; I_0 = \int_0^1 1\,dx = 1$	1
$I_3 = \frac{9}{11}\times\frac{3}{4}\times\frac{3}{5}\times 1$	
$= \frac{81}{220}$	1
	2

Solution	Marks
Q15	
c) i)	



$x = a\sec\theta \quad y = b\tan\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta \quad \frac{dy}{d\theta} = b\sec^2\theta$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

$$m_T = \frac{b\sec\theta}{a\tan\theta}$$

$$m_N = \frac{-a\tan\theta}{b\sec\theta} = -\frac{a\sin\theta}{b} \qquad 1$$

Equation of normal:

$$y - b\tan\theta = \frac{-a\tan\theta}{b\sec\theta}(x - a\sec\theta) \qquad 1$$

$$yb\sec\theta - b^2\tan\theta\sec\theta = -ax\tan\theta + a^2\tan\theta\sec\theta$$

$$yb\sec\theta + ax\tan\theta = (a^2+b^2)(\tan\theta \times \sec\theta)$$

$\div \sec\theta$ $\quad yb + ax\frac{\sin\theta}{\cos\theta}\cdot\cos\theta = (a^2+b^2)\tan\theta \qquad 1$

$$yb + ax\sin\theta = (a^2+b^2)\tan\theta$$

(3)

ii) For G, when $y=0$, $x = \frac{(a^2+b^2)\tan\theta}{a\sin\theta} \qquad 1$

From $b^2 = a^2(e^2-1)$

$b^2 = a^2e^2 - a^2$

$a^2 + b^2 = a^2e^2$ $\qquad x_G = \frac{e^2a^2}{a\cos\theta} \qquad 1$

and $ON = x_N = a\sec\theta \qquad 1$

$$= e^2 a\sec\theta$$

$$OG = e^2 . ON$$

(3)

Solution	Mark

Q16

a) i) Solve $y = mx + c$ and $\frac{x^2}{16} + \frac{y^2}{9} = 1$ simultaneously:

$$\frac{x^2}{16} + \frac{(mx+c)^2}{9} = 1$$ 1

$$9x^2 + 16(m^2x^2 + 2mxc + c^2) = 144$$

$$9x^2 + 16m^2x^2 + 32mcx + 16c^2 = 144$$

$$x^2(9 + 16m^2) + 32mcx + 16c^2 - 144 = 0$$

For the line to be a tangent $\Delta = 0$

$$32^2m^2c^2 - 4(9+16m^2)(16c^2 - 144) = 0$$ } 1

Some students differentiated but were able to change to this

$$1024m^2c^2 - 4(144c^2 - 1296 + 256m^2c^2 - 2304m^2) = 0$$

$$\cancel{1024m^2c^2} - 576c^2 + 5184 - \cancel{1024m^2c^2} + 9216m^2 = 0$$ } 1

3

$$576c^2 = 9216m^2 + 5184$$

$$\div 576 \quad c^2 = 16m^2 + 9$$

ii) Let the tangents have the equation

$y = mx + c$ so at $(3, 4)$

$$4 = 3m + c$$

$$c = 4 - 3m \quad \text{--- (1)}$$

Condition for tangents is $c^2 = 16m^2 + 9$ --- (2)

Some students had problems forming the equation in m and c.

Sub (1) into (2)

$$(4 - 3m)^2 = 16m^2 + 9$$ 1

$$16 - 24m + 9m^2 = 16m^2 + 9$$

$$7m^2 + 24m - 7 = 0$$

Roots of this quadratic equation in m are gradients of the two tangents. So

Most student found the product of the gradients rather than the product of the roots

Product of roots $= \frac{c}{a}$

$$= \frac{-7}{7}$$ 1

$$= -1 \quad \therefore \text{tangents are perpendicular}$$

2

Solution

Q16

b) i) $P(x) = (x^2 - a^2)Q(x) + (px+q)$

$P(a) = pa + q$ — — (1) | Well done by most students

$P(-a) = -pa + q$ — — (2)

(1) − (2)

$P(a) - P(-a) = 2 \times pa$ | 1

$\therefore p = \frac{1}{2a}(P(a) - P(-a))$

(1) + (2)

$P(a) + P(-a) = 2q$ | 1 | 2

$\therefore q = \frac{1}{2}(P(a) + P(-a))$

ii) if $P(x) = x^n - a^n$

(α) when n is even then $P(a) = 0$ & $P(-a) = 0$

$\therefore p = 0, q = 0 \therefore px + q = 0$ | 1 | 1

(β) when n is odd then

$P(a) = 0$ and $P(-a) = -2a^n$ | Many students had difficulty with this question

$\therefore p = \frac{1}{2a}(0 + 2a^n)$ $q = \frac{1}{2}(0 - 2a^n)$

$R = px + q$

$= \frac{2a^n}{2a}x - a^n$

$R = a^{n-1}x - a^n$ | 1 | 1

Solution	Marks
Q16	
c) i) $\ddot{x} = \frac{F}{v} - kv^2$	1 mark
ii) For maximum velocity $\ddot{x} = 0$	1 mark
$0 = \frac{F}{v} - kv^2$	
$kv^3 = F$	1 mark
$k = \frac{F}{V^3}$	2
	Most students did not understand that F in the question was a constant
iii) $\ddot{x} = v\frac{dv}{dx} = \frac{F}{v} - kv^2$	
$\frac{dv}{dx} = \frac{F}{v^2} - kv$	1 mark
$= \frac{F - kv^3}{v^2}$	
$\frac{dx}{dv} = \frac{v^2}{F - kv^3}$	A lot of students used F = ma or $F = m\ddot{x}$ but F was a constant
$x = \int_{V/4}^{V/2} \frac{v^2}{F - kv^3}\, dv$	1 mark
$= -\frac{1}{3k}\int_{V/4}^{V/2} \frac{-3kv^2}{F - kv^3}\, dv$	
$= \frac{-1}{3k}\left[\ln\left(F - kv^3\right)\right]_{V/4}^{V/2}$	
$= -\frac{1}{3}\cdot\frac{1}{F/V^3}\left[\ln\left(F - k\cdot\frac{V^3}{8}\right) - \ln\left(F - \frac{kV^3}{64}\right)\right]$	
$= \frac{-V^3}{3F}\ln\left[\frac{F - \frac{kV^3}{8}}{F - \frac{kV^3}{64}}\right]$	½ mark
$= \frac{V^3}{3F}\ln\left[\frac{F - \frac{kV^3}{64}}{F - \frac{kV^3}{8}}\right]$	
$= \frac{V^3}{3F}\ln\left[\frac{\frac{64kV^3 - kV^3}{64}}{\frac{8kV^3 - kV^3}{8}}\right]$	½ mark
$= \frac{V^3}{3F}\ln\frac{63\cancel{kv^3}}{8\times 7\cancel{kv^3}} = \frac{V^3}{3F}\ln\frac{9}{8}$	