STELLA MARIS COLLEGE

2005

TRIAL HIGHER SCHOOL CERTIFICATE

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question.

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value.

Question 1 – 15 Marks

Marks

(a) Evaluate $\displaystyle\int_0^1 \frac{2}{\sqrt{1+3x}}\,dx$ 2

(b) By using integration by parts, find $\displaystyle\int x\sin 2x\,dx$ 2

(c) Use the substitution $u=\sqrt{x}$ to evaluate $\displaystyle\int_1^{25} \frac{1}{x+\sqrt{x}}\,dx$, expressing your answer in simplest exact form. 3

(d) Using $t=\tan\dfrac{x}{2}$, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$ 4

(e) (i) Find real constants A, B and C such that

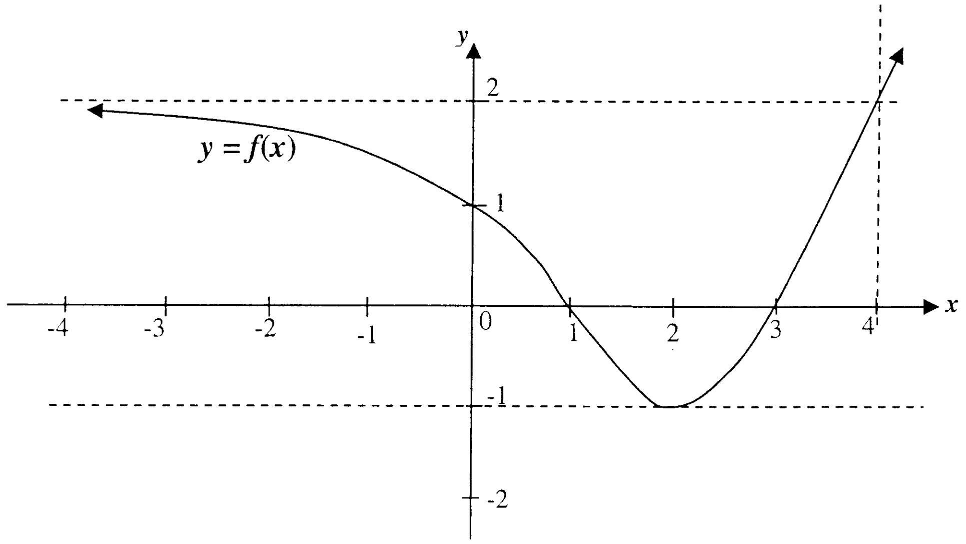
$$\frac{x+4}{x(x^2+4)} \equiv \frac{A}{x}+\frac{Bx+C}{x^2+4}$$ 2

(ii) Hence find $\displaystyle\int \frac{x+4}{x(x^2+4)}\,dx$ 2

Question 2 – 15 marks – Start on a new sheet of paper

Marks

(a) The diagram shows the graph of $y = f(x)$



On separate diagrams, sketch the following, showing essential features.

(i) $y = f(|x|)$ **2**

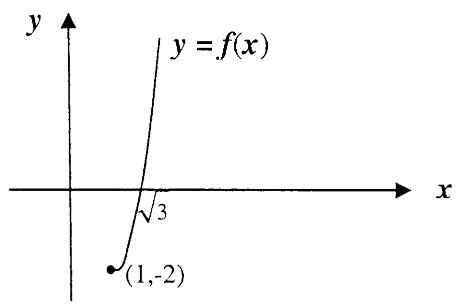
(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \left(f(x)\right)^2$ **2**

(iv) $y^2 = f(x)$ **2**

(v) $y = \ln(f(x))$ **2**

(b) The sketch shows the graph of $y = f(x)$, where $f(x) = x^3 - 3x, \quad x \geq 1.$



Copy the diagram.
On your diagram, sketch the graph of the inverse function $y = f^{-1}(x)$ showing any intercepts with the coordinate axes, the coordinates of any endpoints and the coordinates of the point of intersection of $y = f(x)$ and $y = f^{-1}(x)$ **2**

(c) Find the equation of the tangent to the curve $x^3 + y^3 - 3xy = 3$ at the point (1,2) on the curve. **3**

Question 3 – 15 marks – Start on a new sheet of paper

Marks

(a) Let $\alpha = 1 - 2i$ and $\beta = 3 + i$
Find, in the form $x + iy$,

(i) $\alpha\beta$ 1

(ii) $\dfrac{\alpha}{\beta}$ 1

(b) Let $z = -\sqrt{3} - i$

(i) Express z in modulus-argument form. 2

(ii) Hence or otherwise, find z^{10}, expressing your answer in the form $x + iy$ 2

(c) Let $z_1 = 4(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$ and $z_2 = 2(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$

(i) On an Argand diagram, draw the vectors $\vec{OA}$, $\vec{OB}$ and $\vec{OC}$ representing z_1, z_2 and $z_1 + z_2$ respectively. 2

(ii) Hence find $|\ z_1 + z_2\ |$ in simplest exact form. 2

(d) (i) On an Argand diagram shade the region where both $|z - 1 - i| \le \sqrt{2}$ and $0 \le \arg(z) \le \dfrac{\pi}{2}$ 3

(ii) Find the exact area of the shaded region. Justify your answer. 2

Question 4 – 15 marks – Start on a new sheet of paper

Marks

(a) Sketch the graph of the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ showing the intercepts on the axes, the coordinates of the focii and the equations of the directrices. **4**

(b) The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $a>b>0$, has eccentricity e.

