Student Number:	
Teacher:	

St George Girls High School

Mathematics Extension 2

2020 **Trial HSC Examination**

General Instructions

- Reading time – 10 minutes
- Working Time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in **Section I,** use the multiple-choice answer sheet provided
- For questions in **Section II:**
  - Start each question in a new writing booklet
  - Show relevant mathematical reasoning and/or calculations
  - Extra writing booklets are provided if needed
  - Marks may not be awarded for incomplete or poorly presented solutions

Total marks: 100

Section I – 10 marks (pages 3 – 6)

- Attempt Questions 1 - 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7 – 12)

- Attempt Questions 11 - 16
- Allow about 2 hours and 45 minutes for this section

Q1 – Q10	/10
Q11	/15
Q12	/15
Q13	/15
Q14	/15
Q15	/15
Q16	/15
Total	**/100**
	%

Section I

10 marks
Attempt Questions 1–10

On the multiple choice answer sheet circle the letter corresponding to the most correct answer for questions 1–10.

1. In modular-argument form, the complex number $i - 1$ is :

(A) $\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$

(B) $\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$

(C) $2\text{cis}\left(-\frac{5\pi}{4}\right)$

(D) $2\text{cis}\left(\frac{3\pi}{4}\right)$

2. Which of the following is a primitive of $\dfrac{\sec^2 x}{\tan^3 x}$?

(A) $\frac{1}{2}\cot^2 x$ (B) $-\frac{1}{2}\cot^2 x$ (C) $\frac{1}{4}\cot^4 x$ (D) $-\frac{1}{4}\cot^4 x$

3. A lawyer for a person under investigation for a bank robbery states

> IF MY CLIENT WAS NOT IN THE SUBURB AT THE TIME OF THE ROBBERY, THE ROBBER CANNOT BE MY CLIENT.

Which of the following statements is logically equivalent to the statement made by the lawyer?

(A) If my client was in the suburb, then he was the robber.

(B) If my client was the robber, then my client was in the suburb.

(C) If my client was not the robber, then he was not in the suburb.

(D) If my client was not the robber, then my client was in the suburb.

4. Which of the following is the vector equation of the line segment joining the points $\begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -3 \\ 4 \end{bmatrix}$?

(A) $\underset{\sim}{r} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ -3 \\ 4 \end{bmatrix}$ for $0 \le \lambda \le 1$

(B) $\underset{\sim}{r} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$ for $0 \le \lambda \le 1$

(C) $\underset{\sim}{r} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ for $0 \le \lambda \le 1$

(D) $\underset{\sim}{r} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 15 \\ 6 \\ 12 \end{bmatrix}$ for $0 \le \lambda \le 1$

5. Which of the following is equivalent to $\sqrt{i^3}$?

(A) $e^{\frac{i\pi}{4}}$

(B) $e^{\frac{i\pi}{2}}$

(C) $e^{\frac{3i\pi}{4}}$

(D) $e^{\frac{-i\pi}{2}}$

6. If $\frac{5}{(2x+1)(2-x)} = \frac{A}{2x+1} + \frac{B}{2-x}$, then A and B have values of:

   (A) $A = -1,\ B = 2$

   (B) $A = 1,\ B = -2$

   (C) $A = 2, B = -1$

   (D) $A = 2,\ B = 1$

7. The equation $z^5 = 1$ has roots $1,\ \omega,\ \omega^2,\ \omega^3,\ \omega^4$ where $\omega = e^{\frac{2\pi}{5}i}$.
   What is the value of $(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4)$?

   (A) -5 (B) -4 (C) 4 (D) 5

8. The negation of 'P and not Q' is :

   (A) 'not Q or not P' .

   (B) 'Q or not P' .

   (C) 'Q and not P' .

   (D) 'not Q and not P' .

9. The lines ℓ_1 and ℓ_2 are given by the equations

$$\ell_1:\ \underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + s\begin{pmatrix} -3 \\ -2 \\ a \end{pmatrix} \qquad \ell_2:\ \underset{\sim}{r} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + t\begin{pmatrix} a+2 \\ a^2 \\ -5 \end{pmatrix}$$

Where $s \in \mathbb{R}$, $t \in \mathbb{R}$ and a is a constant.

Given that ℓ_1 and ℓ_2 are perpendicular what are the values of a ?

(A) $a = 1,\ a = 3$

(B) $a = -1,\ a = 3$

(C) $a = 1,\ a = -3$

(D) $a = -1,\ a = -3$

10. Without evaluating the integrals, which one of the following integrals is greater than zero?

(A) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1 + \cos x}{x^3}\, dx$

(B) $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin^{-1}(x^5)\, dx$

(C) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^3 x}{x^5}\, dx$

(D) $\int_{-1}^{1} (x^2 - 4)e^{-x^2}\, dx$

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section.

Marks

Question 11 (15 marks) Begin a new booklet.

(a) Find $\int \frac{dx}{\sqrt{6+4x-x^2}}$. **3**

(b) (i) Write $1+i\sqrt{3}$ in modulus argument form. **2**

(ii) Find the value of $\left(1+i\sqrt{3}\right)^6$. **2**

(iii) Is it possible for $\left(1+i\sqrt{3}\right)^n$, where n is an integer,
to be purely imaginary? Give reasons for your answer. **2**

(c) Rewrite $e^{1+\frac{i\pi}{6}}$ in

(i) Polar form **1**

(ii) Cartesian form **1**

(d) Consider the statement: $\forall x \in \mathbf{R}, \exists y \in \mathbf{R}$ such that $x^2+y^2=2$. **1**

Is the statement true or false? Justify your answer.

(e) Use the substitution $t=\tan\frac{x}{2}$, to find $\int \frac{1}{1+\cos x-\sin x}\,dx$. **3**

Question 12 (15 marks) Begin a new booklet.

(a) (i) Find the numbers A, B and C such that:

$$\frac{1-x}{(1+x)(1+x^2)} \equiv \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$$ **2**

(ii) Hence find the exact value of $\int_0^1 \frac{1-x}{(1+x)(1+x^2)}\,dx$. **3**

(b) The vertices of a triangle ABC are defined by the position vectors

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ -6 \\ -2 \end{pmatrix} \qquad \overrightarrow{OB} = \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix} \qquad \text{and } \overrightarrow{OC} = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}$$

(i) Show that $\cos\angle BAC = \frac{1}{2}$ **3**

(ii) Find the exact area of triangle ABC. **2**

(c) Consider the point $P\ (2, 1, -6)$ and the line L with equation **2**

$$\underset{\sim}{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

Find the shortest distance from the point P to the line L.

(d) Prove by contradiction that if n is a positive integer then

$\sqrt{8n+6}$ is always irrational. **3**

Question 13 (15 marks) Begin a new booklet.

(a) The points P and Q have vector positions:

$\overrightarrow{OP} = -2\underset{\sim}{i} + 14\underset{\sim}{j} - 5\underset{\sim}{k}$ and $\overrightarrow{OQ} = -\underset{\sim}{i} + 12\underset{\sim}{j} - 2\underset{\sim}{k}$

(i) Show that the equation of the line ℓ_1 that passes through P and Q is

$\underset{\sim}{r_1} = (-2+s)\underset{\sim}{i} + (14-2s)\underset{\sim}{j} + (-5+3s)\underset{\sim}{k}$ where s is a real number. **2**

(ii) Consider the line ℓ_2 with equation

$\underset{\sim}{r_2} = (2+at)\,\underset{\sim}{i} + (27 + (a+1)t)\underset{\sim}{j} + (1 + (a+2)t)\underset{\sim}{k}$

where a is a constant and t is a real number. The line ℓ_2 intersects ℓ_1 at the point R. Find the coordinates of R. **3**

(b) Evaluate $\int_0^3 u\sqrt{u+1}\, du$. **3**

(c) (i) For a complex number z, shade the region of the Argand Plane where the inequalities

$|z+1+i| \le 1$ and $-\frac{\pi}{4} \le \arg(z+1+i) \le \frac{\pi}{4}$ hold simultaneously. **3**

(ii) The complex number z satisfies $|z+1+i| = 1$.

