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MANLY
IN OMNIBUS GLORIFICETUR DEUS

2009
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time- 5 minutes
- Working Time – 3 hours
- Write using a blue or black pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Use a separate answer booklet for each question

Total marks - 120

- Attempt Questions 1-8
- All questions are of equal value

<u>Question 1</u> – 15 marks – Use a separate writing booklet

Marks

(a) Find $\int \dfrac{\cos\theta}{\sin^3\theta}\, d\theta$ **2**

(b) Use completing the square to find $\int \dfrac{1}{\sqrt{x^2+4x+13}}\, dx$ **2**

(c) Use integration by parts to find $\int_0^1 \tan^{-1} x\, dx$ **3**

(d) Find real numbers A and B so that

$$\frac{4x^2-2x+9}{(x+1)(x^2+4)} \equiv \frac{A}{x+1}+\frac{Bx-3}{x^2+4}$$ **2**

Hence find $\int \dfrac{4x^2-2x+9}{(x+1)(x^2+4)}\, dx$ **2**

(e) Use the substitution $x = 2\sin\theta$ to find $\int_0^1 \dfrac{1}{(4-x^2)^{\frac{3}{2}}}\, dx$ **4**

<u>Question 2</u> – 15 marks – Use a separate writing booklet

Marks

(a) Let $z = 2 + i$

(i) Find z^3 in the form $a + ib$ **1**

(ii) Find $\dfrac{5}{z}$ in the form $a + ib$ **1**

(b) Find all pairs of real numbers x and y such that $(x + iy)^2 = 3 + 4i$. **2**

(c) Let $\alpha = 1 - i$

(i) Express α in modulus-argument form. **2**

(ii) Hence evaluate α^{10} in the form $a + ib$ **2**

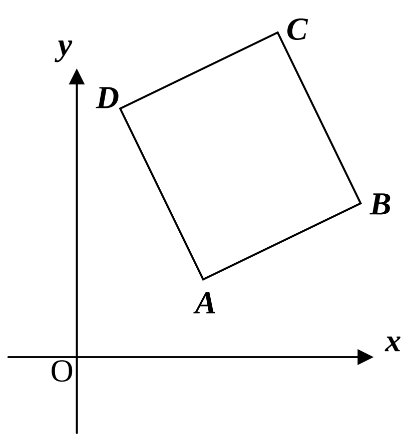
(d) On separate Argand diagrams , sketch the region satisfying the following inequalities:

(i) $1 \le |z| \le 2$ **1**

(ii) $\dfrac{\pi}{4} \le \arg(z) \le \dfrac{3\pi}{4}$ **1**

(iii) $0 \le \mathrm{Re}(z) \le 3$ ***and*** $1 \le \mathrm{Im}(z) \le 2$ **2**

(e)



On the Argand diagram shown, $ABCD$ is a square.
The points A and B represent complex numbers α and β respectively.
Hence AB represents the complex number $\beta - \alpha$.
Find, in terms of α and β,

(i) the complex number represented by AD **1**

(ii) the complex number represented by BD **1**

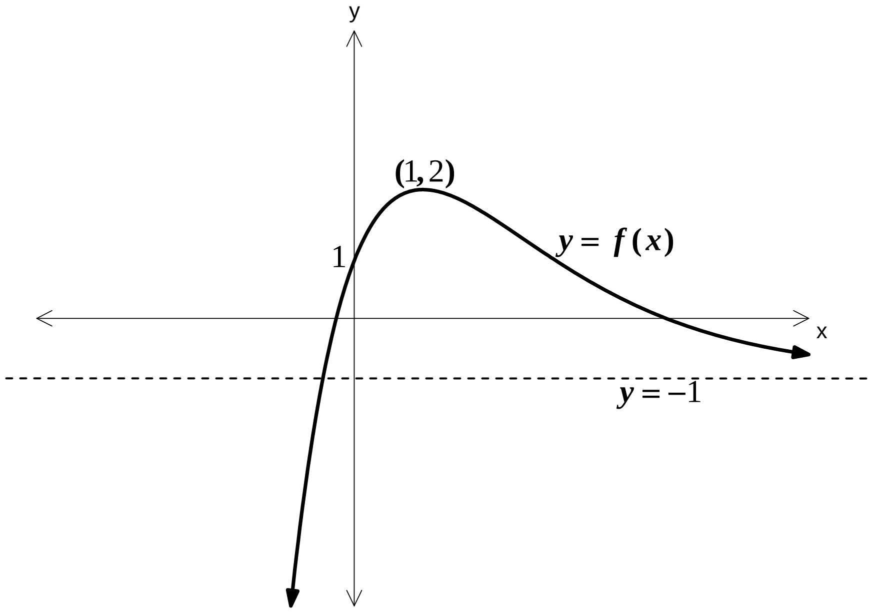
(iii) the complex number that is represented by the point C **1**

