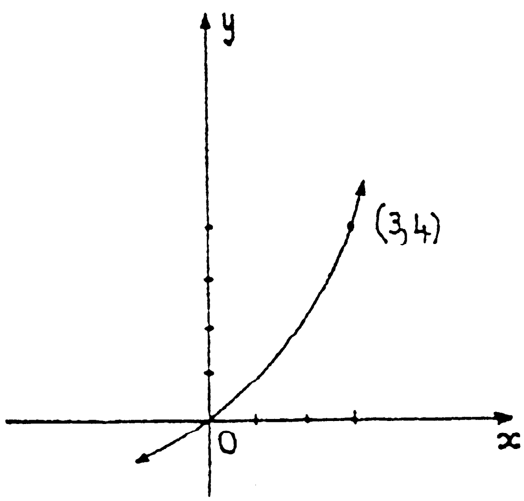
Sydney Boys High School

4 unit mathematics

Trial HSC Examination 1990

1. **(a)**

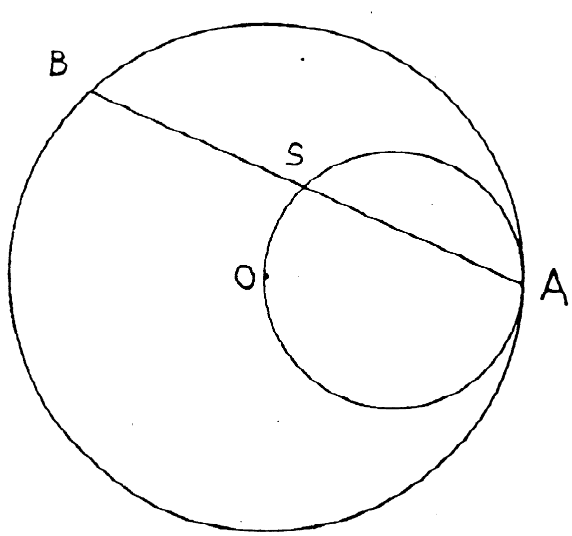


The diagram shows part of the graph $y = g(x)$. Given that the point $(3, 4)$ lies on the curve, find the value of $\int_0^3 y\ dx + \int_0^4 x\ dy$.
(b) Find the derivative of $y = 3^x$ and hence, or otherwise, find $\int 3^x\ dx$.
(c) Find the exact value of
(i) $\int_1^e x \ln x\ dx$
(ii) $\int_0^1 \frac{e^{2x}}{e^x+1}\ dx$
(d) By using a suitable substitution, or otherwise, evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{5+4\cos 2\theta}\ d\theta$ correct to 3 significant figures.

2. **(a)** Draw a neat sketch of the function $f(x) = x^2 - c^2$ where c is a positive constant. State the coordinates of its vertex and its points of intersection with both coordinate axes.
(b) Hence draw separate, neat, sketches of the following curves. Clearly indicate turning points but do not use the calculus.
(i) $y = \frac{1}{x^2-c^2}$
(ii) $y = |\frac{1}{x^2-c^2}|$
(iii) $y^2 = \frac{1}{x^2-c^2}$
(c) Find the equation of the tangent to the curve $x^3 + y^3 - 8y + 7 = 0$ at the point $(1, 2)$.

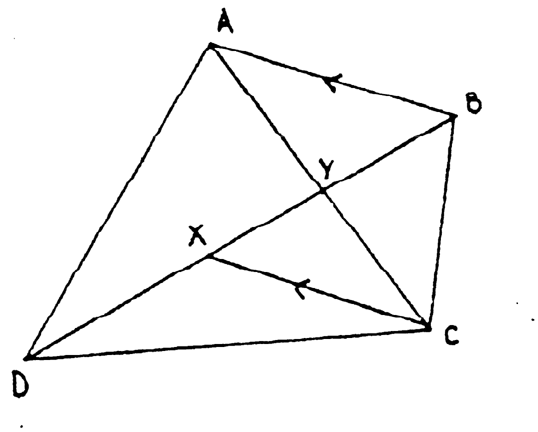
3. **(a)** If $z = 3 + i$, show that $|z|^2 = z\overline{z}$
(b) The complex number w, where $w \neq 1$, is a root of the equation $z^3 - 1 = 0$
(i) Show that $1 + w + w^2 = 0$
(ii) Show that $1 + (1 + w)^3 = 0$
(iii) Evaluate $1 + \frac{1}{w} + \frac{1}{w^2}$
(c) The quadratic equation $z^2 + (1 + i)z + k = 0$ has $1 - 2i$ as a root. Find, in the form $a + ib$, the other root and the value of k.
(d) In the Argand diagram, the point P which represents the complex number z, moves on a curve defined by the equation $|z - 1| = 2|z - i|$. Show that the curve is a circle. Find the radius of this circle and the complex number represented by its centre.