What is the smallest distance that z can be from the real number -2 on an Argand diagram? **1**

(d) Use contrapositive proof to prove that if $x^2(y^2-4y)$ is odd then the integers x and y are odd. **3**

Question 14 (15 marks) Begin a new booklet.

(a) Prove by induction that $\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$ for integers $n \geq 1$. **3**

(b) Find $\int \cos^5 x \, dx$. **3**

(c) Let $\underset{\sim}{v}$ be the position vector of a variable point on the surface of a sphere with centre $P(1, 3, 1)$ and radius 14 units.

(i) Write down the vector equation of the sphere. **1**

(ii) The point Q (2, 1, 4) lies on the surface of the sphere. Find the **Cartesian** equation of the tangent to the sphere at Q. (You may assume that the radius drawn to the point of contact of the tangent is perpendicular to the tangent.) **3**

(d) Let $I_n = \int_1^e x^3 \ (\ln x)^n \, dx,$ where n is an integer $n \geq 0$.

(i) Show that $I_n = \frac{e^4}{4} - \frac{n}{4} I_{n-1}$. **3**

(ii) Show that $I_2 = \frac{5e^4}{32} - \frac{1}{32}$. **2**

Question 15 (15 marks) Begin a new booklet.

(a) A sequence is given by the recurrence relation

$u_1 = 7,\ u_{n+1} = 2u_n + 3$ for $n \geq 1$.

Prove by induction that the general formula for the sequence is

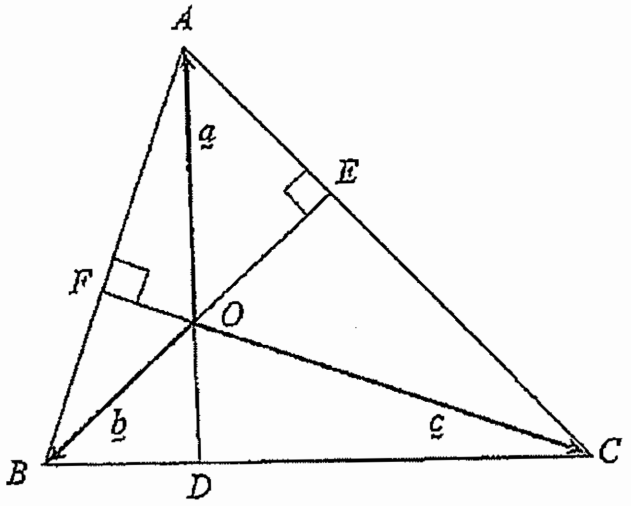
$u_n = 5(2^n) - 3$. **4**

(b) (i) Use De Moivre's Theorem to express $\cos 5\theta$ and $\sin 5\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$. **2**

(ii) Write an expression for $\tan 5\theta$ in terms of t, where $t = \tan\theta$. **1**

(iii) By solving $\tan 5\theta = 0$, deduce that: $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$. **3**

(c)



ABC is an acute angled triangle. The altitudes BE and CF intersect at O. The line AO produced meets BC at D. Relative to O the position vectors of A, B and C are $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively.

(i) Show that $\underset{\sim}{b}\cdot(\underset{\sim}{c} - \underset{\sim}{a}) = 0$ and $\underset{\sim}{c}\cdot(\underset{\sim}{b} - \underset{\sim}{a}) = 0$. **2**

(ii) Hence show that $AD \perp BC$. **2**

(iii) What geometrical property of the triangle has been proved? **1**

Question 16 (15 marks) Begin a new booklet.

(a) Use mathematical induction to prove that $4^{n+1} + 6^n$ is divisible by 10 when n is even. **3**

(b) Recall that $x + \frac{1}{x} \geq 2$ for any real number $x > 0$. (DO NOT PROVE THIS RESULT)

(i) Prove that $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$

for any real numbers $a > 0, b > 0, c > 0$. **2**

(ii) Hence prove that $\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \geq \frac{3}{2}$

for any real numbers $a > 0, b > 0, c > 0$. **3**

(c) The function $F(p)$ is defined as $F(p) = \lim_{t \to \infty} \int_0^t x^{p-1} e^{-x}\, dx$, for $p > 0$.

(i) Show that $F(1) = 1$. **2**

(ii) Use integration by parts to show $F(p + 1) = pF(p)$. **3**

(iii) Hence find $F(n)$ for integers $n \geq 1$. **2**

End of Examination

MATHEMATICS EXTENSION 2 – QUESTION MC

Ext 2 Trial 2020

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
1. $r = \sqrt{1+(-1)^2}$ $= \sqrt{2}$ $\arg(i-1) = \arg(-1+i)$  $\theta = \tan^{-1}\left(\frac{-1}{-1}\right)$ $\theta_{Acute} = \pi/4$ $\therefore \theta = \pi - \frac{\pi}{4}$ $= 3\pi/4$ $\therefore z = \sqrt{2} \text{ cis } 3\pi/4$	B	
2. $\int \frac{\sec^2 x}{\tan^3 x}\,dx$ Let $u = \tan x$ $\frac{du}{dx} = \sec^2 x$ $du = \sec^2 x\,dx$ $= \int u^{-3}\,du$ $= \frac{u^{-2}}{-2} + c$ $= -\frac{1}{2u^2} + c$ $= -\frac{1}{2\tan^2 x} + c$ $= -\frac{1}{2}\cot^2 x + c$	B	
3.	B	

MATHEMATICS EXTENSION 2 – QUESTION MC

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
4. $\underset{\sim}{r} = \begin{pmatrix}3\\-2\\3\end{pmatrix} + \lambda\begin{pmatrix}5-3\\-3--2\\4-3\end{pmatrix}$ $= \begin{pmatrix}3\\-2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$	C	
5. $\sqrt{i^3} = i^{3/2}$ $= (0+i)^{3/2}$ $= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{3/2}$ $= \left(e^{i\frac{\pi}{2}}\right)^{3/2}$ $= e^{i\frac{3\pi}{4}}$	C	
6. $5 = A(2-x) + B(2x+1)$ if $x=2$ if $x=0$ $5 = B(4+1)$ $5 = 2A + B$ $5B = 5$ $5 = 2A + 1$ $B = 1$ $2A = 4$ $A = 2$	D	
7. $z^5 - 1 = (z-1)(z^4+z^3+z^2+z+1)$ and $z^5 - 1 = (z-1)(z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4)$ $\cancel{(z-1)}(z^4+z^3+z^2+z+1) = \cancel{(z-1)}(z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4)$ $z^4+z^3+z^2+z+1 = (z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4)$ but $z=1$ gives $1^4+1^3+1^2+1+1 = (1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4)$ $5 = (1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4)$	D	
8. B		

SUGGESTED SOLUTIONS | MARKS | MARKER'S COMMENTS

9. Dot product of their direction vector is zero

$$\begin{pmatrix}-3\\-2\\a\end{pmatrix}\cdot\begin{pmatrix}a+2\\a^2\\-5\end{pmatrix}=0$$

$$-3(a+2)-2a^2-5a=0$$

$$-3a-6-2a^2-5a=0$$

$$2a^2+8a+6=0$$

$$a^2+4a+3=0$$

$$(a+3)(a+1)=0$$

$a=-3$ or $a=-1$. D

10. Recall $\int_{-a}^{a} f(x)\,dx=0$ if $f(x)$ is odd

and $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if $f(x)$ is even

A. $1+\cos x$ is even, x^3 is odd

$$\frac{f(-x)}{g(-x)} = \frac{-f(x)}{g(x)} = -\left(\frac{f(x)}{g(x)}\right)$$

$\therefore \frac{1+\cos x}{x^3}$ is odd.