<u>Question 3</u> – 15 marks . Use a separate writing booklet

Marks

(a) Find the gradient of the tangent to the curve with equation $x^2 - 3xy + y^2 = 5$ at the point (4,1) on the curve. **2**

(b)



The diagram shows the graph of $y = f(x)$

Make separate, one-third page sketches of the following:

(i) $y = f(|x|)$ **1**

(ii) $y = \frac{1}{f(x)}$ **2**

(iii) $y = \left(f(x)\right)^2$ **2**

(iv) $y = \sqrt{f(x)}$ **2**

(v) $y = 2^{f(x)}$ **2**

(c) Consider the graph of the function $y = \frac{x^2+1}{x^2-9}$.

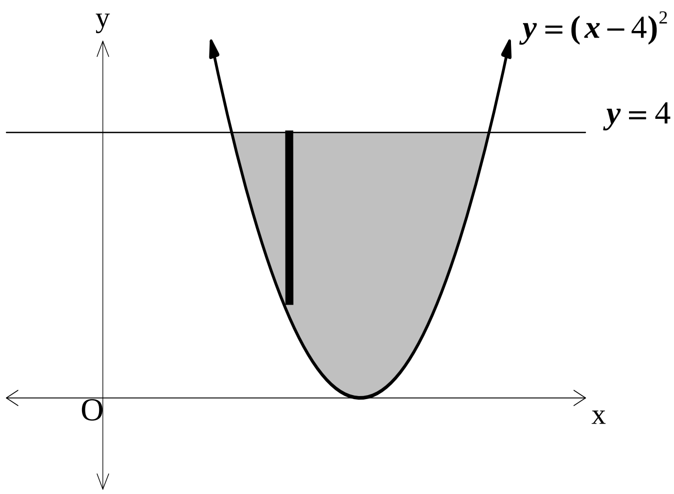
(i) Write down the equations of the vertical and horizontal asymptotes. **2**

(ii) Make a neat sketch of the graph of $y = \frac{x^2+1}{x^2-9}$ clearly showing the asymptotes and any intercepts with the coordinate axes. **2**

Question 4 – 15 marks . Use a separate writing booklet

Marks

(a)

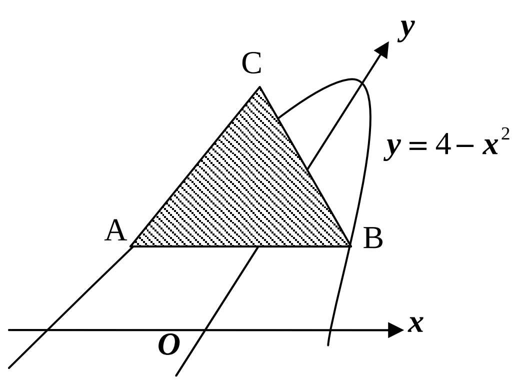


The region bound by $y=(x-4)^2$ and $y=4$ is rotated about the y-axis to form a solid.

(i) Use the method of cylindrical shells to form an integral whose value will give the volume of the solid **3**

(ii) Evaluate this integral to find the volume of the solid. **2**

(b)



The base of a solid is the region bound by $y=4-x^2$ and the x-axis. Vertical cross-sections parallel to the x-axis are equilateral triangles. A typical cross-section ABC is shown.

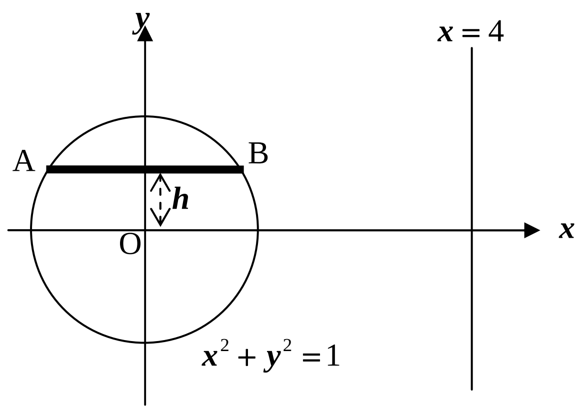
(i) Show that the area of an equilateral triangle with side s units is given by $A=\frac{\sqrt{3}}{4}s^2$ **1**

(ii) Form an integral whose value will give the volume of the solid and evaluate this integral to find the volume. **4**

<u>Question 4</u> (Continued)

Marks

(c)



A solid is formed by rotating the circle $x^2 + y^2 = 1$ about the line $x = 4$.
Any cross-section perpendicular to the line $x = 4$ will be an annulus.