(i) Show that the line through the focus S(ae,0) that is perpendicular to the asymptote $y=\frac{bx}{a}$ has equation $ax+by-a^2e=0$ **1**

(ii) Show that this line meets the asymptote at a point on the corresponding directrix. **3**

(c) $P(p,\frac{1}{p})$ and $Q(q,\frac{1}{q})$ are two variable points on the rectangular hyperbola $xy = 1$ such that the chord PQ passes through the point A(0,2). M is the midpoint of PQ.

(i) Show that PQ has equation $x + pqy - (p+q) = 0$. **3**
Hence deduce that $p + q = 2pq$

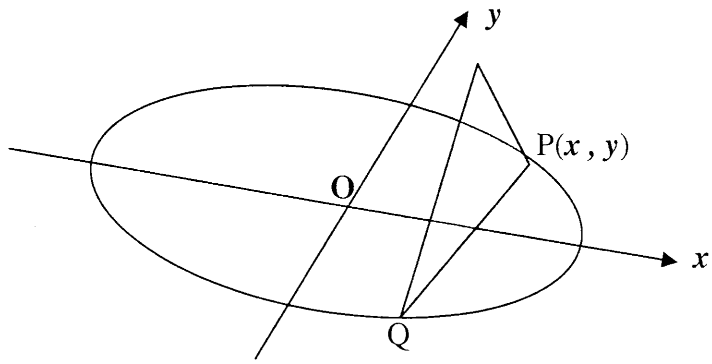
(ii) Deduce that the tangent drawn from the point A to the rectangular hyperbola touches the curve at the point (1,1) **1**

(iii) Sketch the rectangular hyperbola showing the points P,Q,A and M. **3**
Find the equation of the locus of M and state any restrictions on the domain of this locus.

Question 5 – 15 marks – Start on a new sheet of paper

Marks

(a)



The base of a certain solid is the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$

Every cross-section perpendicular to the x-axis is an equilateral triangle. The shaded cross-section is thus an equilateral triangle with base PQ.

(i) Show that the shaded cross-sectional area is given by **1**

$$A=\sqrt{3}y^2$$

(ii) Find the cross-sectional area as a function of x **1**

(iii) Find the volume of the solid. **2**

(b) (i) The polynomial equation $P(x) = 0$ has a double root α. Show that α is also a root of $P'(x) = 0$ **2**

(ii) The line $y = mx$ is a tangent to the curve $y = 3-\frac{1}{x^2}$

Explain why the equation $mx^3-3x^2+1=0$ has a double root. **1**

(iii) Hence find all values of m **2**

(c) (i) Show that the equation $x^3+13x-16=0$ has exactly one real root, $x=\alpha$, and that $1<\alpha<2$ **2**

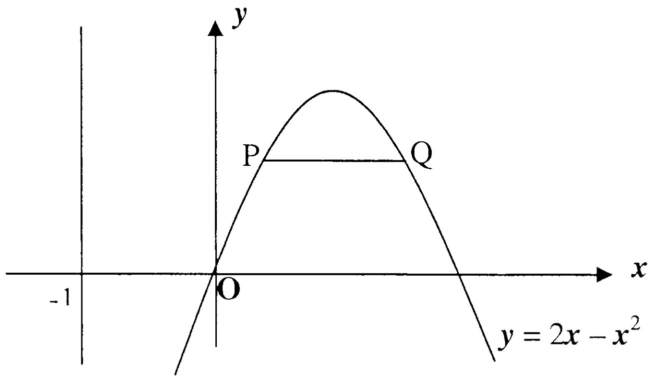
(ii) If $x=\beta$ is one of the non-real roots of the equation in part (i), show that **4**

$$-1<\textbf{Re}(\beta)<-\frac{1}{2} \quad \text{and} \quad 2\sqrt{2}<|\beta|<4$$

Question 6 – 15 marks – Start on a new sheet of paper

Marks

(a)



A solid is formed by rotating the region bounded by $y = 2x - x^2$ and the x-axis about the line $x = -1$.