B. $\sin^{-1}(x^5)$ is odd $\therefore \int_{-\pi/3}^{\pi/3} \sin^{-1}(x^5)\,dx=0$

C. $\tan^3 x$ is odd as $f(-x)=-f(x)$

x^5 is odd as $g(-x)=-g(x)$

$$\frac{\tan^3 x}{x^5} = \frac{-f(x)}{-g(x)} = \frac{f(x)}{g(x)}$$

$\therefore$ even

$$\therefore \int_{-\pi/4}^{\pi/4} \frac{\tan^3 x}{x^5}\,dx = 2\int_{0}^{\pi/4} \frac{\tan^3 x}{x^5}\,dx$$

but $\tan^3 x$ and x^5 are both positive from 0 to $\pi/4$

$\therefore$ the integral will be positive

D. $f(x) = x^2 - 4$ $\quad g(x) = e^{-x^2}$

$f(-x) = (-x)^2 - 4$ $\quad g(-x) = e^{-(-x)^2}$

$= x^2 - 4$ $\quad = e^{-x^2}$

$\therefore f(x)$ is even $\quad = g(x)$

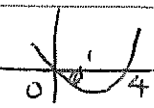
$\therefore$ even

$\therefore (x^2-4).e^{-x^2}$ is an even functn

$\therefore \int_{-1}^{1}(x^2-4).e^{-x^2}\,dx = 2\int_0^1 (x^2-4)e^{-x^2}\,dx$

but between 0 and 1

$x^2 - 4 < 0$



and $e^{-x^2} > 0 \quad \forall x$

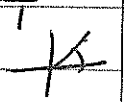
$\therefore (x^2-4)(e^{-x^2}) < 0$

$\therefore$ the integral is negative

$\therefore$ Answer is C

c

MATHEMATICS EXTENSION 2 – QUESTION 11

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) $\int \frac{dx}{\sqrt{6+4x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-4x-6)}}$		This part was generally done well.
$-(x^2-4x-6)$	1	
$= -\left[(x^2-4x+4)-6-4\right]$		
$= -\left[(x-2)^2-10\right]$		
$= 10-(x-2)^2$ -- ①		
$= \int \frac{dx}{\sqrt{10-(x-2)^2}}$ from ①		
$= \int \frac{du}{\sqrt{10-u^2}}$ Let $u=x-2$ $du=dx$	1	
$= \sin^{-1}\frac{u}{\sqrt{10}}+c$		
$= \sin^{-1}\frac{(x-2)}{\sqrt{10}}+c$	1	③
Alternate method:		
b) i) For $1+i\sqrt{3}$		
$r=\sqrt{1^2+3}$ $\arg(1+i\sqrt{3})=\tan^{-1}\frac{\sqrt{3}}{1}$		
$=2$ $\theta=\frac{\pi}{3}$ 		
$\therefore$ $2\,\text{cis}\,\frac{\pi}{3}$	2	②
ii) $(1+i\sqrt{3})^6 = \left(2\,\text{cis}\,\frac{\pi}{3}\right)^6$		
$= 64\,\text{cis}\,\frac{6\pi}{3}$		
$= 64\,\text{cis}\,2\pi$	1	
$= 64(\cos 2\pi + i\sin 2\pi)$		
$= 64(1+0)$		②
$=64$	1	

MATHEMATICS EXTENSION 2 – QUESTION 11

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
b) iii) From (i) & (ii) $(1+i\sqrt{3})^n = 2^n(\text{cis}\frac{n\pi}{3})$		
For $(1+i\sqrt{3})^n$ to be purely imagnery would mean that: $\arg(1+i\sqrt{3})^n$, which is $\frac{n\pi}{3}$, should be $\frac{\pi}{2}+k\pi$, for k any integer.	1	
So $\frac{n\pi}{3} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$		
but there are no such integers n for which $\frac{n\pi}{3} = \frac{\pi}{2}+k\pi$	1	②
c) i) $e^{1+\frac{i\pi}{6}}$ $= e^1 \times e^{i\pi/6}$ $= e(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$	1	①
ii) $= e(\frac{\sqrt{3}}{2} + i \times \frac{1}{2})$ $= \frac{e}{2}(\sqrt{3}+i)$ $= \frac{\sqrt{3}}{2}e + \frac{e}{2}i$	1	①
d) If the set of all counterexamples, $x \in \mathbb{R}$, $x^2 > 2$, then $x > 2$. If $x = 3$, then there does not exist a real y such that $x^2+y^2=2$ i.e no $y \in \mathbb{R}$ exists that satisfies $x^2+y^2=2$ ∴ false statement.	1	Students need to be more explicit when stating what values of x being used and what values of y are disatisfied because the quantifyer is in terms of $x + y$ not x^2+y^2.

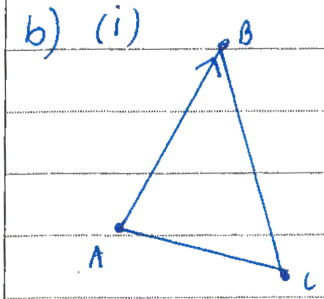
MATHEMATICS EXTENSION 2 – QUESTION 11

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
e) $\int \frac{1}{1+\cos x - \sin x}\,dx$		
$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}} \times \frac{2dt}{1+t^2}$		
$= \int \frac{1+t^2}{1+\cancel{t^2}+1-\cancel{t^2}-2t} \times \frac{2dt}{1+t^2}$	$1\frac{1}{2}$	
$= \int \frac{1+\cancel{t^2}}{2-2t} \times \frac{2dt}{1+t^2}$		
$= \int \frac{2}{2-2t}\,dt$	$\frac{1}{2}$	
$= -\int \frac{1}{1-t}\,dt$		
$= -\ln\|1-t\| + c$		
$= -\ln\left\|1-\tan\frac{x}{2}\right\| + c$	1	③

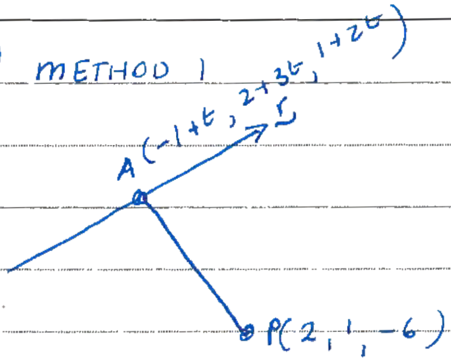
MATHEMATICS EXTENSION 2 – QUESTION 12

SUGGESTED SOLUTIONS			MARKS	MARKER'S COMMENTS
a)(i) $\frac{1-x}{(1+x)(1+x^2)} \equiv \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$				
$1-x \equiv A(1+x^2)+(Bx+C)(1+x)$				
When $x=-1$	When $x=0$	$x=1$	1	1 mark to set up identity and attempt to solve
$2=2A$	$1=A+C$	$0=2A+2B$		
$A=1$	$1=1+C$	$0=2(1)+2B$		
	$C=0$	$2B=-2$		
		$B=-1$		
$\therefore A=1,\ B=-1,\ C=0$			1	Correct values.
(ii) $\int_0^1 \frac{1-x}{(1+x)(1+x^2)}\,dx$				
$=\int_0^1 \left(\frac{1}{1+x}+\frac{-x}{1+x^2}\right)dx$				
$=\int_0^1 \frac{1}{1+x}\,dx - \int_0^1 \frac{x}{1+x^2}\,dx$				
$=\int_0^1 \frac{1}{1+x}\,dx - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$			1	Use of identity
$=\left[\ln\lvert 1+x\rvert - \frac{1}{2}\ln\lvert 1+x^2\rvert\right]_0^1$			1	Correct Integration
$=\ln 2-\frac{1}{2}\ln 2-\left(\ln 1-\frac{1}{2}\ln 1\right)$				
$=\ln 2-\frac{1}{2}\ln 2$				
$=\frac{1}{2}\ln 2$ also accepted $\ln\sqrt{2}$, $\ln 2^{1/2}$			1	Correct answer