(i) Consider the annulus formed when the chord AB , at $y = h$, is rotated about the line $x = 4$. **2**

Show that the area of this annulus is given by A = $16\pi\sqrt{1-h^2}$

(ii) Form an integral whose value will give the volume of the solid and evaluate it to find the volume of the solid. **3**

<u>Question 5</u> – 15 marks . Use a separate writing booklet

Marks

(a) The polynomial $\boldsymbol{P(x)} = 2\boldsymbol{x}^4 + 9\boldsymbol{x}^3 + 6\boldsymbol{x}^2 - 20\boldsymbol{x} - 24$ has a triple zero. **3**
Find all the zeros of $\mathrm{P}\boldsymbol{(x)}$.

(b) The cubic equation $\boldsymbol{x}^3 - \boldsymbol{x}^2 + 4\boldsymbol{x} - 2 = 0$ has roots α, β and γ.

(i) Find the cubic polynomial with integer coefficients whose roots are α^2, β^2 and γ^2. **3**

(ii) Find the value $\alpha^2 + \beta^2 + \gamma^2$ **1**

(iii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$ **1**

(c) It is given that $1 + 2\boldsymbol{i}$ is a root of $\boldsymbol{x}^3 + \boldsymbol{ax}^2 + \boldsymbol{bx} + 10 = 0$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are real.

(i) Find all the roots of $\boldsymbol{x}^3 + \boldsymbol{ax}^2 + \boldsymbol{bx} + 10 = 0$. **2**

(ii) Hence express $\boldsymbol{x}^3 + \boldsymbol{ax}^2 + \boldsymbol{bx} + 10$ as a product of linear and quadratic factors with real coefficients. **1**

(d) Let $\boldsymbol{z} = \cos\theta + \boldsymbol{i}\sin\theta$

(i) Find the roots of $\boldsymbol{z}^5 = -1$, expressing the complex roots in modulus-argument form. **2**

(ii) Hence show that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$ **2**

<u>Question 6</u> – 15 marks . Use a separate writing booklet

Marks

(a) Consider the ellipse whose equation is $\frac{x^2}{9}+\frac{y^2}{4}=1$.

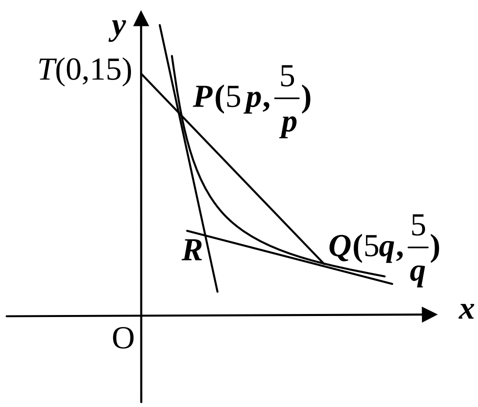
(i) Find the eccentricity of the ellipse. **1**

(ii) Write down the coordinates of the focii and the equations of the directrices. **2**

(iii) Make a neat sketch of the ellipse, clearly showing the focii and the directrices. **2**

(iv) Let P be any point on the ellipse. Show that $PS + PS' = 6$, where S and S' are the focii of the ellipse. **2**

(b)



$P(5p,\frac{5}{p})$, $p>0$ and $Q(5q,\frac{5}{q}), q>0$ are two points on the hyperbola $xy = 25$.

(i) Show that the equation of the chord PQ is $x + pqy = 5(p+q)$ **2**

(ii) Show that the equation of the tangent at P is $x + p^2y = 10p$ **2**

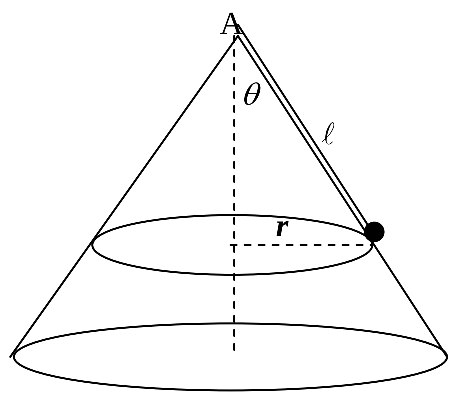
(iii) The tangents at P and Q intersect at R. Show that the coordinates of R are $(\frac{10pq}{p+q},\frac{10}{p+q})$ **2**

(iv) PQ passes through the point T(0,15). Find the equation of the locus of R. **2**

<u>Question 7</u> – 15 marks . Use a separate writing booklet

Marks

(a)



A body of mass $\boldsymbol{m}$ kg is attached by a light, inextensible string of lenth ℓ metres to the vertex of a smooth, inverted cone whose semi-vertical angle is θ°.
The body remains in contact with the surface of the cone and rotates as a conical pendulum with angular velocity ω radians per second and radius $\boldsymbol{r}$ metres..
The forces acting on the body are tension in the string ($\boldsymbol{T}$), the normal reaction of the cone ($\boldsymbol{N}$) and the gravitational force $\boldsymbol{mg}$.