4. **(a)** Two circles touch internally at A. O is the centre of the larger circle. B is a point on the larger circle and chord AB cuts the smaller circle at S.



(i) Copy the diagram.
(ii) Prove that AB is bisected at S.
(b) The complex numbers Z_1 and Z_2 are given by $Z_1 = \frac{1+i\sqrt{3}}{2}$ and $Z_2 = i$.
(i) Represent the numbers Z_1, Z_2 and $Z_1 + Z_2$ on the Argand diagram.
(ii) By using your diagram, show that $\tan \frac{5\pi}{12} = 2 + \sqrt{3}$

(c) $ABCD$ is a quadrilateral. $CX||AB$. X lies on diagonal BD. $XC = DX$ and $\angle BXC = 2\angle ACX$.



(i) Copy the diagram.
(ii) Prove that $ABCD$ is a cyclic quadrilateral.

5. P is an arbitrary point on the ellipse $4x^2 + 5y^2 = 20$. The curve has S and S' as its foci.
(a) State the eccentricity, the coordinates of S and S' and the equations of the directrices of this curve.
(b) Sketch the curve indicating its important features.
(c) Show that
(i) the sum of the distances from P to the directrices is independent of the position of P
(ii) that the sum of the distances from P to the foci is independent of the position of P
(iii) the normal at P bisects $\angle SPS'$
(iv) if the tangent at P meets the corresponding directrix at T, then $\angle PST = 90°$.

6. (a) (i) Show that, for x real, $x(1-x) \leq \frac{1}{4}$.
(ii) The real numbers x and y satisfy the equation $x+y=1$. Show that $x^2+y^2 \geq \frac{1}{2}$.
(b) Find the general solution of the equation $\cos 3\theta = \sin\theta$, expressing your answer in radian measure.
(c) Two marksmen, A and B, fire simultaneously at a target. If A is twice as likely to hit the target as B, and if the probability that the target does get hit is $\frac{1}{2}$ find the probability of A hitting the target. (Answer correct to 3 significant figures)

7. (a) α and $-\alpha$ are both roots of the equation $x^4 + px^3 + qx + r = 0, r \neq 0$. Prove that $q^2 + p^2 r = 0$.
(b) Given that $\sin^{-1} x, \cos^{-1} x$ and $\sin^{-1}(1-x)$ are all acute,
(i) show that $sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1$
(ii) solve the equation $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(1-x)$

(c) Prove that the curve $y = x^2e^{-x}$ has a minimum turning point at $(0, 0)$ and a maximum turning point at $(2, \frac{4}{e^2})$. Sketch the curve. The roots of the equation $x^2e^{2-x} - 4 = 0$ are $x = 2, \alpha$. In terms of the root(s), determine the subdomain of values of x for which $x^2e^{2-x} - 4$ is positive. (You are not required to find the values of α)

8. **(a)** How many different ways are there of seating four married couples at a circular table with men and women in alternative positions and no wife next to her husband. (Two seating arrangements are the same if each person has the same left and right hand neighbours)
(b) By considering the integral $\int_0^3 (1 + x)^n \, dx$, prove that $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}3^{k+1} = \frac{1}{n+1}(4^{n+1} - 1)$
(c) The points A, B, C and D lie in a horizontal straight line. The line ABC, when produced, passes through the foot of a vertical tower of height h at D. E is the point at the top of the tower. From A, B and C, the angles of elevation of the top of the tower are $\theta, 2\theta, 3\theta$ respectively. $AB = a$ and $BC = b$.
(i) Draw a diagram illustrating this situation
(ii) Prove that
(α) $h = s \sin 2\theta$
(β) $a = b(1 + 2\cos 2\theta)$
(γ) $h = \frac{a}{2b}[(b + a)(3b - a)]^{\frac{1}{2}}$