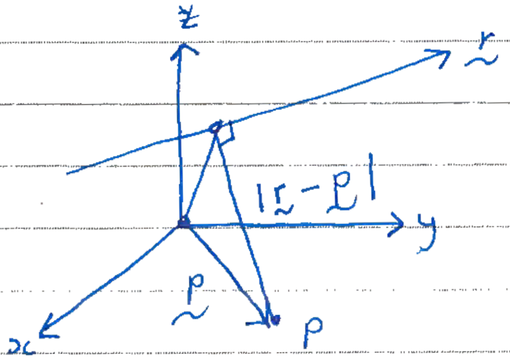
MATHEMATICS EXTENSION 2 – QUESTION 12

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
b) (i)  $\vec{AB} = \begin{pmatrix} 0-1 \\ -4--6 \\ -1--2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$	½	Find $\vec{AB}$
$\vec{AC} = \begin{pmatrix} 3-1 \\ -4--6 \\ 2--2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$	½	Find $\vec{AC}$
$\vec{AB} \cdot \vec{AC} = \lvert\vec{AB}\rvert\lvert\vec{AC}\rvert \cos\angle CAB$ $\cos\angle CAB = \dfrac{\vec{AB}\cdot\vec{AC}}{\lvert\vec{AB}\rvert\lvert\vec{AC}\rvert}$ $= \dfrac{-1\times 2 + 2\times 2 + 1\times 4}{\sqrt{(-1)^2+(2)^2+(1)^2} \times \sqrt{(2)^2+2^2+4^2}}$ $= \dfrac{-2+4+4}{\sqrt{6}\times\sqrt{24}}$ $= \dfrac{6}{\sqrt{6}\times 2\sqrt{6}}$ $= \dfrac{6}{12}$ $= \dfrac{1}{2}$ as required.		
(ii) $A = \frac{1}{2}ab\sin C$ $= \frac{1}{2}\times\sqrt{6}\times\sqrt{24}\sin\frac{\pi}{3}$ as $\cos\angle BAC = \frac{1}{2}$ $\therefore \angle BAC = \frac{\pi}{3}$  $= \frac{1}{2}\times 12 \times \frac{\sqrt{3}}{2}$	1	set up area with correct angle
$= 3\sqrt{3}$	1	correct answer

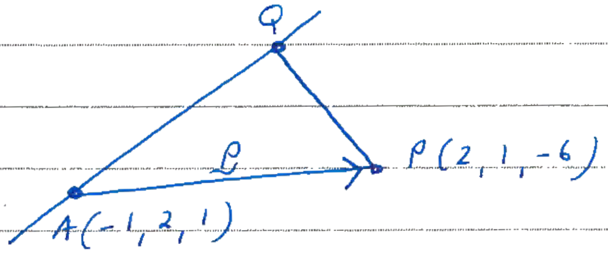
MATHEMATICS EXTENSION 2 – QUESTION 12

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) METHOD 1  $\underset{\sim}{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ This has the direction vector $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \underset{\sim}{b}$		
Let $A\begin{pmatrix} -1+t \\ 2+3t \\ 1+2t \end{pmatrix}$ which is ANY point or $\underset{\sim}{r}$	½	Correct start and finding A.
$\overrightarrow{PA} = \begin{pmatrix} -1+t-2 \\ 2+3t-1 \\ 1+2t+6 \end{pmatrix} = \begin{pmatrix} t-3 \\ 3t+1 \\ 2t+7 \end{pmatrix}$	½	find $\overrightarrow{PA}$
We want A to be the closest point to P $\therefore$ $\overrightarrow{PA} \perp \underset{\sim}{b}$ $\therefore \overrightarrow{PA} \cdot \underset{\sim}{b} = 0$ $1 \times (t-3) + 3(3t+1) + 2(2t+7) = 0$ $t - 3 + 9t + 3 + 4t + 14 = 0$ $14t + 14 = 0$ $t = -1$ $\therefore \overrightarrow{PA} = \begin{pmatrix} -1-3 \\ 3(-1)+1 \\ 2(-1)+7 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ 5 \end{pmatrix}$ $\lvert \overrightarrow{PA} \rvert = \sqrt{(-4)^2 + (-2)^2 + 5^2}$ $= \sqrt{45}$ $= 3\sqrt{5}$	1	Correct answer

MATHEMATICS EXTENSION 2 – QUESTION 12

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) METHOD 2		
		
$\underset{\sim}{r} = \begin{pmatrix}-1\\2\\1\end{pmatrix} + t\begin{pmatrix}1\\3\\2\end{pmatrix}$		
$\underset{\sim}{r} = \begin{pmatrix}-1+t\\2+3t\\1+2t\end{pmatrix}$ and $\underset{\sim}{p} = \begin{pmatrix}2\\1\\6\end{pmatrix}$	½	Correct start
$\underset{\sim}{r}-\underset{\sim}{p} = \begin{pmatrix}t-3\\3t+1\\2t+7\end{pmatrix}$	½	find $\underset{\sim}{r}-\underset{\sim}{p}$
$\lvert\underset{\sim}{r}-\underset{\sim}{p}\rvert = \sqrt{(t-3)^2+(3t+1)^2+(2t+7)^2}$		
$= \sqrt{t^2-6t+9+9t^2+6t+1+4t^2+28t+49}$		
$= \sqrt{14t^2+28t+59}$		
$= \sqrt{14(t^2+2t+1)+45}$		
$= \sqrt{14(t+1)^2+45}$		
$\therefore$ Minimum distance occurs when $t=-1$		
$\therefore \lvert\underset{\sim}{r}-\underset{\sim}{p}\rvert = \sqrt{45}$		
$= 3\sqrt{5}$	1	Correct answer.

MATHEMATICS EXTENSION 2 – QUESTION 12

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) METHOD 3		
		
$\underset{\sim}{p} = \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$ $\overrightarrow{AQ} = \begin{pmatrix} -1+t \\ 2+3t \\ 1+2t \end{pmatrix}$	½	Correct start $\overrightarrow{AQ}$
$\underset{\sim}{p} = \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ $\underset{\sim}{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$		
$= \begin{pmatrix} 3 \\ -1 \\ -7 \end{pmatrix}$	½	$\underset{\sim}{p}$
$\lvert \underset{\sim}{b} \rvert^2 = 1+9+4$ $= 14$		
$\lvert \overrightarrow{PQ} \rvert = \lvert \text{proj}_{\underset{\sim}{b}} \underset{\sim}{p} - \underset{\sim}{p} \rvert$		
$= \left\lvert \dfrac{\underset{\sim}{p} \cdot \underset{\sim}{b}}{\lvert \underset{\sim}{b} \rvert^2} \times \underset{\sim}{b} - \underset{\sim}{p} \right\rvert$		
$= \left\lvert \dfrac{3-3-14}{14} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ -7 \end{pmatrix} \right\rvert$		
$= \left\lvert \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ -7 \end{pmatrix} \right\rvert$		
$= \left\lvert \begin{pmatrix} -4 \\ -2 \\ 5 \end{pmatrix} \right\rvert$		
$= \sqrt{(-4)^2 + (-2)^2 + 5^2}$		
$= \sqrt{16+4+25}$	1	Correct answe
$= \sqrt{45}$		
$= 3\sqrt{5}$		