(i) Copy the diagram and show the forces that are acting on the body. **1**

(ii) By resolving forces vertically and horizontally, show that **2**

$$\boldsymbol{T}\cos\theta + \boldsymbol{N}\sin\theta = \boldsymbol{mg}$$

and $$\boldsymbol{T}\sin\theta - \boldsymbol{N}\cos\theta = \boldsymbol{mr}\omega^2$$

(iii) Show that $\boldsymbol{T} = \boldsymbol{mg}\cos\theta + \boldsymbol{mr}\omega^2\sin\theta$ and find a similar expression for N. **2**

(iv) The angular velocity is increased until the body is about to lose contact with the cone. Find an expression for this value of ω in terms of $\boldsymbol{g}$, $\boldsymbol{r}$ and θ. **2**

<u>Question 7</u> (continued)

Marks

(b) A body of mass $\boldsymbol{m}$ kg is projected vertically under the influence of gravity in a medium whose resistance is $\boldsymbol{mkv}^2$ where $\boldsymbol{v}$ is its velocity and $\boldsymbol{k}$ is a constant. Its initial velocity is $\boldsymbol{U}$ metres per second.

(i) Make a neat sketch showing the forces acting on the body as it is moving up. **1**

(ii) Show that its acceleration $\ddot{\boldsymbol{x}}$ is given by $\ddot{\boldsymbol{x}} = -(\boldsymbol{g} + \boldsymbol{kv}^2)$ **1**

(iii) Show that the displacement of the body is given by $\boldsymbol{x} = \dfrac{1}{2\boldsymbol{k}} \log_e \left(\dfrac{\boldsymbol{g} + \boldsymbol{kU}^2}{\boldsymbol{g} + \boldsymbol{kv}^2} \right)$ **3**

(iv) Find the maximum height reached by the body. **1**

(v) The body now starts falling back towards the ground. **1**
Make a neat sketch showing the forces acting on the body and hence write an equation for the acceleration of the body.

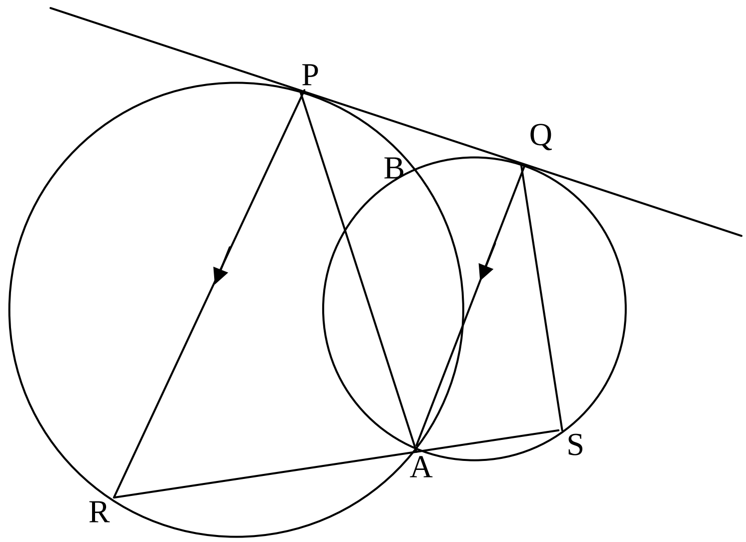
(vi) Determine the terminal velocity of the body as it falls. **1**

<u>Question 8</u> – 15 marks . Use a separate writing booklet

Marks

(a) Use mathematical induction to prove that $\mathbf{sin}(x + n\pi) = (-1)^n \mathbf{sin}\, x$ where $\boldsymbol{n}$ an integer ≥ 1 **3**

(b) Two circles intersect at A and B and a common tangent touches them at P and Q as shown.



A chord PR is drawn parallel to QA.
RA is produced to meet the other circle at S.
Copy the diagram onto your own paper.

(i) Explain why $\angle PQA = \angle QSA$. **1**

(ii) Prove that PQSR is a cyclic quadrilateral **3**

(iii) Hence prove that PA is parallel to QS **2**

Question 8 (continued)

Marks

(c) Let $I_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx$ for $n = 0,1,2,3....$

(i) By writing $\int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx$ as $\int_0^{\frac{\pi}{2}} \cos^{2n-1} x \cos x \, dx$ and using integration by parts, show that $I_n = \frac{2n-1}{2n} I_{n-1}$ for $n = 1,2,3...$ **4**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^6 x \, dx$ **2**

END OF EXAMINATION