(i) When the segment PQ of the region is rotated about $x = -1$, it will form an annulus. **2**
Show that the area of this annulus is given by $A = 8\pi\sqrt{1-y}$

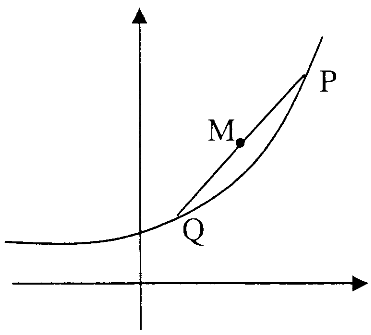
(ii) Find the volume of the solid. **3**

(b) (i) Sketch the region containing all points that simultaneously satisfy the following

$x \le 1$, $y \ge 1$ and $y \le e^x$ **2**

(ii) The region in part (i) is rotated through one complete revolution about the line x = 2
Use the method of cylindrical shells to find the volume of the solid that is formed. **4**

(c)



$P(a, e^a)$ and $Q(b, e^b)$, where a > b , are two points on the curve $y = e^x$
M is the midpoint of ***PQ.***

(i) Use the diagram to show that $e^a + e^b > 2e^{\frac{1}{2}(a+b)}$ **2**

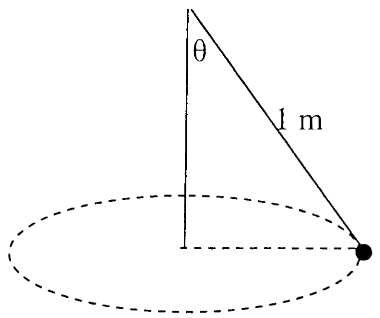
(ii) Hence show that if a > b > c > d then $e^a + e^b + e^c + e^d > 4e^{\frac{1}{4}(a+b+c+d)}$ **2**

Question 7 – 15 marks – Start on a new sheet of paper

Marks

(a) A conical pendulum consists of a mass of M kg hanging at the end of a light, inextensible string of length 1 metre attached from a fixed point O

The mass rotates in a circle and moves with a period of S seconds. Therefore, its angular velocity (ω) is $\frac{2\pi}{S}$ radians per second.
The string makes an angle θ to the vertical.



(i) Use a sketch to illustrate the forces acting on the mass. **1**

(ii) By resolving the forces on the mass, show that $S = 2\pi\sqrt{\frac{\cos\theta}{g}}$ where g is the acceleration due to gravity. **3**

(iii) The string can just support a stationary mass of 5M kg hanging vertically. **2**

Find the smallest period that the conical pendulum can have, leaving your answer in terms of ***g***.

(b) A body of mass ***m*** kg falls from rest and moves under gravity.
The air resistance on the body is ***kv*** newtons when the speed of the body is ***v*** ms^{-1}

(i) Make a neat sketch showing the forces acting on the body during its fall. **1**

(ii) Show that the equation of motion of the body is given by **1**

$$\frac{dv}{dt} = \frac{mg - kv}{m}$$

(iii) Hence, find the terminal velocity V of the system, stating your answer in terms of ***m, g*** and ***k***. **1**

(iv) Show that the time elapsed since the beginning of the motion is given by **3**

$$t = \frac{m}{k}\ln\left(\frac{mg}{mg - kv}\right)$$

(v) If the body has attained a speed equal to half the terminal speed, show that the time elapsed is equal to **3**

$$\frac{V}{g}\ln 2$$

where V is the terminal velocity.

Question 8 – 15 marks – Start on a new sheet of paper

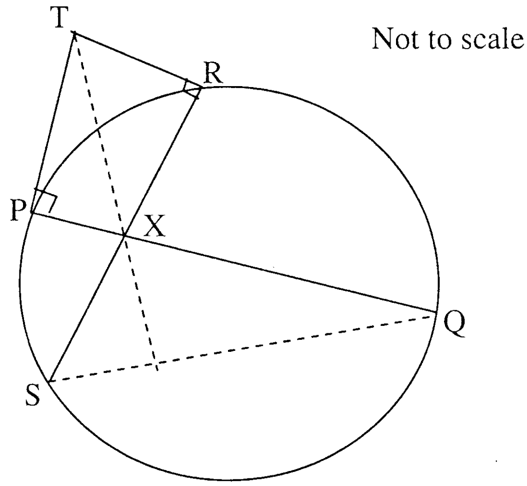
Marks

(a) (i) With the aid of a diagram, show that $\int_1^{\sqrt{u}} \frac{1}{x}\,dx < \sqrt{u} - 1$ for $u > 1$ **2**

(ii) Hence show that $0 < \ln u < 2(\sqrt{u} - 1)$, for $u > 1$ **2**

(iii) Hence show that $\frac{\ln u}{u} \to 0$ as $u \to \infty$ **1**

(b)



In the diagram, PQ and RS are two chords of the circle intersecting at X.

TR and TP are perpendicular to RS and PQ respectively.

Prove that the line through T and X is perpendicular to SQ. **3**

(c) (i) Show that $\sin\frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$ and find a similar expression for $\cos\frac{\pi}{12}$ **2**

(ii) Expand $(x - iy)^3$ **1**

(iii) Hence or otherwise, find all real numbers x and y satisfying : **4**

$$x^3 - 3xy^2 = 1$$
$$y^3 - 3x^2y = 1$$

END OF QUESTIONS

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$