MATHEMATICS EXTENSION 2 – QUESTION 12

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
d) Assume $\sqrt{8n+6}$ is rational ie Assume $n \in \mathbb{N}$ such that $\sqrt{8n+6} = \frac{p}{q}$ where p and q are integers and HCF $= 1$, $q \neq 0$	①	
$\sqrt{8n+6} = \frac{p}{q}$ square both sides $8n+6 = \frac{p^2}{q^2}$ $2(4n+3)q^2 = p^2$ $p^2 = 2(4n+3)$ $\therefore$ p^2 is divisible by 2 $\therefore$ p is divisible by 2 Let $p = 2k$	①	correctly showing p was divisible by 2
$\therefore (2k)^2 = 2(4n+3)q^2$ $4k^2 = 2(4n+3)q^2$ $2k^2 = (4n+3)q^2$ $q^2 = 2 \times \frac{k^2}{4n+3}$ $\therefore$ q^2 is divisible by 2 $\therefore$ q is divisible by 2 ANOTHER WAY OR $2k^2 = (4n+3)q^2$ $4n+3$ is odd and since $2k^2$ is even q^2 must be even (odd x even is even) $\therefore$ q must be divisible by 2. etc . . . As both p and q are divisible by 2, this is a contradiction of the condition that p and q have HCF $= 1$. Hence $\sqrt{8n+6}$ is not rational $\therefore$ $\sqrt{8n+6}$ is irrational	①	this caused issues for some solutions. Correctly showing q was divisible by 2 and hence HCF of p and $q \neq 1$ leading to proof by contradiction that $\sqrt{8n+6}$ is irrational

MATHEMATICS EXTENSION 2 – QUESTION 13

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$ $\quad Q = \begin{pmatrix}-1\\12\\-2\end{pmatrix}$		
$P = \begin{pmatrix}-2\\14\\-5\end{pmatrix}$		Many students
$= \begin{pmatrix}-1+2\\12-14\\-2+5\end{pmatrix}$		found the three equations
$\Rightarrow$ direction vector		but found it difficult to
$= \begin{pmatrix}1\\-2\\3\end{pmatrix}$		find $s=-2$.
$\underset{\sim}{r_1} = \begin{pmatrix}-2\\14\\-5\end{pmatrix} + s\begin{pmatrix}1\\-2\\3\end{pmatrix}$		
$\underset{\sim}{r_1} = \begin{pmatrix}-2+s\\14-2s\\-5+3s\end{pmatrix}$		
$\underset{\sim}{r} = (-2+s)\underset{\wedge}{i} + (14-2s)\underset{\wedge}{j} + (-5+3s)\underset{\sim}{k}$		
ii) $-2+s = 2+at \quad ---①$		
$27+at+t = 14-2s \quad ---②$	1	
$-5+3s = 1+(a+2)t \quad ---③$		
From ① $\quad at = s-4 \quad --④$		
From ②		
$14-2s = 27+(s-4)+t$		
$3s+t = -9 \quad --⑤$		
From ③		
$-5+3s = 1+at+2t$		
$-5+3s = 1+s-4+2t$		
$s-t = 1 \quad --⑥$		
① + ② $\quad 4s = -8$		
$s = -2$	1	

MATHEMATICS EXTENSION 2 – QUESTION 13

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
$\therefore \underset{\sim}{r} = (-2+-2)\underset{\sim}{i} + (14-2(-2))\underset{\sim}{j} + (-5+3\cdot(-2))\underset{\sim}{k}$		
$= -4\underset{\sim}{i} + 18\underset{\sim}{j} - 11\underset{\sim}{k}$ $\therefore R = (-4, 18, -11)$	1	③
13 (b) $I = \int_0^3 u\sqrt{u+1}\,du$ Let $x = u+1$ --① $\frac{dx}{du} = 1$ $dx = du$ From ① $u = x-1$ when $u=3$, $x=4$ $u=0$, $x=1$		
$I = \int_1^4 (x-1)\sqrt{x}\,dx$ $= \int_1^4 x^{3/2} - x^{1/2}\,dx$	1	
$= \left[\frac{2x^{5/2}}{5} - \frac{2x^{3/2}}{3}\right]_1^4$	1	
$= \left[\left(\frac{2(4)^{5/2}}{5} - \frac{2(4)^{3/2}}{3}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right]$ $= \frac{2}{5}(32) - \frac{2}{5}\times 8 - \left(-\frac{4}{15}\right)$ $= \frac{64}{5} - \frac{16}{3} + \frac{4}{15}$		
$= \frac{116}{15}$	1	

MATHEMATICS EXTENSION 2 – QUESTION 13

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS

Alternative solution to 13(b)

b) $\int_0^3 u\sqrt{u+1}\,du$

$= \int_0^3 (u+1)\sqrt{u+1}\,du - \int_0^3 \sqrt{u+1}\,du$

$= \int_0^3 (u+1)^{3/2} - (u+1)^{1/2}\,du$

$= \left[\frac{2(u+1)^{5/2}}{5}\right]_0^3 - \left[\frac{2(u+1)^{3/2}}{3}\right]_0^3$

$= \left(\frac{2(4)^{5/2}}{5} - \frac{2}{5}\right) - \left(\frac{2(4)^{3/2}}{3} - \frac{2}{3}\right)$

$= \frac{2(32)}{5} - \frac{2}{5} - \left(\frac{2(8)}{3} - \frac{2}{3}\right)$

$= \frac{64}{5} - \frac{16}{3} - \frac{2}{5} + \frac{2}{3}$

$= \frac{116}{15}$

c) $|z + 1 + i| \leq 1$

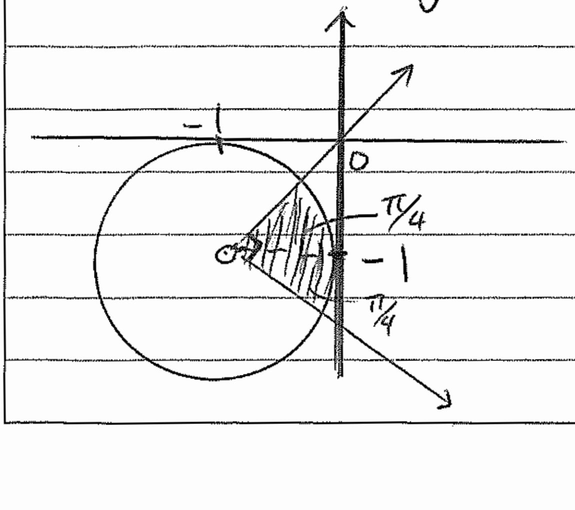
$|x + iy + 1 + i| \leq 1$

$|(x+1) + i(y+1)| \leq 1$

$\sqrt{(x+1)^2 + (y+1)^2} \leq 1$

$(x+1)^2 + (y+1)^2 \leq 1$

A few students did not have an open circle at $(-1, -1)$.
→ $\frac{1}{2}$ mark off.



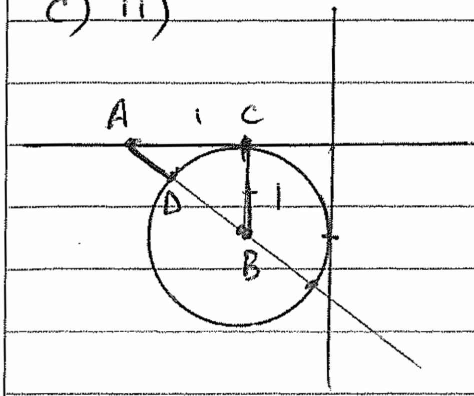
1 – circle / arc

1 – arms

1 – shading.

$\boxed{3}$

MATHEMATICS EXTENSION 2 – QUESTION 13

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) ii)		
		
In ΔABC		
$AB = \sqrt{1^2+1^2}$		
$= \sqrt{2}$		
The shortest distance is AD		
$AD = AB - DB$		
$= \sqrt{2} - 1$ units.	1	①
d) The contrapositive statement is:	$\frac{1}{2}$	
if x <u>or</u> y are not odd (even)		Many students
then $x^2(y^2-4y)$ is not odd (i.e even)		put in '<u>and</u>'.
There are three ways to prove		
this contrapositive statement.		
Method 1: Prove by 2 cases.		Many students wrote
Case 1		the contrapositive
Let x be even		statement as:
i.e Let $x = 2k$, $k \in \mathbb{Z}$		'if x and y are
$x^2(y^2-4y) = (2k)^2[y^2-4y]$		even then $x^2(y^2-4y)$
$= 4k^2(y^2-4y)$		is even'.
$= 2\times 2k^2(y^2-4y)$		This statement is
$= 2[2k^2(y^2-4y)]$	1	not logically
$= 2M$ as $M, k, y \in \mathbb{Z}$		equivalent to the
$\therefore$ even		original.
Case 2: Let y be even		Using this statement
i.e Let $y = 2m$, $m \in \mathbb{Z}$		only one case
$x^2(y^2-4y) = x^2[(2m)^2-4(2m)]$		was shown
$= x^2[4m^2-8m]$		(x even and y even)
$= 2x^2(2m^2-4m)$	1	$\therefore$ 1 mark <u>only</u>
$= 2[x^2(2m^2-4m)]$		was given.

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
$= 2J$, where $J, \in \mathbb{Z}$		
since $x, m \in \mathbb{Z}$.		
∴ divisible by 2		
∴ even		
∴ If x or y are even then		
$x^2(y^2-4y)$ is even		
∴ the by cont	$\frac{1}{2}$	
if $x^2(y^2-4)$ is odd then		③
x and y are odd.		
Method 2		
Proof by 3 cases:		
1. Let x be even and y is odd		
2. Let x be odd and y is even		
3. Let x be even and y is even		
Case 1: Let $x = 2m$ and $y = 2m+1$, $m\in\mathbb{Z}$		
Now $x^2(y^2-4y) = (2m)^2[(2m+1)^2-4(2m+1)]$		
$= 4m^2[4m^2+4m+1-8m-4]$		
$= 2[2m^2(4m^2-4m-3)]$		
$= 2N$ where $N\in\mathbb{Z}$	1	
since $2m^2(4m^2-4m-3)\in\mathbb{Z}, m\in\mathbb{Z}$		
∴ div. by 2 ∴ even.		
Case 2: Let $x = 2k+1$ and $y = 2k$		
Now $x^2(y^2-4y) = (2k+1)^2[(2k)^2-4(2k)]$		
$= (2k+1)^2[4(k^2-2k)]$		
$= 4(2k+1)^2(k^2-2k)$		
$= 2\times 2(2k+1)^2(k^2-2k)$		
$= 2M$ where $M\in\mathbb{Z}$		
since $2(2k+1)^2(k^2-2k)\in\mathbb{Z}$	1	
as $k\in\mathbb{Z}$		
∴ divisible by 2		
∴ even		

Case 3 : Let $x = 2k$ and $y = 2n$

where $k, n \in \mathbb{Z}$

$$x^2(y^2-4y) = (2k)^2\left[(2n)^2 - 4(2n)\right]$$

$$= 4k^2(4n^2 - 8n)$$

$$= 4k^2 \times 4(n^2 - 2n)$$

$$= 16h^2(n^2 - 2n)$$

$$= 2 \times 8h^2(n^2 - 2n)$$

$$= 2P \quad \text{where } P \in \mathbb{Z}$$ 1

since $8k^2(n^2-2n) \in \mathbb{Z}$

as $h, n \in \mathbb{Z}$

∴ even

etc.

Method 3

If x or y are even

then xy is even

So $x^2(y^2-4y) = x^2y^2 - 4x^2y$

$$= xy(xy - 4y)$$

$$= xy(x(y-4))$$

∴ $x^2(y^2-4y)$ has a factor of xy which is even

∴ it has a factor of 2

∴ even.

MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) Step 1 – Base case – Prove true for $n=1$ $LHS = \sum_{r=1}^{n} \frac{r}{(r+1)!}$ $RHS = 1 - \frac{1}{(n+1)!}$ $= \sum_{r=1}^{1} \frac{r}{(r+1)!}$ $= 1 - \frac{1}{(1+1)!}$ $= \frac{1}{(1+1)!}$ $= 1 - \frac{1}{2!}$ $= \frac{1}{2!}$ $= 1 - \frac{1}{2}$ $= \frac{1}{2}$ $= \frac{1}{2}$ ∴ LHS = RHS ∴ true for $n=1$	½	Prove base case correctly
Step 2 – Inductive Hypothesis. Assume true for $n=k$. , $k \geq 1$, $n \in \mathbb{Z}$ ie $\sum_{r=1}^{k} \frac{r}{(r+1)!} = 1 - \frac{1}{(k+1)!}$	½	Correct assumption
Step 3 – Inductive step Prove true for $n=k+1$ ie prove $\sum_{r=1}^{k+1} \frac{r}{(r+1)!} = 1 - \frac{1}{(k+1+1)!}$ $= 1 - \frac{1}{(k+2)!}$ $LHS = \sum_{r=1}^{k+1} \frac{r}{(r+1)!}$	1	Correct set up of proof.
$= \sum_{r=1}^{k} \frac{r}{(r+1)!} + \frac{k+1}{(k+2)!}$ $= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$ By the assumption $= 1 - \left[\frac{1}{(k+1)!} - \frac{k+1}{(k+2)!} \right]$		Note – ½ lost if "by the assumption" (or similar was not written.

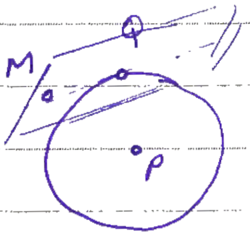
MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) LHS $= 1 - \left[\frac{k+2-(k+1)}{(k+2)!}\right]$ $= 1 - \frac{1}{(k+2)!}$ $= RHS$	1	Correct working to prove result.
step 4 - conclusion ∴ true for $n=k+1$ since it is true for $n=k$. Since it is true for $n=1$, then it is true for $n=1+1=2$ and so true for all integer n, $n \geq 1$.		

MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
b) $\int \cos^5 x \, dx$		More experience needed to think of the quickest method. Integration by part was not required and much time was wasted here.
$= \int \cos^4 x \cos x \, dx$		
$= \int (1-\sin^2 x)^2 \cos x \, dx$	①	
$= \int (1 - 2\sin^2 x + \sin^4 x) \cos x \, dx$		
$= \int \cos x \, dx - 2\int \sin^2 x \cos x \, dx + \int \sin^4 x \cos x \, dx$	①	
let $u = \sin x$		
$\frac{du}{dx} = \cos x$		
$du = \cos x \, dx$		
$= \sin x - 2\int u^2 du + \int u^4 du$		
$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^4 x + C$	①	

MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) (i) $\lvert \underset{\sim}{v} - \underset{\sim}{c} \rvert = r$		
$\left\lvert \underset{\sim}{v} - \begin{pmatrix}1\\3\\1\end{pmatrix} \right\rvert = 14$	1	Correct answer in vector form. Many students mixed up "vector form" and "Cartesian form"
(ii) $\overrightarrow{OP} = \begin{pmatrix}1\\3\\1\end{pmatrix}$ $\quad \overrightarrow{OQ} = \begin{pmatrix}2\\1\\4\end{pmatrix}$		
$\overrightarrow{PQ} = \begin{pmatrix}2\\1\\4\end{pmatrix} - \begin{pmatrix}1\\3\\1\end{pmatrix}$ $= \begin{pmatrix}1\\-2\\3\end{pmatrix}$ 	1	
Let another point on tangent be M		
$\overrightarrow{OM} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ $\quad OQ = \begin{pmatrix}2\\1\\4\end{pmatrix}$		Very poorly done.
$\overrightarrow{MQ} = \begin{pmatrix}2-x\\1-y\\4-z\end{pmatrix}$		Full marks awarded if one tangent was found.
$\overrightarrow{PQ} \cdot \overrightarrow{MQ} = 0$	1	
$\begin{pmatrix}1\\-2\\3\end{pmatrix} \cdot \begin{pmatrix}2-x\\1-y\\4-z\end{pmatrix} = 0$		
$2 - x - 2(1-y) + 3(4-z) = 0$ $2 - x - 2 + 2y + 12 - 3z = 0$		
$x - 2y + 3z = 12$	1	

MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
d) $I_n = \int_1^e x^3 (\ln x)^n \, dx$		
$u = (\ln x)^n \qquad v' = x^3$ $u' = n(\ln x)^{n-1} \cdot \frac{1}{x} \qquad v = \frac{x^4}{4}$ $I_n = uv - \int vu' \, dx$ $I_n = \left[(\ln x)^n \times \frac{x^4}{4} \right]_1^e - \int_1^e n(\ln x)^{n-1} \times \frac{1}{x} \times \frac{x^4}{4} \, dx$	②	1 mark for uv 1 mark $\int vu' dx$
$= \left[(\ln e)^n \times \frac{e^4}{4} - (\ln 1)^n \times \frac{1}{4} \right] - \frac{n}{4} \int_1^e (\ln x)^{n-1} x^3 \, dx$ $= \frac{e^4}{4} - 0 - \frac{n}{4} \int_1^e (\ln x)^{n-1} x^3 \, dx$	①	SHOW substitution. You are convincing the marker that you know how to get to the answer.
$I_n = \frac{e^4}{4} - \frac{n}{4} I_{n-1}$ as required		
(ii) $I_2 = \frac{e^4}{4} - \frac{1}{2} I_1$		
$I_1 = \frac{e^4}{4} - \frac{1}{4} I_0$	①	Evaluating I_2, I_1, I_0
$I_0 = \int_1^e x^3 \, dx$ $= \left[\frac{x^4}{4} \right]_1^e$ $= \frac{e^4}{4} - \frac{1}{4}$		
$\therefore I_2 = \frac{e^4}{4} - \frac{1}{2} \left[\frac{e^4}{4} - \frac{1}{4} \left(\frac{e^4}{4} - \frac{1}{4} \right) \right]$ $= \frac{e^4}{4} - \frac{e^4}{8} + \frac{1}{8} \left(\frac{e^4}{4} - \frac{1}{4} \right)$ $= \frac{e^4}{8} + \frac{e^4}{32} - \frac{1}{32}$	①	Showing this obtained correct answer

$= \frac{5e^4}{32}$

MATHEMATICS EXTENSION 2 – QUESTION 15

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
(a) Prove true for $n=1$		This proof was generally done well.
For $u_n = 5(2^n)-3$, $u_1 = 7$ when $n=1$		
$u_1 = 5(2^1)-3$ $= 10 - 3$ $= 7$	1	
∴ true for $n=1$		
Assume the statement is true for $n=k$.		
ie given $u_{k+1} = 2u_k + 3$	1	
then $u_k = 5(2^k)-3$		
Inductive step, prove true for $n=k+1$		
i.e. prove that $u_{k+1} = 5(2^{k+1})-3$		
LHS $= u_{k+1}$ $= 2u_k + 3$ $= 2[5(2^k)-3]+3$ by the assumption $= 5(2\times 2^k) - 6 + 3$ $= 5(2^{k+1}) - 6 + 3$		2.
$= 5(2^{k+1}) - 3$		
∴ true for $n=k+1$, since it is true for $n=k$. Since it is true for $n=1$ then it is true for $n=1+1=2$ and so true for all integer n, $n \geq 1$.		④

MATHEMATICS EXTENSION 2 – QUESTION 15

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
b) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$.		
i)		
$LHS = \cos^5\theta + 5i\cos^4\theta\sin\theta + 10\cos^3\theta\, i^2\sin^2\theta +$		
$10\cos^2\theta\, i^3\sin^3\theta + 5\cos\theta\, i^4\sin^4\theta + i^5\sin^5\theta$		
$= \cos^5\theta + 5\cos^4\theta\, i\sin\theta - 10\cos^3\theta\sin^2\theta$		
$- 10\cos^2\theta\, i\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$		
Equating real part		
$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$	1	
Equating imaginary part		
$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$	1	
ii)		
$\frac{\sin 5\theta}{\cos 5\theta} = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$		
$\div\cos^5\theta$		
$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$		
$\tan 5\theta = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ ①	1	
iii) If $\tan 5\theta = 0$		
$5\theta = 0 + k\pi$		
$\theta = \frac{k\pi}{5}, \quad k = 0, 1, 2, 3, 4.$		
If $k=0, \ \theta = 0$		
$k=1, \quad \theta = \frac{\pi}{5}$		
$k=2, \quad \theta = \frac{2\pi}{5}$	1	
$k=3, \quad \theta = \frac{3\pi}{5}$		
$k=4, \quad \theta = \frac{4\pi}{5}$		

MATHEMATICS EXTENSION 2 – QUESTION 15

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
b) ii)		
Also if $\tan 5\theta = 0$ from ①		
$\frac{5t-10t^3+t^5}{1-10t^2+5t^4} = 0$		
$5t-10t^3+t^5=0$		
$t(5-10t^2+t^4)=0$		
$t=0$ or $5-10t^2+t^4=0$	1	
Now the solution of $t^4-10t^2+5=0$ are the solutions to $\tan 5\theta = 0$ but not where $t=0$ as $\theta=0$.		
So if $t=\tan\theta$, the solutions of $t^4-10t^2+5=0$ are the roots:		
$\tan\frac{\pi}{5}$, $\tan\frac{2\pi}{5}$, $\tan\frac{3\pi}{5}$, $\tan\frac{4\pi}{5}$		
Product of roots $= \frac{e}{a}$		
i.e. $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = \frac{5}{1}$	1	
$= 5$		
		③

MATHEMATICS EXTENSION 2 – QUESTION 15

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) i) $\overrightarrow{OB} = \underset{\sim}{b}, \quad \overrightarrow{OA} = \underset{\sim}{a} \quad \overrightarrow{OC} = \underset{\sim}{e}$		
$\therefore \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$		
$= \underset{\sim}{c} - \underset{\sim}{a}$		
Now $\overrightarrow{EB} \perp \overrightarrow{AC}$ (given)	$\frac{1}{2}$	
but $\overrightarrow{OB} = k\overrightarrow{EB}$		
$\therefore \overrightarrow{OB} \perp \overrightarrow{AC}$	$\frac{1}{2}$	
So $\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$		
$\therefore \underset{\sim}{b} \cdot (\underset{\sim}{c} - \underset{\sim}{a}) = 0$ --- ① #		
Also		
$\overrightarrow{FC} \perp \overrightarrow{AB}$ (given)	$\frac{1}{2}$	
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$		
$= \underset{\sim}{b} - \underset{\sim}{a}$		
but $\overrightarrow{OC} = k\overrightarrow{FC}$		
$\therefore \overrightarrow{OC} \perp \overrightarrow{AB}$	$\frac{1}{2}$	
So $\overrightarrow{OC} \cdot \overrightarrow{AB} = 0$		
i.e. $\underset{\sim}{c} \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = 0$ --- ② #		②
ii) Using the results from (i)		
From ① $\underset{\sim}{b} \cdot (\underset{\sim}{c} - \underset{\sim}{a}) = 0$		
$\underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{b} \cdot \underset{\sim}{a} = 0$		
$\underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{b} \cdot \underset{\sim}{a}$ --- ③		
From ② $\underset{\sim}{c} \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = 0$		
$\underset{\sim}{c} \cdot \underset{\sim}{b} - \underset{\sim}{c} \cdot \underset{\sim}{a} = 0$		
$\underset{\sim}{c} \cdot \underset{\sim}{b} = \underset{\sim}{c} \cdot \underset{\sim}{a}$ --- ④		
③ = ④ $\therefore \underset{\sim}{b} \cdot \underset{\sim}{a} = \underset{\sim}{c} \cdot \underset{\sim}{a}$	1	
$\underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{c} \cdot \underset{\sim}{a} = 0$		
$\underset{\sim}{a} \cdot (\underset{\sim}{b} - \underset{\sim}{c}) = 0$	1	②
$\therefore AD \perp CB$		
iii) the 3 perpendicular lines from each vertex		
go through a common point O.	1	①

$\therefore$ all altitudes are concurrent

MATHEMATICS EXTENSION 2 – QUESTION 16

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
METHOD 1		
a) Step 1 - Base case - Prove true for $n=2$		
$4^{n+1}+6^{n}$		
$=4^{2+1}+6^{2}$		
$=4^{3}+6^{2}$		
$=64+36$		
$=100$		
$=10(10)$ which is divisible by 10	½	Prove base case
$\therefore$ true for $n=2$		
Step 2 - Inductive hypothesis		
Assume true for $n=k$, k is even, $k\in\mathbb{Z}$		
ie $4^{k+1}+6^{k}=10P$ where P is an integer		Correct assumption
or $4^{k+1}=10P-6^{n}$	½	
Step 3 - Inductive step		
Prove true for $n=k+2$, k is even, $k\in\mathbb{Z}$		
ie prove $4^{k+3}+6^{k+2}=10Q$ where Q		
is an integer	1	Correct Inductive
$LHS=4^{k+3}+6^{k+2}$		step with
$=4^{k+1}\cdot4^{2}+6^{k}\cdot6^{2}$		correct assumption.
$=4^{2}(10P-6^{k})+6^{2}\cdot6^{k}$ By the assumption	1	
$=160P-16(6^{k})+36(6^{k})$		
$=160P+20(6^{k})$		Correct working
$=10(16P+2(6^{k}))$ where $P\in\mathbb{Z}$		to obtain
$=10Q$ $\therefore Q$ is an integer.		10Q.
Alternative for Step 3		
$LHS=4^{2}\cdot4^{k+1}+6^{2}(10P-4^{k+1})$		
$=4^{2}\cdot4^{k+1}+360P-6^{2}\cdot4^{k+1}$		
$=360P-20(4^{k+1})$		
$=10(36P-2(4^{k+1}))$ etc		
Step 4 $\therefore$ true for $n=k+2$ since it is		

true for $n=k$. Since it is true for $n=2$ then it is true for $n=2+2=4$ and so true for all even integer n, $n\geq2$.

MATHEMATICS EXTENSION 2 – QUESTION 16

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
b) (i) LHS $= (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ $= 1+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+1+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+1$ $= 3+\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}$		
Now $x+\frac{1}{x}\geq 2$ (given) $x \Rightarrow \frac{a}{b}$ $\quad x \Rightarrow \frac{a}{c}$ $\quad x \Rightarrow \frac{b}{c}$ $\frac{a}{b}+\frac{b}{a}\geq 2$ $\quad \frac{a}{c}+\frac{c}{a}\geq 2$ $\quad \frac{b}{c}+\frac{c}{b}\geq 2$	½	Correct use of inequalities
∴ From LHS $= 3+\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}$ $\geq 3+2+2+2$ $= 9$	1 ½	Correct expansion + Correct grouping to use inequalities ½ replacing with 2 and writing correct inequality
∴ $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$		
(ii) $a \rightarrow a+b$ $\quad b \Rightarrow b+c$ $\quad c \rightarrow c+a$ ∴ From (i) $(a+b+b+c+c+a)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)\geq 9$ $2(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)\geq 9$ $(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)\geq \frac{9}{2}$	1	recognising replacements.
$\frac{a}{a+b}+\frac{a}{b+c}+\frac{a}{c+a}+\frac{b}{a+b}+\frac{b}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{c}{b+c}+\frac{c}{c+a}\geq\frac{9}{2}$	1	Correct application of replacements
$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\underline{\frac{a}{a+b}+\frac{b}{a+b}}+\underline{\frac{a}{c+a}+\frac{c}{c+a}}+\underline{\frac{b}{b+c}+\frac{c}{b+c}}\geq\frac{9}{2}$ $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+1+1+1\geq\frac{9}{2}$ $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{9}{2}-3$ $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$	1	Correct grouping that led to answer

MATHEMATICS EXTENSION 2 – QUESTION 16

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c)(i) $F(p) = \lim_{t\to\infty}\int_0^t x^{p-1}e^{-x}\,dx \qquad p>0$		
$F(1) = \lim_{t\to\infty}\int_0^t x^{1-1}e^{-x}\,dx$		
$= \lim_{t\to\infty}\int_0^t e^{-x}\,dx$	①	
$= \lim_{t\to\infty}\left[-e^{-x}\right]_0^t$		
$= \lim_{t\to\infty}\left[-e^{-t}+e^{0}\right]$	①	Some half marks lost if there was no mention of "zero". Either the zero or As $t\to\infty$ $-\frac{1}{e^t}\to 0$ needs to be mentioned
$= \lim_{t\to\infty}\left(-\frac{1}{e^t}+1\right)$ As $t\to\infty$ $-\frac{1}{e^t}\to 0$		
$=0+1$		
$=1$ Write this		
(ii) $F(p+1) = \lim_{t\to\infty}\int_0^t x^{p+1-1}e^{-x}\,dx$		
$= \lim_{t\to\infty}\int_0^t x^{p}e^{-x}\,dx$		
$u = x^p \qquad v' = e^{-x}$		① correct application of Integration by parts
$u' = px^{p-1} \qquad v = -e^{-x}$		Some ½ marks lost due to no brackets – the limit is applied to BOTH.
$F(p+1) = \lim_{t\to\infty}\left\{\left[-x^pe^{-x}\right]_0^t - \int_0^t px^{p-1}(-e^{-x})\,dx\right\}$		
$= \lim_{t\to\infty}\left\{-t^pe^{-t}+0^pe^{0}+p\int_0^t x^{p-1}e^{-x}\,dx\right\}$ ①		① for correct integration and limits
As $t\to 0$ $\frac{1}{e^t}\to 0$ $\quad = \lim_{t\to\infty}\left(-t^p\times\frac{1}{e^t}\right) - \lim_{t\to\infty}p\int_0^t x^{p-1}e^{-x}\,dx$		
$= 0 + p\,F(p)$	①	correct working that led to answer.
$\therefore F(p+1) = pF(p)$		

MATHEMATICS EXTENSION 2 – QUESTION 16

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) (iii) $F(p+1) = pF(p)$		
Replace $p+1 \to n$		
$\therefore p \to n-1$		
$F(n-1+1) = (n-1)F(n-1)$	①	obtaining F(n)
$F(n) = (n-1)F(n-1)$		
$= (n-1)[(n-2)F(n-2)]$		
$= (n-1)(n-2)F(n-2)$		Students found
$= (n-1)(n-2)[(n-3)F(n-3)]$	①	this part of
		the question
$= (n-1)(n-2)(n-3)\cdots 1\cdot F(1)$		very challenging.
$= (n-1)(n-2)(n-3)\cdots 1$ from (i)		
$\therefore F(n) = (n-1)!$		